INVERSE PROBLEM FOR EULER–POISSON–DARBOUX ABSTRACT DIFFERENTIAL EQUATION

A. V. Glushak and V. A. Popova

Abstract. For the nonhomogeneous Euler–Poisson–Darboux equation in a Banach space, we consider the problem of determination of a parameter on the right-hand side of the equation by the excessive final condition. This problem can be reduced to the inversion of some operator represented in a suitable form and related to the operator solving the Cauchy problem for the homogeneous Euler–Poisson–Darboux equation. As the final result, we show that the solvability of the problem considered depends on the distribution of zeroes of some analytic function. In addition, we give a simple sufficient condition ensuring the unique solvability of the problem.

Let E be a Banach space, A be a closed linear operator in E whose domain $D(A)$ is dense in E, and $t_1 > 0$. We seek a function $u(t) \in C^2([0,t_1],E)$ with values from $D(A)$ for $t \in [0,t_1]$ and a parameter $p \in E$ satisfying the relations

$$u''(t) + \frac{k}{t}u'(t) = Au(t) + f(t) + p, \quad 0 < t \le t_1, \tag{1.1}$$

$$u(0) = u_0, \quad u'(0) = 0, \quad u(t_1) = u_1, \tag{1.2}$$

where $k \in (0,2]$, $u_0,\ u_1 \in D(A)$. As for $f(t)$, we suppose that the following condition holds.

Condition 1.1. *The function $f(t)$ takes values from $D(A)$ and belongs to $C([0,t_1],E)$ together with $Af(t)$.*

Taking into account the final condition $u(t_1) = u_1$ in (1.2), we call problem (1.1), (1.2) the *inverse problem for the abstract Euler–Poisson–Darboux equation* or the *problem of determination of a parameter*. Problems of this kind relate to the theory of ill-posed problems.

The inverse problem for the equation

$$u^{(n)}(t) = Au(t) + f(t) + p \tag{1.3}$$

with $n = 1$ or $n = 2$ under various restrictions for the operator A was considered by many authors. The survey can be found in [11]. In that paper, it was also shown that the uniqueness in the inverse problem for Eq. (1.3) depends (for any $n \in N$) only on the location of eigenvalues of the operator A on the complex plane and is related to the distribution of zeroes of a function of the Mittag–Leffler type, while the solvability is much more delicate.

Problems of form (1.1), (1.2) with bounded operators A were studied in [8]. It was proved there that problem (1.1), (1.2) with a bounded operator A has a unique solution if and only if the inequality

$$\frac{1}{z}\left({}_0F_1(k/2+1/2;t_1^2z/4) - 1\right) \ne 0, \quad z \in \sigma(A),$$

holds on the spectrum $\sigma(A)$ of the operator A, where ${}_0F_1(a;z) = \sum\limits_{j=0}^{\infty} \dfrac{\Gamma(a)\,z^j}{\Gamma(a+j)\,j!}$, while $\Gamma(\cdot)$ denotes the gamma-function.

In the present paper, we establish conditions of the unique solvability of problem (1.1), (1.2) in the case of an unbounded operator A. First let us describe the set of admissible operators A.

By G_k, $k > 0$, denote the set of all operators A such that the direct Cauchy problem

$$w''(t) + \frac{k}{t}w'(t) = Aw(t), \quad w(0) = w_0, \quad w'(0) = 0 \tag{1.4}$$

is uniformly well defined; the corresponding solving operator to (1.4) (the Bessel operator-function (BOF)) is denoted by $Y_k(t)$. Thus, if $A \in G_k$, then (1.4) has a unique solution depending continuously on the initial data. Moreover, $w(t) = Y_k(t)w_0$ and

$$\|Y_k(t)\| \le Me^{\omega t}, \quad M \ge 1, \quad \omega \ge 0. \tag{1.5}$$

A criterion of the uniform well-posedness of problem (1.4) and properties of the BOF $Y_k(t)$ are given in [3]. In particular, it is proved in [3] that

(1) if $A \in G_k$ and $m > k$, then $A \in G_m$;
(2) if $A \in G_k$ and $\operatorname{Re}\lambda > \omega$, then λ^2 is a regular point for A, i.e., there exists a bounded inverse operator $R(\lambda^2) = \left(\lambda^2 I - A\right)^{-1}$ and

$$\left\|\lambda^{1-k/2} R(\lambda^2)\right\| \le \frac{M(k)\Gamma(k/2+1)}{(\operatorname{Re}\lambda - \omega)^{1+k/2}}. \tag{1.6}$$

To study problem (1.1), (1.2), we need the BOFs $Y_k(t)$ and $Y_{2-k}(t)$. Thus, we impose the following restrictions on A.

Condition 1.2. *If $k \in (0,1]$, then $A \in G_k$; if $k \in (1,2)$, then $A \in G_{2-k}$; finally, if $k = 2$, then $A \in G_0$, where G_0 is the set of generators of cosine operator-functions $C(t, A)$.*

If $k \in (0,1) \cup (1,2]$ and Conditions 1.1 and 1.2 are fulfilled, then problem (1.1), (1.2) is equivalent (see [4]) to the problem of finding a function $u(t)$ and a parameter p satisfying the relations

$$u(t) = Y_k(t)u_0 + \frac{1}{1-k}\left(t^{1-k}Y_{2-k}(t)\int_0^t \tau^k Y_k(\tau)f(\tau)\,d\tau - Y_k(t)\int_0^t \tau Y_{2-k}(\tau)f(\tau)\,d\tau\right)$$
$$+ \frac{1}{1-k}\left(t^{1-k}Y_{2-k}(t)\int_0^t \tau^k Y_k(\tau)\,p\,d\tau - Y_k(t)\int_0^t \tau Y_{2-k}(\tau)\,p\,d\tau\right), \tag{1.7}$$
$$u(t_1) = u_1.$$

It follows from these relations that problem (1.1), (1.2) is uniquely solvable if and only if the equation $B_k p = q$ is uniquely solvable for all $q \in D(A)$, where

$$B_k p = \frac{1}{1-k}\int_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)p - \tau Y_k(t_1)Y_{2-k}(\tau)p\right)\,d\tau. \tag{1.8}$$

In other words, we must determine whether the point $\lambda = 0$ is a resolvent point of the operator B_k or not. Expression (1.8) seems to be useless for this purpose; therefore, the main part of the present paper is devoted to obtaining a more convenient representation.

Theorem 1.1. *Let $x \in D(A)$, $k \in (0;1) \cup (1;2)$, and Condition 1.2 be fulfilled. Then the operator B_k defined by (1.8) can be represented as*

$$B_k x = \frac{1}{i\pi}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{1}{\lambda}\left({}_0F_1(k/2+1/2;\, t_1^2\lambda^2/4) - 1\right) R(\lambda^2)x\,d\lambda, \quad \sigma_0 > \omega \ge 0. \tag{1.9}$$

Proof. It follows from [3] that under the restrictions imposed on the operator A, one has

$$Y_k(t) = \frac{2^{k/2-1/2}\Gamma(k/2+1/2)}{i\pi t^{k/2-1/2}}\int_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \lambda^{(3-k)/2} I_{k/2-1/2}(t\lambda)R(\lambda^2)\,d\lambda, \quad \sigma_1 > \omega, \tag{1.10}$$

$$Y_{2-k}(t) = \frac{2^{1/2-k/2}\Gamma(3/2-k/2)}{i\pi t^{1/2-k/2}} \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \xi^{(1+k)/2} I_{1/2-k/2}(t\xi) R(\xi^2)\, d\xi, \quad \sigma_2 > \omega, \tag{1.11}$$

for the BOFs $Y_k(t)$ and $Y_{2-k}(t)$, where $I_\nu(z)$ is the modified Bessel function.

Transform each term in (1.8) separately, using representations (1.10) and (1.11).

Let $x \in D(A)$, $\omega < \operatorname{Re}\mu < \sigma_2 < \sigma_1$, and μ^2 be a regular point for the operator A. Then $x = R(\mu^2)y$, $y \in E$. Therefore, applying the Hilbert identity

$$(\mu^2 - \lambda^2) R(\mu^2) R(\lambda^2) = R(\lambda^2) - R(\mu^2) \tag{1.12}$$

and representation (1.10), we get

$$\int\limits_0^{t_1} \tau^k Y_k(\tau) x d\tau = \frac{2^{k/2-1/2}\Gamma(k/2+1/2)}{i\pi} \int\limits_0^{t_1} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \tau^{k/2+1/2} \lambda^{3/2-k/2} I_{k/2-1/2}(\tau\lambda) \frac{R(\lambda^2)y}{\mu^2-\lambda^2}\, d\lambda\, d\tau$$
$$- \frac{2^{k/2-1/2}\Gamma(k/2+1/2)}{i\pi} \int\limits_0^{t_1} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \tau^{k/2+1/2} \lambda^{3/2-k/2} I_{k/2-1/2}(\tau\lambda) \frac{R(\mu^2)y}{\mu^2-\lambda^2}\, d\lambda\, d\tau. \tag{1.13}$$

Taking into account the integral representation

$$I_\nu(z) = \frac{2^{1-\nu} z^\nu}{\sqrt{\pi}\Gamma(\nu+1/2)} \int\limits_0^1 (1-s^2)^{\nu-1/2} \cosh zs\; ds, \quad |\arg\ z| < \pi, \quad \operatorname{Re}\nu > -1/2, \tag{1.14}$$

for the modified Bessel function and the estimate

$$\int\limits_0^{t_1} \int\limits_{-\infty}^{+\infty} \int\limits_0^1 \left\| \tau^k (1-s^2)^{k/2-1} e^{\pm(\sigma_1+i\rho)\tau s} \frac{(\sigma_1+i\rho) R((\sigma_1+i\rho)^2) y}{\mu^2 - (\sigma_1+i\rho)^2} \right\| ds\; d\rho\; d\tau$$
$$\le \frac{M(k)\Gamma(k/2+1)\, \|y\|}{(\sigma_1-\omega)^{k/2+1}} \int\limits_0^{t_1} \int\limits_{-\infty}^{+\infty} \int\limits_0^1 \frac{\tau^k (1-s^2)^{k/2-1} e^{\pm\sigma_1 \tau s}\, |\sigma_1+i\rho|^{k/2}}{|\mu^2 - (\sigma_1+i\rho)^2|}\; ds\; d\rho\; d\tau < \infty, \quad 0 < k < 2,$$

valid by virtue of (1.6), we change the integration order in the first term on the right-hand side of (1.13). Using [10, 1.11.1.5], we get

$$\frac{2^{k/2-1/2}\Gamma(k/2+1/2)}{i\pi} \int\limits_0^{t_1} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \tau^{k/2+1/2} \lambda^{3/2-k/2} I_{k/2-1/2}(\tau\lambda) \frac{R(\lambda^2)y}{\mu^2-\lambda^2}\, d\lambda\, d\tau$$
$$= \frac{2^{k/2-1/2} t_1^{k/2+1/2} \Gamma(k/2+1/2)}{i\pi} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{1/2-k/2} I_{k/2+1/2}(t_1\lambda) \frac{R(\lambda^2)y}{\mu^2-\lambda^2}\, d\lambda. \tag{1.15}$$

To change the second term in (1.13), use the equality

$$\frac{1}{i\pi} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} I_\nu(t_1\lambda) \frac{\lambda^{1-\nu}}{\lambda^2-\lambda_0^2}\, d\lambda = \lambda_0^{-\nu} I_\nu(t_1\lambda_0), \quad 1+\nu > |\nu|, \quad \sigma_1 > \operatorname{Re}\lambda_0 > 0, \tag{1.16}$$

following from the properties of the Meyer transform [1, formula 10.1.1]. We have

$$\frac{2^{k/2-1/2}}{i\pi}\Gamma(k/2+1/2)\int\limits_0^{t_1}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\tau^{k/2+1/2}\lambda^{3/2-k/2}I_{k/2-1/2}(\tau\lambda)\frac{R(\mu^2)y}{\mu^2-\lambda^2}\,d\lambda\,d\tau$$
$$=-2^{k/2-1/2}(t_1/\mu)^{k/2+1/2}\Gamma(k/2+1/2)I_{k/2+1/2}(t_1\mu)R(\mu^2)y. \quad (1.17)$$

Since $y=(\mu^2I-A)x$, it follows from (1.15)–(1.17) that

$$\int\limits_0^{t_1}\tau^kY_k(\tau)x\,d\tau=\frac{2^{k/2-1/2}t_1^{k/2+1/2}\Gamma(k/2+1/2)}{i\pi}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{1/2-k/2}I_{k/2+1/2}(t_1\lambda)$$
$$\times\frac{R(\lambda^2)}{\mu^2-\lambda^2}(\mu^2I-A)x\,d\lambda+2^{k/2-1/2}(t_1/\mu)^{k/2+1/2}\Gamma(k/2+1/2)I_{k/2+1/2}(t_1\mu)x$$
$$=\frac{2^{k/2-1/2}}{i\pi}t_1^{k/2+1/2}\Gamma(k/2+1/2)\left(\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{1/2-k/2}I_{k/2+1/2}(t_1\lambda)R(\lambda^2)x\,d\lambda\right.$$
$$\left.+\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{1/2-k/2}I_{k/2+1/2}(t_1\lambda)\frac{x\,d\lambda}{\mu^2-\lambda^2}\right)+2^{k/2-1/2}(t_1/\mu)^{k/2+1/2}\Gamma(k/2+1/2)I_{k/2+1/2}(t_1\mu)x$$
$$=\frac{2^{k/2-1/2}}{i\pi}t_1^{k/2+1/2}\Gamma(k/2+1/2)\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{1/2-k/2}I_{k/2+1/2}(t_1\lambda)R(\lambda^2)x\,d\lambda. \quad (1.18)$$

Using (1.11), (1.18), and (1.12), rewrite the first term of (1.8) in the form

$$\frac{1}{1-k}t_1^{1-k}Y_{2-k}(t_1)\int\limits_0^{t_1}\tau^kY_k(\tau)x\,d\tau=-\frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\xi^{1/2+k/2}\lambda^{1/2-k/2}I_{1/2-k/2}(t_1\xi)$$
$$\times I_{1/2+k/2}(t_1\lambda)\frac{R(\xi^2)x}{\lambda^2-\xi^2}\,d\lambda\,d\xi+\frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\xi^{1/2+k/2}\lambda^{1/2-k/2}I_{1/2-k/2}(t_1\xi)$$
$$\times I_{1/2+k/2}(t_1\lambda)\frac{R(\lambda^2)x}{\lambda^2-\xi^2}\,d\lambda\,d\xi, \quad \sigma_2<\sigma_1. \quad (1.19)$$

The inner integral in the first term of (1.19) is calculated with the help of (1.16), while we change the integration order in the second term. By (1.14) and the residue theorem, for the second term of (1.19), we have

$$\frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\xi^{1/2+k/2}\lambda^{1/2-k/2}I_{1/2+k/2}\left(t_1\lambda\right)I_{1/2-k/2}\left(t_1\xi\right)\frac{R(\lambda^2)\,x}{\lambda^2-\xi^2}\,d\lambda\,d\xi$$
$$=\frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{1/2-k/2}I_{1/2+k/2}\left(t_1\lambda\right)R(\lambda^2)\,x\,d\lambda\int\limits_0^1\left(1-s^2\right)^{-k/2}\left(\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{\xi\,e^{-t_1s\xi}}{\lambda^2-\xi^2}\,d\xi\right.$$
$$\left.+\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{\xi\,e^{-t_1s\xi}}{\lambda^2-\xi^2}\,d\xi\right)ds=-\frac{t_1}{2i\ \cos(\pi k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{1/2-k/2}I_{1/2+k/2}(t_1\lambda)d\lambda$$

$$\times \int_0^1 (1-s^2)^{-k/2}(e^{-t_1 s\lambda}+e^{-t_1 s\lambda})\,ds = 0.$$

Hence,

$$\frac{1}{1-k} t_1^{1-k} Y_{2-k}(t_1) \int_0^{t_1} \tau^k Y_k(\tau) x\,d\tau = \frac{t_1}{2\pi\ i\cos(\pi k/2)} \int_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_{1/2-k/2}(t_1\xi) I_{1/2+k/2}(t_1\xi) R(\xi^2) x\,d\xi. \quad (1.20)$$

Thus, we obtain the desired representation for the first term of (1.8) and turn to the second term of (1.8).

As before, let $x \in D(A)$, $x = R(\mu^2)y$, $y \in E$, $\omega < \operatorname{Re}\mu < \sigma_2 < \sigma_1$, and μ^2 be a regular point of the operator A. Applying the Hilbert identity (1.12) and equality (1.11), we have

$$\frac{1}{1-k}\int_0^{t_1} \tau Y_{2-k}(\tau) x d\tau = \frac{2^{1/2-k/2}\Gamma(3/2-k/2)}{i\pi(1-k)} \int_0^{t_1}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \tau^{1/2+k/2}\xi^{1/2+k/2} I_{1/2-k/2}(\tau\xi)\frac{R(\xi^2)y}{\mu^2-\xi^2}\ d\xi\ d\tau$$
$$- \frac{2^{1/2-k/2}\Gamma(3/2-k/2)}{i\pi(1-k)} \int_0^{t_1}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \tau^{1/2+k/2}\xi^{1/2+k/2} I_{1/2-k/2}(\tau\xi)\frac{R(\mu^2)y}{\mu^2-\xi^2}\ d\xi\ d\tau. \quad (1.21)$$

As in (1.13), we can change the integration order in the first term of (1.21). Using [10, 1.11.1.5], we get

$$\frac{2^{1/2-k/2}\Gamma(3/2-k/2)}{i\pi(1-k)} \int_0^{t_1}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \tau^{1/2+k/2}\xi^{1/2+k/2} I_{1/2-k/2}(\tau\xi)\frac{R(\xi^2)y}{\mu^2-\xi^2}\ d\xi\ d\tau$$
$$= \frac{2^{1/2-k/2} t_1^{1/2+k/2}\Gamma(3/2-k/2)}{i\pi(1-k)} \int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \xi^{k/2-1/2} I_{-k/2-1/2}(t_1\xi)\frac{R(\xi^2)y}{\mu^2-\xi^2}\ d\xi$$
$$- \frac{1}{i\pi}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{R(\xi^2)y}{\xi(\mu^2-\xi^2)}\ d\xi. \quad (1.22)$$

To transform the second term of (1.21), we use (1.16) and get

$$\frac{2^{1/2-k/2}\Gamma(3/2-k/2)}{i\pi(1-k)} \int_0^{t_1}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} \tau^{1/2+k/2}\xi^{1/2+k/2} I_{1/2-k/2}(\tau\xi)\frac{R(\mu^2)y}{\mu^2-\xi^2}\ d\xi\ d\tau$$
$$= -\frac{2^{1/2-k/2} t_1^{1/2+k/2}\mu^{k/2-3/2}}{1-k}\Gamma(3/2-k/2) I_{-k/2-1/2}(t_1\mu) R(\mu^2)y + \mu^{-2}R(\mu^2)y. \quad (1.23)$$

We will need the equality

$$\frac{1}{i\pi}\int_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_\nu(t_1\lambda)\frac{\lambda^{-1-\nu}d\lambda}{\lambda^2-\lambda_0^2} = \lambda_0^{-\nu-2} I_\nu(t_1\lambda_0) - \frac{2^{-\nu}t_1^\nu}{\lambda_0^2\Gamma(1+\nu)}, \quad 2+\nu > |\nu|, \quad \sigma_2 > \operatorname{Re}\lambda_0 > 0, \quad (1.24)$$

later on. It can be obtained from (1.16) and [1, 10.1.11]. Indeed,

$$\frac{1}{i\pi}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_\nu(t_1\lambda)\frac{\lambda^{-1-\nu}d\lambda}{\lambda^2-\lambda_0^2} = \frac{1}{i\pi\lambda_0^2}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_\nu(t_1\lambda)\frac{\lambda^{1-\nu}d\lambda}{(\lambda^2-\lambda_0^2)} - \frac{1}{i\pi\lambda_0^2}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_\nu(t_1\lambda)\lambda^{-1-\nu}d\lambda$$

$$= \lambda_0^{-\nu-2}I_\nu(\lambda_0 t_1) - \frac{2^{-\nu}t_1^\nu}{\lambda_0^2\Gamma(1+\nu)}.$$

Further, since $y=(\mu^2 I-A)x$, from (1.21)–(1.24) it follows that

$$\frac{1}{1-k}\int\limits_0^{t_1}\tau Y_{2-k}(\tau)x\,d\tau = \frac{2^{1/2-k/2}t_1^{1/2+k/2}\Gamma(3/2-k/2)}{i\pi(1-k)}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\xi^{k/2-1/2}I_{-k/2-1/2}(t_1\xi)\frac{R(\xi^2)y}{\mu^2-\xi^2}\,d\xi$$

$$-\frac{1}{i\pi}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{R(\xi^2)y}{\xi(\mu^2-\xi^2)}\,d\xi - \frac{R(\mu^2)y}{\mu^2\Gamma(1/2-k/2)}$$

$$+\frac{2^{1/2-k/2}t_1^{1/2+k/2}\mu^{k/2-3/2}\Gamma(3/2-k/2)}{1-k}I_{-k/2-1/2}(t_1\mu)R(\mu^2)y$$

$$= \frac{2^{1/2-k/2}t_1^{1/2+k/2}\Gamma(3/2-k/2)}{i\pi(1-k)}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\xi^{k/2-1/2}I_{-k/2-1/2}(t_1\xi)R(\xi^2)x\,d\xi - \frac{1}{i\pi}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\frac{R(\xi^2)x}{\xi}\,d\xi. \quad (1.25)$$

Using representations (1.10), (1.25), and the Hilbert identity (1.12), we can write the second term of (1.8) in the form

$$\frac{1}{1-k}Y_k(t_1)\int\limits_0^{t_1}\tau Y_{2-k}(\tau)x\,d\tau = \frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\lambda^{3/2-k/2}\xi^{k/2-1/2}I_{k/2-1/2}(t_1\lambda)$$

$$\times I_{-k/2-1/2}(t_1\xi)\frac{R(\lambda^2)x}{\xi^2-\lambda^2}\,d\xi\,d\lambda$$

$$-\frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\lambda^{3/2-k/2}\xi^{k/2-1/2}I_{k/2-1/2}(t_1\lambda)I_{-k/2-1/2}(t_1\xi)\frac{R(\xi^2)x}{\xi^2-\lambda^2}\,d\xi\,d\lambda$$

$$-\frac{2^{k/2-1/2}t_1^{1/2-k/2}}{\pi\cos(\pi k/2)\Gamma(1/2-k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\lambda^{3/2-k/2}\xi^{-1}I_{k/2-1/2}(t_1\lambda)\frac{R(\lambda^2)x}{\xi^2-\lambda^2}\,d\xi\,d\lambda$$

$$+\frac{2^{k/2-1/2}t_1^{1/2-k/2}}{\pi\cos(\pi k/2)\Gamma(1/2-k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\lambda^{3/2-k/2}\xi^{-1}I_{k/2-1/2}(t_1\lambda)\frac{R(\xi^2)x}{\xi^2-\lambda^2}\,d\xi\,d\lambda. \quad (1.26)$$

Now let us transform the first term of (1.26). By the formula $I_{\nu-1}(z)-I_{\nu+1}(z)=\frac{2\nu}{z}I_\nu(z)$, representation (1.14) for the Bessel function, and integral [9, 2.2.3.1], we have

$$\frac{t_1}{2\pi\cos(\pi k/2)}\int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty}\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty}\lambda^{3/2-k/2}\xi^{k/2-1/2}I_{k/2-1/2}(t_1\lambda)I_{-k/2-1/2}(t_1\xi)\frac{R(\lambda^2)x}{\xi^2-\lambda^2}\,d\xi\,d\lambda$$

$$= \frac{1-k}{2\pi\cos(\pi k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\, d\lambda \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \xi^{k/2-3/2} I_{1/2-k/2}(t_1\xi) \frac{d\xi}{\xi^2-\lambda^2}$$

$$+ \frac{t_1}{2\pi\cos(\pi k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\, d\lambda \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \xi^{k/2-1/2} I_{3/2-k/2}(t_1\xi) \frac{d\xi}{\xi^2-\lambda^2}$$

$$= \frac{(1-k)\ 2^{k/2-3/2}\ t_1^{1/2-k/2}}{\pi^{3/2}\cos(\pi k/2)\Gamma(1-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x d\lambda \int\limits_0^1 (1-s^2)^{-k/2}$$

$$\times \left(\int\limits_{\sigma_2-i\infty}^{\sigma_2-i\infty} \frac{e^{t_1\xi\, s} d\xi}{\xi(\xi^2-\lambda^2)} + \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{e^{-t_1\xi\, s} d\xi}{\xi(\xi^2-\lambda^2)} \right) ds + \frac{(t_1/2)^{5/2-k/2}}{\pi^{3/2} cos(\pi k/2)\Gamma(2-k/2)}$$

$$\times \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\ d\lambda \int\limits_0^1 (1-s^2)^{1-k/2} \left(\int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{\xi e^{t_1\xi s} d\xi}{(\xi^2-\lambda^2)} + \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{\xi e^{-t_1\xi s} d\xi}{(\xi^2-\lambda^2)} \right) ds$$

$$= \frac{(1-k)(t_1/2)^{1/2-k/2} i\pi}{\sqrt{\pi}\cos(\pi k/2)\Gamma(1-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\, d\lambda \int\limits_0^1 (1-s^2)^{-k/2} \left(\frac{e^{-t_1\lambda s}}{2\lambda^2} - \frac{1}{\lambda^2} \right.$$

$$\left. - \frac{e^{-t_1\lambda s}}{2\lambda^2} \right) ds - \frac{(t_1/2)^{5/2-k/2} i\pi}{\sqrt{\pi}\cos(\pi k/2)\Gamma(2-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\ d\lambda \int\limits_0^1 (1-s^2)^{1-k/2}$$

$$\times (e^{-t_1\lambda s} - e^{-t_1\lambda s})\ ds = \frac{(k-1)2^{k/2-1/2} t_1^{-1/2-k/2}}{\sqrt{\pi}\Gamma(1-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{-1/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x$$

$$\times \int\limits_0^1 (1-s^2)^{-k/2}\ ds\ d\lambda = \frac{(k-1)2^{k/2-1/2}\ t_1^{-1/2-k/2} i\pi}{\Gamma(3/2-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{-1/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\ d\lambda. \quad (1.27)$$

The third term of (1.26) can be transformed with the help of the residue theorem. We have

$$\frac{(t_1/2)^{1/2-k/2}}{\pi\cos(\pi k/2)\Gamma(1/2-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \lambda^{3/2-k/2} \xi^{-1} I_{k/2-1/2}(t_1\lambda) \frac{R(\lambda^2)x}{\xi^2-\lambda^2}\ d\xi\ d\lambda$$

$$= \frac{(t_1/2)^{1/2-k/2}}{\pi\cos(\pi k/2)\Gamma(1/2-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{3/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \frac{1}{\xi(\xi^2-\lambda^2)}\ d\xi\ d\lambda$$

$$= \frac{(t_1/2)^{1/2-k/2}}{i\cos(\pi k/2)\Gamma(1/2-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{-1/2-k/2} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\, d\lambda. \quad (1.28)$$

Change the integration order in the second and fourth terms of (1.26) and use integral (1.16). We get

$$\frac{t_1}{2\,\pi\cos(\pi k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \lambda^{3/2-k/2} \xi^{k/2-1/2} I_{k/2-1/2}(t_1\lambda) I_{-k/2-1/2}(t_1\xi) \frac{R(\xi^2)x}{\xi^2-\lambda^2}\ d\xi\ d\lambda$$

$$= \frac{t_1}{2\,i\cos(\pi k/2)} \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} I_{k/2-1/2}(t_1\xi) I_{-k/2-1/2}(t_1\xi) R(\xi^2)\, x\, d\xi, \quad (1.29)$$

$$\frac{(t_1/2)^{1/2-k/2}}{\pi\cos(\pi k/2)\Gamma(1/2-k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \lambda^{3/2-k/2}\xi^{-1} I_{k/2-1/2}(t_1\lambda) \frac{R(\xi^2)x}{\xi^2-\lambda^2}\, d\xi\, d\lambda$$

$$= \frac{(t_1/2)^{1/2-k/2}}{i\cos(\pi k/2)\Gamma(1/2-k/2)} \int\limits_{\sigma_2-i\infty}^{\sigma_2+i\infty} \xi^{-1/2-k/2} I_{k/2-1/2}(t_1\xi) R(\xi^2) x\, d\xi. \quad (1.30)$$

Thus, taking into account formulas (1.26)–(1.30), we come to the following representation for the second term of (1.8):

$$\frac{1}{1-k} Y_k(t_1) \int\limits_0^{t_1} \tau Y_{2-k}(\tau) x\, d\tau = \frac{t_1}{2\pi\cos(\pi k/2)} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} I_{k/2-1/2}(t_1\lambda) I_{-k/2-1/2}(t_1\xi) R(\lambda^2) x\, d\lambda$$

$$- \frac{(1-k)(t_1/2)^{1/2-k/2}\Gamma(k/2+1/2)}{i\pi} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{-(1/2+k/2)} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\, d\lambda. \quad (1.31)$$

Using (1.20) and (1.31) in (1.8), we get

$$B_k x = \frac{t_1}{2\cos(\pi k/2) i} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \left(I_{1/2-k/2}(t_1\lambda) I_{1/2+k/2}(t_1\lambda) - I_{k/2-1/2}(t_1\lambda) I_{-1/2-k/2}(t_1\lambda) \right) R(\lambda^2) x\, d\lambda$$

$$+ \frac{\Gamma(k/2+1/2)(t_1/2)^{1/2-k/2}}{i\pi} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \lambda^{-(1/2+k/2)} I_{k/2-1/2}(t_1\lambda) R(\lambda^2) x\, d\lambda.$$

Application of the formula $I_\nu(z) I_{1-\nu}(z) - I_{-\nu}(z) I_{\nu-1}(z) = -\dfrac{2}{\pi z} \sin \nu\pi$ (see [13, 3.7.3]) yields

$$B_k x = \frac{1}{i\pi} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{1}{\lambda} \left(\Gamma(k/2+1/2)(t_1\lambda/2)^{1/2-k/2} I_{k/2-1/2}(t_1\lambda) - 1 \right) R(\lambda^2)\ x\ d\lambda$$

$$= \frac{1}{i\pi} \int\limits_{\sigma_1-i\infty}^{\sigma_1+i\infty} \frac{1}{\lambda} \left({}_0F_1(k/2+1/2; t_1^2\lambda^2/4) - 1 \right) R(\lambda^2)\ x\ d\lambda.$$

Theorem 1.1 is proved. □

To change the integration order in the first term of (1.13), we had to assume that $0 < k < 2$. Show now that representation (1.9) remains valid in the case $k = 2$.

Theorem 1.2. *Let $x \in D(A)$, $k = 2$, and Condition* 1.2 *be fulfilled. Then the operator B_2 defined by* (1.8) *can be represented as*

$$B_2 x = \frac{1}{i\pi} \int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{1}{\lambda} \left({}_0F_1(3/2;\ t_1^2\lambda^2/4) - 1 \right) R(\lambda^2) x\, d\lambda. \quad (1.32)$$

Proof. In the case considered, equality (1.8) takes the form

$$B_2x = \frac{1}{t_1}\int_0^{t_1} \tau\left(C(\tau)S(t_1) - C(t_1)S(\tau)\right)x\,d\tau = \frac{1}{t_1}\int_0^{t_1} \tau S(t_1-\tau)x\,d\tau.$$

As is known from [12], for $S(t)$, one has

$$S(t)x = \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} e^{\lambda t}R(\lambda^2)x\,d\lambda, \qquad x\in D(A).$$

Therefore,

$$B_2x = \frac{1}{2\pi t_1 i}\int_0^{t_1}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \tau e^{\lambda(t_1-\tau)}R(\lambda^2)x\,d\lambda\,d\tau. \tag{1.33}$$

Let $x\in D(A)$, $\omega < \operatorname{Re}\mu < \sigma_0$, and μ^2 be a regular point of the operator A. Then $x = R(\mu^2)y$, $y\in E$. Applying the Hilbert identity (1.12) in (1.33), we have

$$B_2x = \frac{1}{2\pi t_1 i}\int_0^{t_1}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \tau e^{\lambda(t_1-\tau)}\frac{R(\lambda^2)y}{\mu^2-\lambda^2}\,d\lambda\,d\tau - \frac{1}{2\pi t_1 i}\int_0^{t_1}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \tau e^{\lambda(t_1-\tau)}\frac{R(\mu^2)y}{\mu^2-\lambda^2}\,d\lambda\,d\tau. \tag{1.34}$$

Transform each term of (1.34) separately. As in the proof of Theorem 1.1, we just interchange the integration order in the first term of (1.34). So,

$$\frac{1}{2\pi\ t_1 i}\int_0^{t_1}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \tau\ e^{\lambda(t_1-\tau)}\frac{R(\lambda^2)y}{\mu^2-\lambda^2}\,d\lambda\,d\tau = \frac{1}{2\pi\ t_1 i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{e^{\lambda t_1}}{\lambda^2(\mu^2-\lambda^2)}R(\lambda^2)y\,d\lambda$$
$$- \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{R(\lambda^2)y}{\lambda(\mu^2-\lambda^2)}\,d\lambda - \frac{1}{2\pi t_1 i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{R(\lambda^2)y}{\lambda^2(\mu^2-\lambda^2)}\,d\lambda. \tag{1.35}$$

Calculating the second term of (1.34) by means of residues, we get

$$\frac{1}{2\pi t_1 i}\int_0^{t_1}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \tau e^{\lambda(t_1-\tau)}\frac{R(\mu^2)y}{\mu^2-\lambda^2}\,d\lambda\,d\tau = \frac{1}{2\pi t_1 i}\int_0^{t_1}\tau R(\mu^2)y\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{e^{\lambda(t_1-\tau)}}{\mu^2-\lambda^2}\,d\lambda\,d\tau$$
$$= \frac{R(\mu^2)y}{2t_1\mu}\left(\int_0^{t_1}\tau e^{-\mu(t_1-\tau)}d\tau - \int_0^{t_1}\tau e^{\mu(t_1-\tau)}d\tau\right) = \frac{1}{\mu^2}R(\mu^2)y - \frac{\sinh(\mu t_1)}{t_1\mu^3}R(\mu^2)y. \tag{1.36}$$

Using (1.35) and (1.36) in (1.34), we have

$$B_2x = \frac{1}{2\pi t_1 i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{e^{\lambda t_1}R(\lambda^2)y\,d\lambda}{\lambda^2(\mu^2-\lambda^2)} - \frac{1}{2\pi i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{R(\lambda^2)y\,d\lambda}{\lambda(\mu^2-\lambda^2)}$$
$$- \frac{1}{2\pi\ t_1 i}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\frac{R(\lambda^2)y\,d\lambda}{\lambda^2(\mu^2-\lambda^2)} - \frac{1}{\mu^2}R(\mu^2)y + \frac{\sinh(\mu\ t_1)}{t_1\mu^3}R(\mu^2)y. \tag{1.37}$$

Since $y = (\mu^2 I - A)x$, it follows from (1.37) that

$$B_2x = \frac{1}{2\pi t_1 i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{e^{\lambda t_1}}{\lambda^2} R(\lambda^2)x\ d\lambda + \frac{x}{2\pi t_1 i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{e^{\lambda t_1} d\lambda}{\lambda^2(\mu^2-\lambda^2)} - \frac{1}{2\pi t_1 i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{R(\lambda^2)x}{\lambda^2}\ d\lambda$$

$$- \frac{1}{2\pi i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{R(\lambda^2)x}{\lambda}\ d\lambda - \frac{x}{2\pi t_1 i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{d\lambda}{\lambda^2(\mu^2-\lambda^2)} - \frac{x}{2\pi i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{d\lambda}{\lambda(\mu^2-\lambda^2)}$$

$$- \frac{x}{\mu^2} + \frac{\sinh(\mu t_1)x}{t_1\mu^3} = \frac{1}{2\pi t_1 i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{\left(e^{\lambda t_1} - t_1\lambda - 1\right) R(\lambda^2)x\ d\lambda}{\lambda^2}$$

$$= \frac{1}{2\pi\ t_1 i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{\left(e^{\lambda t_1} - e^{-\lambda t_1} - 2t_1\lambda\right)}{\lambda^2} R(\lambda^2)x\, d\lambda = \frac{1}{i\pi}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{1}{\lambda}\left(\frac{\sinh t_1\lambda}{t_1\lambda} - 1\right) R(\lambda^2)x\, d\lambda$$

$$= \frac{1}{i\pi}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{1}{\lambda}\left({}_0F_1(3/2;\ t_1^2\lambda^2/4) - 1\right) R(\lambda^2)x\, d\lambda;$$

here we used the obvious equalities

$$\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{e^{-\lambda t_1}\, R(\lambda^2)\, x\, d\lambda}{\lambda^2} = \int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{R(\lambda^2)\, x\, d\lambda}{\lambda^2} = 0,$$

and the equality

$$\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{R(\lambda^2)\, x\, d\lambda}{\lambda} = 0 \tag{1.38}$$

valid by virtue of [6, Theorem 6.3.1]. Note that integral (1.38) in representation (1.9) guarantees the analyticity of the function $\frac{1}{\lambda}\left({}_0F_1(k/2+1/2;\ t_1^2\lambda^2/4) - 1\right)$. The theorem is proved. □

Remark 1.1. Similarly to Theorem 1.2, it can be proved that representation (1.9) for problem (1.1), (1.2) remains valid in the case where $k = 0$, $A \in G_0$, and

$$B_0x = \int\limits_0^{t_1} S(t_1 - s)\, x\, ds = \frac{1}{2\pi\, i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{1}{\lambda}\left(e^{\lambda t_1} - 1\right)\ R(\lambda^2)\, x\, d\lambda$$

$$= \frac{1}{\pi\, i}\int\limits_{\sigma_0-i\infty}^{\sigma_0+i\infty} \frac{1}{\lambda}\left({}_0F_1(1/2; t_1^2\lambda^2/4) - 1\right)\ R(\lambda^2)\, x\, d\lambda$$

for $x \in D(A)$.

Finally, consider problem (1.1), (1.2) for $k = 1$. In this case, equality (1.7) should be replaced (see [4]) by the relation

$$u(t) = Y_1(t)u_0 + \int\limits_0^t \tau\left(Z_1(t)Y_1(\tau) - Y_1(t)Z_1(\tau)\right) f(\tau)\ d\tau + \int\limits_0^t \tau\left(Z_1(t)Y_1(\tau) - Y_1(t)Z_1(\tau)\right) p\ d\tau,$$

where

$$Z_1(t)x = \lim_{m\to 1} \frac{1}{m-1} \left(Y_m(t) - t^{1-m} Y_{2-m}(t)\right) x - 2\ln 2\, Y_1(t)$$

$$= \frac{2}{\pi} \int\limits_0^1 \left(1-s^2\right)^{-1/2} \ln\left(t(1-s^2)\right) Y_0(ts)x\, ds, \qquad x \in D(A), \quad (1.39)$$

and the operator B_1 is defined by

$$B_1 p = \int\limits_0^{t_1} \tau \left(Z_1(t_1)Y_1(\tau) - Y_1(t_1)Z_1(\tau)\right) p\, d\tau. \quad (1.40)$$

Theorem 1.3. *Let $x \in D(A)$, $k = 1$, and Condition 1.2 be fulfilled. Then the operator B_1 defined by (1.40) can be represented as*

$$B_1 x = \frac{1}{i\pi} \int\limits_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} \frac{1}{\lambda} \left({}_0F_1(1; t_1^2\lambda^2/4) - 1\right) R(\lambda^2)x\, d\lambda, \quad \sigma_0 > \omega \geq 0. \quad (1.41)$$

Proof. Since representation (1.39) holds for $Z_1(t)x$ and $\lim_{m\to 1} Y_m(t)\, x = Y_1(t)\, x$ uniformly over $t \in [0,\, t_1]$ (see [2, 7]), relation (1.40) takes the form

$$B_1 x = \lim_{m\to 1} \frac{1}{m-1} \Bigg(Y_m(t_1) \int\limits_0^{t_1} \tau Y_1(\tau)x\, d\tau - t_1^{1-m} Y_{2-m}(t_1) \int\limits_0^{t_1} \tau Y_1(\tau)x\, d\tau - Y_1(t_1) \int\limits_0^{t_1} \tau Y_m(\tau)x\, d\tau$$

$$+ Y_1(t_1) \int\limits_0^{t_1} \tau^{2-m} Y_{2-m}(\tau)x\, d\tau \Bigg). \quad (1.42)$$

As in the proof of Theorem 1.1, we establish the following relations:

$$Y_m(t_1) \int\limits_0^{t_1} \tau Y_1(\tau)x\ d\tau$$

$$= \frac{2(2t_1)^{3/2-k/2}\Gamma(m/2+1/2)}{i\pi} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \lambda^{1/2-m/2} I_{m/2-1/2}(t_1\lambda) \times I_1(t_1\lambda) R(\lambda^2)x\, d\lambda, \quad (1.43)$$

$$t_1^{1-m} Y_{2-m}(t) \int\limits_0^{t_1} \tau Y_1(\tau)x\ d\tau$$

$$= \frac{2(2t_1)^{3/2-k/2}\Gamma(3/2-k/2)}{i\pi} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \lambda^{m/2-1/2} I_{1/2-m/2}(t_1\lambda) \times I_1(t_1\lambda) R(\lambda^2)x\, d\lambda, \quad (1.44)$$

$$Y_1(t_1) \int\limits_0^{t_1} \tau Y_m(\tau)x\ d\tau = \frac{2(t_1/2)^{3/2-k/2}\Gamma(m/2+1/2)}{i\pi}$$

$$\times \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \lambda^{1/2-m/2} I_{m/2-3/2}(t_1\lambda) I_0(t_1\lambda) R(\lambda^2)x\, d\lambda + \frac{m-1}{i\pi} \int\limits_{\sigma_1 - i\infty}^{\sigma_1 + i\infty} \frac{1}{\lambda} I_0(t_1\lambda) R(\lambda^2)x\, d\lambda, \quad (1.45)$$

$$Y_1(t_1)\int_0^{t_1}\tau^{2-m}Y_{2-m}(\tau)x\,d\tau$$
$$=\frac{(2t_1)^{3/2-m/2}\Gamma(3/2-k/2)}{2\,i\pi}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\lambda^{m/2-1/2}I_{3/2-m/2}(t_1\lambda)\times I_0(t_1\lambda)R(\lambda^2)x\,d\lambda. \quad (1.46)$$

Using (1.43)–(1.46) in relation (1.42), we obtain the following representation:

$$B_1x=\frac{t_1}{i\pi}\int_{\sigma_0-i\infty}^{\sigma_0+i\infty}\Bigg(I_1(t_1\lambda)\lim_{m\to 1}\frac{I_{m/2-1/2}(t_1\lambda)-I_{1/2-m/2}(t_1\lambda)}{m-1}$$
$$+I_0(t_1\lambda)\lim_{m\to 1}\frac{I_{3/2-m/2}(t_1\lambda)-I_{m/2-1/2}(t_1\lambda)}{m-1}\Bigg)R(\lambda^2)x\,d\lambda+\frac{1}{i\pi}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{1}{\lambda}I_0(t_1\lambda)R(\lambda^2)x\,d\lambda.$$

Taking into account the representation

$$K_n(z)=\frac{\pi}{2}\lim_{\nu\to n}\frac{I_{-\nu}(z)-I_\nu(z)}{\sin\ \nu\pi}$$

for the MacDonald function of integer index and the formula

$$I_\nu(z)K_{\nu+1}(z)+I_{\nu+1}(z)K_\nu(z)=\frac{1}{z}$$

(see [6, 3.71.20]), we finally get the relation

$$B_1x=\frac{t_1}{i\pi}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\Bigg(\frac{1}{t_1\,\lambda}I_0(t_1\lambda)-I_1(t_1\lambda)K_0(t_1\lambda)-I_0(t_1\lambda)K_1(t_1\lambda)\Bigg)R(\lambda^2)x\,d\lambda$$
$$=\frac{1}{i\pi}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{1}{\lambda}\left(I_0(t_1\lambda)-1\right)R(\lambda^2)x\,d\lambda=\frac{1}{i\pi}\int_{\sigma_1-i\infty}^{\sigma_1+i\infty}\frac{1}{\lambda}\left({}_0F_1(1,t_1^2\lambda^2/4)-1\right)R(\lambda^2)x\,d\lambda.$$

The theorem is proved. □

Remark 1.2. If A is a bounded operator, then the operator B_k defined by (1.8) for $k\in[0,\,1)\cup(1,\,2]$ and by (1.40) for $k=1$ has the representation (see [8])

$$B_kx=\frac{1}{2\pi\,i}\int_\gamma\frac{1}{z}\left({}_0F_1(k/2+1/2;\,t_1^2z/4)-1\right)R(z)x\,dz,\qquad x\in E,$$

where γ is a contour enveloping the spectrum $\sigma(A)$ of the operator A. Thus, if we introduce the function

$$\chi_k(\lambda)=\frac{1}{\lambda}\left({}_0F_1(k/2+1/2;\,t_1^2\lambda/4)-1\right)$$

important in the study of the solvability to the inverse problem (1.1), (1.2), then we have $B_k=\chi_k(A)$ in the case considered.

Remark 1.3. We restrict ourselves to $k\in[0,\,2]$ because, in general, the BOF $Y_{2-k}(t)$ is no longer a bounded operator for $k>2$ (see [4]), and we should impose some additional restrictions on u_0, u_1, and $f(t)$ to prove the theorems in that case.

Establish a necessary condition for the uniqueness of the solution to problem (1.1), (1.2).

Theorem 1.4. *Let $k \geq 0$ and A be a closed linear operator in E. Suppose that the inverse problem* (1.1), (1.2) *has a solution $(u(t), p)$. If this solution is unique, then no zeroes λ_j of the entire function $\chi_k(\lambda)$ are eigenvalues of the operator A.*

Proof. For $\lambda \in C$, the function $\theta(t) = \dfrac{t_1}{\lambda}\left({}_0F_1(k/2+1/2; t^2\lambda/4) - 1\right)$ is a solution to the scalar problem

$$\theta''(t) + \frac{k}{t}\theta'(t) = \lambda\theta(t) + t_1, \qquad \theta(0) = \theta'(0) = 0.$$

Assume the contrary, i.e., some zero λ_j from the countable set of zeroes of the entire function $\chi_k(\lambda)$ is an eigenvalue of A and $h_j \neq 0$ is a corresponding eigenvector. Then the pair $u_j(t) = \theta_{\lambda_j}(t)h_j$, $p = t_1 h_j$ is a nontrivial solution of the homogeneous inverse problem

$$u''(t) + \frac{k}{t}u'(t) = Au(t) + p, \quad u(0) = u'(0) = u(t_1) = 0,$$

which is a contradiction to the uniqueness of the solution of problem (1.1), (1.2). The theorem is proved. □

Note that for $k = 0$, the absence of zeroes of $\chi_0(\lambda)$ in the spectrum of the unbounded operator A is also sufficient (see [11]) for the uniqueness of the solution, but, as was shown in [5], it is not sufficient for the unique solvability of problem (1.1), (1.2). Thus, in order to obtain sufficient conditions for the unique solvability of problem (1.1), (1.2), one has to impose additional assumptions on A, u_0, u_1, and $f(t)$. The representation of the operator B_0 (see Remark 1.1) and the fact that zeroes of $\chi_0(\lambda)$ can be written out explicitly, $\lambda_j = -\left(2\pi j/t_1\right)^2$, $j = 1, 2, \ldots$, are found to be very important here. So, for further investigations in the case where $k > 0$, we need to know the distribution of zeroes of the entire function $\chi_k(\lambda)$.

Now we pass to a sufficient condition of the unique solvability of problem (1.1), (1.2), starting with the spectral properties of the BOF $Y_k(t)$.

Theorem 1.5. *Suppose that $k > 0$ and $A \in G_k$. Then the following inclusion holds for the spectra of the operators A and $Y_k(t)$:*

$$ {}_0F_1(k/2+1/2;\, t^2\,\sigma(A)/4) \subseteq \sigma(Y_k(t)). \tag{1.47}$$

Proof. Let $\lambda \in {}_0F_1(k/2+1/2;\, t^2\,\sigma(A)/4)$, i.e., there exists $\mu \in \sigma(A)$ such that $\lambda = {}_0F_1(k/2+1/2;\, t^2\,\mu/4)$. Verify that $\lambda \in \sigma(Y_k(t))$. Assuming that $\lambda \in \rho(Y_k(t))$, show that $\mu \in \rho(A)$.

The direct checking yields

$$\frac{2}{\sqrt{\mu}B(k/2,1/2)}\,(\mu\, I - A)\int_0^1 \left(1-\tau^2\right)^{k/2-1} d\tau \int_0^{t\tau} \sinh(\sqrt{\mu}(t\tau - s))\, Y_0(s)x\, ds$$
$$= {}_0F_1(k/2+1/2;\, t^2\,\mu/4)x - Y_k(t)x, \qquad x \in D(A), \tag{1.48}$$

where

$$Y_0(s)x = \frac{d}{ds}\left(s\, Y_2(s)\, x\right) = \frac{d}{ds}\left(\frac{2s}{B\left(k/2+1/2, 1-k/2\right)}\int_0^1 \left(1-\xi^2\right)^{-k/2} \xi^k\, Y_k(\xi s)x\, d\xi\right).$$

Since $D(A)$ is dense in E, it follows from (1.48) (after the integration by parts) that

$$\frac{2}{B(k/2,1/2)}\,(\mu\, I - A)\int_0^1 \left(1-\tau^2\right)^{k/2-1} d\tau \int_0^{t\tau} s\,\cosh(\sqrt{\mu}(t\tau - s))\, Y_2(s)x\, ds = \lambda x - Y_k(t)x, \qquad x \in E,$$

which implies that $\mu \in \rho(A)$. This contradiction concludes the proof. Note that for $k = 0$, inclusion (1.47) was established in [12]. □

Theorem 1.6. *Let $k \in [0,\, 2]$, the operator A obey Condition* 1.2, *and the BOF $Y_k(t)$ be such that*

$$\|Y_k(t_1)\| < 1. \tag{1.49}$$

If $u_0,\ u_1 \in D(A)$ and $f(t)$ obeys Condition 1.1, *then problem* (1.1), (1.2) *has a unique solution $(u(t), p)$ and the following estimates hold:*

$$\|u(t)\| \le M_0 \left(\|u_0\| + \|u_1\| + \max_{0\le t\le t_1} \|f(t)\| \right), \tag{1.50}$$

$$\|p\| \le M_1 \left(\|u_0\| + \|u_1\| + \|A\,u_0\| + \|A\,u_1\| + \max_{0\le t\le t_1} \|f(t)\| + \max_{0\le t\le t_1} \|A\,f(t)\| \right). \tag{1.51}$$

Proof. By virtue of inequality (1.49), the spectrum $\sigma(Y_k(t_1))$ of the operator $Y_k(t_1)$ lies inside the disk $|\lambda| \le \sigma < 1$, while inclusion (1.47) from Theorem 1.5 implies that $\lambda = 0$ does not belong to the spectrum of the operator A. Hence, there exists the bounded inverse A^{-1} defined on the whole space E. Let $p \in E$ and $k \ne 1$. From (1.8), after obvious transformations, we get

$$B_k p = \frac{1}{1-k} \int_0^{t_1} \left(t_1^{1-k} \tau^k Y_{2-k}(t_1) Y_k(\tau) p - \tau Y_k(t_1) Y_{2-k}(\tau) p \right) A\, A^{-1}\, d\tau$$

$$= \frac{t_1^{1-k} Y_{2-k}(t_1)}{1-k} \int_0^{t_1} \left(\tau^k Y_k''(\tau) + k\tau^{k-1} Y_k'(\tau) \right) A^{-1} p\, d\tau$$

$$- \frac{Y_k(t_1)}{1-k} \int_0^{t_1} \left(\left(\tau Y_{2-k}''(\tau) + Y_{2-k}'(\tau)\right) + (1-k) Y_{2-k}'(\tau) \right) A^{-1} p\, d\tau = \frac{t_1^{1-k} Y_{2-k}(t_1)}{1-k} \int_0^{t_1} \left(\tau^k Y_k'(\tau) \right)' A^{-1} p\, d\tau$$

$$- \frac{Y_k(t_1)}{1-k} \int_0^{t_1} \left(\tau Y_{2-k}'(\tau) \right)' A^{-1} p\, d\tau - Y_k(t_1) \int_0^{t_1} Y_{2-k}'(\tau)\, A^{-1} p\, d\tau = \left(\frac{t_1}{1-k} Y_{2-k}(t_1) Y_k'(t_1) \right.$$

$$\left. - \frac{t_1}{1-k} Y_{2-k}'(t_1) Y_k(t_1) - Y_{2-k}(t_1) Y_k(t_1) + Y_k(t_1) \right) A^{-1} p. \tag{1.52}$$

Using the equality

$$Y_{2-k}(t_1) Y_k'(t_1) - Y_k(t_1) Y_{2-k}'(t_1) = \frac{k-1}{t_1} \left(I - Y_{2-k}(t_1) Y_k(t_1) \right) \tag{1.53}$$

proved in [4] in the process of solving the nonhomogeneous Euler–Poisson–Darboux equation, we get

$$B_k p = \left(Y_k(t_1) - I \right) A^{-1} p. \tag{1.54}$$

Thus,

$$B_k^{-1} q = A \left(Y_k(t_1) - I \right)^{-1} q, \quad q \in D(A), \tag{1.55}$$

and, therefore, problem (1.1), (1.2) is uniquely solvable.

Now prove estimates (1.50), (1.51).

By virtue of (1.7) and (1.55), the desired value of p from the equation $B_k p = q$ is given by

$$p = A \left(Y_k(t_1) - I \right)^{-1} \left(u_1 - Y_k(t_1)\, u_0 - \int_0^{t_1} \left(t_1^{1-k} \tau^k Y_{2-k}(t_1) Y_k(\tau) - \tau Y_k(t_1) Y_{2-k}(\tau) \right) f(\tau)\, d\tau \right). \tag{1.56}$$

Since the function $f(t)$ obeys Condition 1.1 and the operator A is closed, it follows from (1.5) that

$$\left\| A \int_0^{t_1} \left(t_1^{1-k} \tau^k Y_{2-k}(t_1) Y_k(\tau) - \tau Y_k(t_1) Y_{2-k}(\tau) \right) f(\tau)\, d\tau \right\|$$

$$\leq M_1(t_1)\int\limits_0^{t_1}\|A\,f(\tau)\|\,d\tau \leq M_2(t_1)\max_{0\leq t\leq t_1}\|A\,f(t)\|\,, \quad (1.57)$$

$$\left\|\int\limits_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)-\tau\,Y_k(t_1)Y_{2-k}(\tau)\right)f(\tau)\,d\tau\right\| \leq M_1(t_1)\max_{0\leq t\leq t_1}\|f(t)\|\,. \quad (1.58)$$

Let λ be a regular point of the operator A. Since the operator $A\,(Y_k(t_1)-I)^{-1}$ is closed, we see that the operator $A\,(Y_k(t_1)-I)^{-1}\,(\lambda\,I-A)^{-1}$ is also closed. Therefore, by the equality $A\,(Y_k(t_1)-I)^{-1} = A\,(Y_k(t_1)-I)^{-1}\,(\lambda\,I-A)^{-1}\,(\lambda\,I-A)$ and estimates (1.56), (1.57), and (1.58), we have

$$\|p\| \leq M_3\left\|(\lambda\,I-A)\left(u_1-Y_k(t_1)\,u_0-\int\limits_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)-\tau Y_k(t_1)Y_{2-k}(\tau)\right)f(\tau)\,d\tau\right)\right\|$$
$$\leq M_3\left(C_1\|u_1\|+C_2\|u_0\|+C_3\max_{0\leq t\leq t_1}\|f(t)\|+C_4\|A\,u_1\|+C_5\|A\,u_0\|+C_6\max_{0\leq t\leq t_1}\|A\,f(t)\|\right)$$
$$\leq M_1\left(\|u_0\|+\|u_1\|+\|A\,u_0\|+\|A\,u_1\|+\max_{0\leq t\leq t_1}\|f(t)\|+\max_{0\leq t\leq t_1}\|A\,f(t)\|\right).$$

An estimate for $u(t)$ containing all the terms from the right-hand side of (1.51) could be obtained on the basis of (1.7) and (1.51). However, the more precise estimate (1.50) actually holds for $u(t)$.

Representation (1.7) is valid for $u(t)$. Therefore, taking (1.56) into account, we get

$$u(t) = Y_k(t)u_0+\frac{1}{1-k}\int\limits_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)-\tau\,Y_k(t_1)Y_{2-k}(\tau)\right)f(\tau)\,d\tau$$
$$+\int\limits_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)-\tau\,Y_k(t_1)Y_{2-k}(\tau)\right)A\,(Y_k(t_1)-I)^{-1}\,d\tau\,(u_1-Y_k(t_1)\,u_0$$
$$\left.-\int\limits_0^{t_1}\left(t_1^{1-k}\tau_1^k Y_{2-k}(t_1)Y_k(\tau_1)-\tau Y_k(t_1)Y_{2-k}(\tau_1)\right)f(\tau_1)\,d\tau_1\right). \quad (1.59)$$

Now let us prove the existence of a number M such that the estimate

$$\left\|\int\limits_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)-\tau Y_k(t_1)Y_{2-k}(\tau)\right)A\,(Y_k(t_1)-I)^{-1}x\,d\tau\right\| \leq M\,\|x\| \quad (1.60)$$

holds for any $t\in[0,t_1]$, $x\in D(A)$.

From (1.54), it follows that

$$B_k A\,(Y_k(t_1)-I)^{-1}\int\limits_0^{t_1}\left(t_1^{1-k}\tau^k Y_{2-k}(t_1)Y_k(\tau)-\tau Y_k(t_1)Y_{2-k}(\tau)\right)A\,(Y_k(t_1)-I)^{-1}x\,d\tau = (1-k)x.$$

This yields (1.60) by virtue of (1.5). Thus, the desired estimate for $u(t)$ follows from (1.5), (1.56), (1.58), (1.60), and (1.59) now.

The case $k=1$ is analyzed analogously. We just note that the equality

$$\left(Y_1(t_1)Z_1'(t_1)-Z_1(t_1)Y_1'(t_1)\right)x = \frac{1}{t_1}x$$

should be used instead of (1.53). The theorem is proved.

REFERENCES

1. H. Bateman and A. Erdelyi, *Tables of Integral Transforms II*, McGraw–Hill Book Co., New York (1954).
2. A. V. Glushak, "On perturbation of abstract Euler–Poisson–Darboux equation," *Math. Notes*, **60**, No. 3, 269–273 (1996).
3. A. V. Glushak, "Bessel operator function," *Dokl. Math.*, **55**, No. 1, 103–105 (1997).
4. A. V. Glushak, V. I. Kononenko, and S. D. Shmulevich, "On a singular abstract Cauchy problem," *Sov. Math.*, **30**, No. 6, 78-81 (1986).
5. A. V. Glushak and V. A. Popova, "Problem of determination of a parameter for higher-order abstract differential equation," *Transactions of Mathematical Faculty of Voronezh State University*, No. 5, 42–52 (2001).
6. E. Hille and R.S. Phillips, *Functional Analysis and Semigroups*, AMS, Providence (1957).
7. V. A. Hondo, "Perturbation problems for abstract Euler–Poisson–Darboux equation," deposited in VINITI, No. 1971–1399, Voronezh, Russia (1999).
8. V. A. Hondo, " Determination of a parameter of abstract Euler–Poisson–Darboux equation with a bounded operator," in: *Mathematical Faculy of Voronezh State University*: *Collection of students' and graduate students' works*, 59–64, (2000).
9. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Elementary Functions. I* [in Russian], Nauka, Moscow (1981).
10. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Special Functions. II* [in Russian], Nauka, Moscow (1983).
11. I. V. Tikhonov and Yu. S. Eidel'man, "Inverse problem for a differential equation in a Banach space and distribution of zeroes of a Mittag–Leffler type entire function," *Differ. Equ.*, **38**, No. 5, 669–677 (2002).
12. V. V. Vasil'ev, S. G. Krein, and S. I. Piskarev, "Semigroups of operators, cosine operator-functions, and differential equations," *J. Sov. Math.*, **54**, No. 4, 1042–1129 (1990).
13. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge Univ. Press, Cambridge (1944).