On the Properties of a Cauchy-Type Problem for an Abstract Differential Equation with Fractional Derivatives

A. V. Glushak*

Belgorod State University

Abstract—We study the relationship between the solutions of abstract differential equations with fractional derivatives and their stability with respect to the perturbation by a bounded operator. Besides, we obtain representations for the solution of an inhomogeneous equation and for an equation containing a fractional power of the generator of a cosine operator function.

Key words: *differential equation with fractional derivatives, Cauchy-type problem, Riemann–Liouville fractional derivative, fractional integral, Mittag-Leffler function.*

INTRODUCTION

In a Banach space X, consider the following Cauchy-type problem with linearly closed and densely defined operator A:

$$D^{\alpha} D^{\beta} u(t) = Au(t), \qquad t > 0, \tag{1}$$

$$\lim_{t\to 0} D^{\beta-1} u(t) = u_0, \qquad \lim_{t\to 0} D^{\alpha-1} D^{\beta} u(t) = 0, \tag{2}$$

where $\alpha \in (0,1)$, $\beta \in (0,1)$,

$$D^{\alpha} u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s)\, ds$$

is the left fractional Riemann–Liouville derivative (see [1, p. 84]), $\Gamma(\,\cdot\,)$ is the Euler gamma function, and

$$D^{\beta-1} u(t) = I^{1-\beta} u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u(s)\, ds$$

is the left fractional Riemann–Liouville integral.

The problem under consideration is particular in that there are two conditions of the form (2) even when $0 < \alpha + \beta < 1$. In the case of an ordinary differential equation of fractional order ($E = \mathbb{R}^1$ and A is the operator of multiplication by a number), this feature can be explained by the equality (see [1, formula (2.68)])

$$D^{\alpha} D^{\beta} u(t) = D^{\alpha+\beta} u(t) - \frac{D^{\beta-1} u(0) t^{-\alpha-1}}{\Gamma(-\alpha)},$$

by which Eq. (1) can be reduced to the inhomogeneous equation

$$D^{\alpha+\beta} u(t) = Au(t) + \frac{D^{\beta-1} u(0) t^{-\alpha-1}}{\Gamma(-\alpha)},$$

*E-mail: aleglu@mail.ru.

and, for the selection of the unique solution, one should use two conditions namely, the condition

$$\lim_{t\to 0} D^{\beta-1}u(t) = u_0,$$

to define the right-hand side of the equation and also another condition to obtain a Cauchy-type problem.

In particular, if $A = 0$, then the general solution of Eq. (1) is of the form

$$u(t) = t^{\beta-1}c_1 + t^{\alpha+\beta-1}c_2$$

with arbitrary constants $c_1, c_2 \in E$.

The method of reduction to an inhomogeneous equation is, apparently, less convenient, because it requires a separate study of each of the cases $0 < \alpha + \beta \le 1$ and $1 < \alpha + \beta < 2$.

In [2], it was proved that if, for $\operatorname{Re}\lambda > \omega$, the operator A has the resolvent

$$R(\lambda^{\alpha+\beta}) = (\lambda^{\alpha+\beta}I - A)^{-1},$$

which satisfies the inequality

$$\left\| \frac{d^n(\lambda^\alpha R(\lambda^{\alpha+\beta}))}{d\lambda^n} \right\| \le \frac{M\Gamma(n+\beta)}{(\operatorname{Re}\lambda - \omega)^{n+\beta}} \tag{3}$$

for all integers $n \ge 0$, then problem (1), (2) is uniformly well posed and its solution is defined by the equality

$$u(t) = \Phi(t)u_0 = D^{1-\beta}\frac{1}{2\pi i}\ \mathrm{v.\,p.}\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\lambda^{\alpha+\beta-1}R(\lambda^{\alpha+\beta})u_0\,d\lambda, \tag{4}$$

where $u_0 \in D(A)$, $\sigma = \max(\omega, 0)$, and, moreover, for any $x \in X$, the following expression is valid:

$$R(\lambda^{\alpha+\beta})x = \lambda^{-\alpha}\int_0^\infty e^{-\lambda t}\Phi(t)x\,dt. \tag{5}$$

Besides, it was established in [2] that inequality (3) is also the necessary condition for problem (1), (2) to be uniformly well-posed.

In the present paper, which is a continuation of [2], we show that if, with the operator A, problem (1), (2) is uniformly well posed, then, with the same operator, the problem

$$D^\delta v(t) = Av(t), \qquad t > 0, \tag{6}$$

$$\lim_{t\to 0} D^{\delta-1}v(t) = v_0 \in D(A) \tag{7}$$

containing the fractional derivative of order $\delta = (\alpha+\beta)/2$ is uniformly well posed as well. Note that the uniform well-posedness of problem (6), (7) and a number of other questions were studied by the author in [3]–[6].

Further, in the present paper, we prove a theorem on the preservation of uniform well-posedness of problem (1), (2) under the perturbation of the operator A by a bounded operator and also derive a formula for solving a related inhomogeneous equation. Finally, we consider a Cauchy-type problem containing a fractional power of the generator of a cosine operator function.

The results obtained are consistent with the corresponding results (see the survey [7]) for abstract differential equations of integral order for the case in which $\alpha = \beta = 1$.

Among the papers concerned with the study of abstract differential equations with fractional derivatives of order α, we note the paper [8] which deals with a weaker Cauchy problem with regularized fractional derivative and also [9] containing a criterion for uniform well-posedness of a Cauchy-type problem. The inhomogeneous differential equation of order $1+\alpha$ with regularized fractional derivative and positive operator A was studied in [10] with the help of the sum method of Da Prato and Grivard.

1. UNIFORM WELL-POSEDNESS OF PROBLEM (6), (7)

In what follows, we need the function of Mittag-Leffler type (see [11, p. 224 (Russian transl.)])

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}.$$

For $\beta = 1$, this function coincides with the Mittag-Leffler function $E_\alpha(z) = E_{\alpha,1}(z)$.

Theorem 1. *Suppose that inequality* (3) *holds and* $\delta = (\alpha + \beta)/2$. *Then problem* (6), (7) *is uniformly well posed and its solution is of the form*

$$v(t) = T_\delta(t)v_0 = \frac{2t^{\delta-1}}{\pi} \int_0^\infty \int_0^\infty s^{\beta-1} E_{1,\delta}(-ts^2) \sin\left(\frac{\pi\beta}{2} - \tau s\right) ds\, \Phi(\tau)v_0\, d\tau. \tag{8}$$

Proof. Let us verify that, for $\operatorname{Re}\mu > \omega_1$, the resolvent $R(\mu)$ of the operator A satisfies the inequality

$$\left\| \frac{d^n R(\mu^\delta)}{d\mu^n} \right\| \le \frac{M\Gamma(n+\delta)}{(\operatorname{Re}\mu - \omega_1)^{n+\delta}} \tag{9}$$

for all integers $n \ge 0$. In view of the results of [3], it follows from inequality (9) that problem (6), (7) is uniformly well posed.

Setting $\lambda = \sqrt{\mu}$ in (5), we obtain

$$R(\mu^\delta)x = \mu^{-\alpha/2} \int_0^\infty e^{-t\sqrt{\mu}} \Phi(t)x\, dt, \tag{10}$$

whence, denoting $z = t\sqrt{\mu}$, we can write

$$\begin{aligned} \frac{d^n R(\mu^\delta)x}{d\mu^n} &= \int_0^\infty \frac{d^n}{d\mu^n}(\mu^{-\alpha/2} e^{-t\sqrt{\mu}}) \Phi(t)x\, dt \\ &= \int_0^\infty t^{2n+\alpha} \left(\frac{d}{d(\mu t^2)}\right)^n ((t\sqrt{\mu})^{-\alpha} e^{-t\sqrt{\mu}}) \Phi(t)x\, dt \\ &= \int_0^\infty \frac{t^{2n+\alpha}}{2^n} \left(\frac{1}{z}\frac{d}{dz}\right)^n \left(z^{1-\alpha}\frac{e^{-z}}{z}\right) \Phi(t)x\, dt. \end{aligned} \tag{11}$$

In (11), let us use the formula (see [12, formula (1.13)])

$$\left(\frac{1}{z}\frac{d}{dz}\right)^n (z^{2p+2n} f(z)) = \sum_{j=0}^{n} \frac{2^{n-j}\binom{n}{j}\Gamma(n+1+p)}{\Gamma(j+1+p)} z^{2p+2j} \left(\frac{1}{z}\frac{d}{dz}\right)^j f(z),$$

setting

$$2p + 2n = 1 - \alpha \qquad \text{and} \qquad f(z) = z^{-1}e^{-z}.$$

We obtain

$$\frac{d^n R(\mu^\delta)x}{d\mu^n} = \sum_{j=0}^{\infty} \frac{\binom{n}{j}\Gamma(3/2 - \alpha/2)}{2^j \Gamma(j - n + 3/2 - \alpha/2)} \int_0^\infty t^{2n+\alpha} z^{1-\alpha-2n+2j} \left(\frac{1}{z}\frac{d}{dz}\right)^j \left(\frac{e^{-z}}{z}\right) \Phi(t)x\, dt. \tag{12}$$

To calculate the derivative of the function under the sign of the integral in (12), we express the function $z^{-1}e^{-z}$ in terms of the MacDonald function

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}$$

and, for $\nu = 1/2$, apply the well-known differentiation formula for the MacDonald function (see [13, p. 729]):

$$\left(\frac{1}{z}\frac{d}{dz}\right)^j (z^{-v} K_v(z)) = (-1)^j z^{-v-j} K_{v+j}(z).$$

We have

$$\begin{aligned}
&\frac{d^n R(\mu^\delta)x}{d\mu^n}\\
&\quad= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{n} \frac{\binom{n}{j}\Gamma(3/2-\alpha/2)}{(-2)^j\Gamma(j-n+3/2-\alpha/2)} \int_0^\infty t^{2n+\alpha} z^{1/2-\alpha-2n+j} K_{1/2+j}(z)\Phi(t)x\,dt\\
&\quad= \sqrt{\frac{2}{\pi}} \sum_{j=0}^{n} \frac{\binom{n}{j}\Gamma(3/2-\alpha/2)\mu^{1/4-\alpha/2-n+j/2}}{(-2)^j\Gamma(j-n+3/2-\alpha/2)} \int_0^\infty t^{j+1/2} K_{1/2+j}(t\sqrt{\mu})\Phi(t)x\,dt. \qquad (13)
\end{aligned}$$

Since (see [14, p. 217]) the function $K_\nu(z)$ has a singularity of the form $z^{-\nu}$, $\nu > 0$, at the point $z = 0$, while, for large $|z|$ and $|\arg z| < \pi/2 - \gamma$, $\gamma > 0$,

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z}\left(1 + O\left(\frac{1}{z}\right)\right),$$

it follows from (13) that, for sufficiently small $\varepsilon > 0$, the following inequality holds:

$$\begin{aligned}
\left\|\frac{d^n R(\mu^\delta)}{d\mu^n}\right\| &\le M_1 \sum_{j=0}^{n} \frac{\binom{n}{j}|\mu|^{(1-2\alpha-4n+2j)/4}}{2^j|\Gamma(j-n+3/2-\alpha/2)|}\\
&\times \left(|\mu|^{-(1+2j)/4}\int_0^{\varepsilon/\sqrt{|\mu|}} \|\Phi(t)\|\,dt + \mu^{-1/4}\int_{\varepsilon/\sqrt{|\mu|}}^\infty t^j e^{-t\,\mathrm{Re}\sqrt{\mu}}\|\Phi(t)\|\,dt\right). \qquad (14)
\end{aligned}$$

For the operator function $\Phi(t)$, it was proved in [2] that the following inequality holds:

$$\|\Phi(t)\| \le M_0 t^{\beta-1}e^{\omega t}. \qquad (15)$$

Thus, from (14), (15) we obtain

$$\begin{aligned}
\left\|\frac{d^n R(\mu^\delta)}{d\mu^n}\right\| &\le M_2 \sum_{j=0}^{n} \frac{\binom{n}{j}}{2^j|\Gamma(j-n+3/2-\alpha/2)|}\\
&\qquad\times\left(\frac{1}{|\mu|^{n+(\alpha+\beta)/2}} + \frac{\Gamma(j+\beta)}{|\mu|^{n+(\alpha-j)/2}(\mathrm{Re}\sqrt{\mu}-\omega)^{j+\beta}}\right)\\
&\le \frac{M_3}{(\mathrm{Re}\,\mu-\omega_1)^{n+(\alpha+\beta)/2}} \sum_{j=0}^{n} \frac{\binom{n}{j}\Gamma(j+\beta)}{2^j|\Gamma(j-n+3/2-\alpha/2)|}\\
&\le \frac{M_4}{(\mathrm{Re}\,\mu-\omega_1)^{n+(\alpha+\beta)/2}} \sum_{j=0}^{n} 2^{-j}\binom{n}{j}\Gamma(j+\beta)\left|\Gamma\left(n-j+\frac{\alpha}{2}-\frac{1}{2}\right)\right| \qquad (16)
\end{aligned}$$

for $\mathrm{Re}\,\mu > \omega_1 > 0$.

Further, let us show that, for all integers $n \ge 0$, the following inequality holds:

$$\sum_{j=0}^{n} 2^{-j}\binom{n}{j}\Gamma(j+\beta)\left|\Gamma\left(n-j+\frac{\alpha}{2}-\frac{1}{2}\right)\right| \le M_5\Gamma\left(n+\frac{(\alpha+\beta)}{2}\right) \qquad (17)$$

with a constant $M_5 > 0$ independent of n.

In the proof, we use the well-known asymptotic equality (see formula (4) in [15, p. 62 (Russian transl.)])

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + \frac{1}{2z}(a-b)(a+b-1) + O(z^{-2})\right).$$

Since, for sufficiently large n,

$$\sum_{j=0}^{n}\frac{\binom{n}{j}\Gamma(j+\beta)|\Gamma(n-j+\alpha/2-1/2)|}{2^j\Gamma(n+(\alpha+\beta)/2)}$$
$$\le M_6\sum_{j=0}^{n}\frac{\binom{n}{j}\Gamma(j+1)}{2^jn^{j+(\beta+1)/2}}=\frac{M_6}{n^{(\beta+1)/2}}\sum_{j=0}^{n}\frac{n(n-1)\cdots(n-j+1)}{2^jn^j}\le\frac{2M_6}{n^{(\beta+1)/2}},$$

inequality (17) obviously holds.

It follows from (16), (17) that the required inequality (9) holds, and hence problem (6), (7) is uniformly well posed.

Let us establish a formula relating the solutions of problems (1) (2) and (6), (7). Taking into account the representation of the solution of problem (6), (7) in terms of the resolvent (see [2]) and also relation (5), after elementary transformations we obtain

$$\begin{aligned}T_\delta(t)v_0&=D^{1-\delta}\frac{1}{2\pi i}\ \mathrm{v.\,p.}\int_{\sigma-i\infty}^{\sigma+i\infty}\mu^{\delta-1}e^{\mu t}R(\mu^\delta)v_0\,d\mu\\&=D^{1-\delta}\frac{1}{2\pi i}\ \mathrm{v.\,p.}\int_{\sigma-i\infty}^{\sigma+i\infty}\mu^{\beta/2-1}e^{\mu t}\,d\mu\int_0^\infty e^{-\tau\sqrt{\mu}}\Phi(\tau)v_0\,d\tau\\&=\int_0^\infty D_t^{1-\delta}\frac{1}{2\pi i}\ \mathrm{v.\,p.}\int_{\sigma-i\infty}^{\sigma+i\infty}\mu^{\beta/2-1}e^{\mu t-\tau\sqrt{\mu}}\,d\mu\,\Phi(\tau)v_0\,d\tau.\end{aligned}\tag{18}$$

As in [16, p. 236], we transform the integral

$$\frac{1}{2\pi i}\ \mathrm{v.\,p.}\int_{\sigma-i\infty}^{\sigma+i\infty}\mu^{\beta/2-1}e^{\mu t-\tau\sqrt{\mu}}\,d\mu=\frac{2}{\pi}\int_0^\infty s^{\beta-1}e^{-ts^2}\sin\left(\frac{\pi\beta}{2}-\tau s\right)ds,\tag{19}$$

and, taking into account the equality (see [1, p. 140])

$$D^{1-\delta}e^{\lambda t}=t^{\delta-1}E_{1,\delta}(\lambda t),$$

from (18), (19) we obtain the representation (8). The theorem is proved. □

Remark 1. To see that the resulting representation (8) is consistent with the well-known one, set $\alpha=\beta=1$ in (19); then the integral (19) can be evaluated and formula (8) transforms into a well-known one (see [7, p. 130]) relating the C_0-semigroup $T(t,A)$ with the cosine operator function $C(t,A)$,

$$T(t,A)v_0=\frac{1}{\sqrt{\pi t}}\int_0^\infty e^{-\tau^2/(4t)}C(\tau,A)v_0\,d\tau.$$

2. PERTURBATION OF THE OPERATOR A IN PROBLEM (1), (2) WITH A BOUNDED OPERATOR

We now pass to establishing the fact that problem (1), (2) under the perturbation of the operator A by a bounded operator P, $P\in B(X)$, is uniformly well posed. Consider the problem of determining the solution of the equation

$$D^\alpha D^\beta u(t)=(A+P)u(t),\qquad t>0,\tag{20}$$

satisfying the initial conditions (2).

Theorem 2. *Suppose that problem* (1), (2) *is uniformly well posed and* $P\in B(X)$. *Then problem* (20), (2) *is also uniformly well posed and, moreover, the following expression for the solving operator* $\Phi(t,A+P)$, *is valid*:

$$\Phi(t,A+P)x=\sum_{n=0}^{\infty}\Phi_n(t)x,\qquad x\in X,$$

where $\Phi_0(t)x = \Phi(t)x$ *(see* (4)*)*, $\Psi_0(t)x = I^\alpha \Phi(t)x$,

$$\Psi_n(t)x = \int_0^t \Psi_0(t-s)P\Psi_{n-1}(s)x\,ds, \qquad n \in \mathbb{N},$$

$$\Phi_n(t)x = D^\alpha \Psi_n(t)x = \int_0^t \Phi(t-s)P\Psi_{n-1}(s)x\,ds, \qquad n \in \mathbb{N}.$$

Proof. For $\operatorname{Re}\lambda > \omega_2 > 0$, it follows from inequality (3) that the resolvent $R(\lambda^{\alpha+\beta}, A+P)$ exists and the following equality holds:

$$R(\lambda^{\alpha+\beta}, A+P)x = R(\lambda^{\alpha+\beta}, A)\sum_{n=0}^{\infty}(PR(\lambda^{\alpha+\beta}, A))^n x. \tag{21}$$

In what follows, we shall need estimates of the norms of the operators introduced above in the statement of the theorem. In [2], it was proved that

$$\|\Phi(t)\| \le M_0 t^{\beta-1}e^{\omega t}; \tag{22}$$

hence we can easily obtain the following inequalities:

$$\|\Psi_0(t)\| \le M_1 t^{\alpha+\beta-1}e^{\omega t}, \qquad M_1 = \frac{M_0\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{23}$$

$$\|\Psi_n(t)\| \le \frac{M_1^{n+1}\|P\|^n\Gamma^{n+1}(\alpha+\beta)}{\Gamma((n+1)(\alpha+\beta))}\, t^{(n+1)(\alpha+\beta)-1}e^{\omega t}, \qquad n \in \mathbb{N}, \tag{24}$$

$$\|\Phi_n(t)\| \le \frac{MM_1^n\|P\|^n\Gamma^n(\alpha+\beta)\Gamma(\beta)}{\Gamma(n(\alpha+\beta)+\beta)}\, t^{(n+1)\beta+n\alpha-1}e^{\omega t}, \qquad n \in \mathbb{N}. \tag{25}$$

Consider the series

$$\widehat{\Psi}(t)x = \sum_{n=0}^{\infty}\Psi_n(t)x, \qquad \widehat{\Phi}(t)x = \sum_{n=0}^{\infty}\Phi_n(t)x,$$

which are absolutely convergent in view of (22)–(25) and admit the estimates

$$\begin{aligned}\|\widehat{\Psi}(t)x\| &\le M_1\Gamma(\alpha+\beta)t^{\alpha+\beta-1}e^{\omega t}\sum_{n=0}^{\infty}\frac{(M_1\|P\|\Gamma(\alpha+\beta)t^{\alpha+\beta})^n}{\Gamma(n(\alpha+\beta)+\alpha+\beta)}\\ &= M_1\Gamma(\alpha+\beta)t^{\alpha+\beta-1}e^{\omega t}E_{\alpha+\beta,\alpha+\beta}(M_1\Gamma(\alpha+\beta)\|P\|t^{\alpha+\beta})\\ &\le M_2 t^{\alpha+\beta-1}e^{\omega_3 t}, \qquad M_2, \omega_3 > 0,\end{aligned} \tag{26}$$

$$\|\widehat{\Phi}(t)x\| \le M_3 t^{\beta-1}e^{\omega_4 t}, \qquad M_3, \omega_4 > 0; \tag{27}$$

here we have used the asymptotic equality

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp(z^{1/\alpha}) + O\left(\frac{1}{|z|}\right), \qquad z \to \infty, \quad |\arg z| \le \frac{\alpha\pi}{2},$$

which is satisfied by a function of Mittag-Leffler type (see formula (22) in [11, p. 224 (Russian transl.)]).

Moreover, since the series are absolutely convergent, the following relation holds:

$$I^\alpha\widehat{\Phi}(t)x = \widehat{\Psi}(t)x. \tag{28}$$

Let us show that, for $x \in X$, $\operatorname{Re}\lambda > \omega_5 = \max(\omega_5, \omega_4)$, and $n \in \mathbb{N}$, the following relation is valid:

$$\int_0^\infty e^{-\lambda t}\Psi_n(t)x\,dt = R(\lambda^{\alpha+\beta}, A)P\int_0^\infty e^{-\lambda t}\Psi_{n-1}(t)x\,dt. \tag{29}$$

Indeed, in view of (5), after some obvious transformations, we obtain

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\Psi_n(t)x\,dt &= \int_0^\infty e^{-\lambda t}\,dt\int_0^t \Psi_0(t-s)P\Psi_{n-1}(s)x\,ds\\ &= \int_0^\infty ds\int_s^\infty e^{-\lambda t}\Psi_0(t-s)P\Psi_{n-1}(s)x\,dt\\ &= \int_0^\infty ds\int_0^\infty e^{-\lambda(s+\tau)}\Psi_0(\tau)P\Psi_{n-1}(s)x\,d\tau\\ &= \int_0^\infty e^{-\lambda s}\,ds\int_0^\infty e^{-\lambda\tau}\Psi_0(\tau)P\Psi_{n-1}(s)x\,d\tau\\ &= \int_0^\infty e^{-\lambda s}\,ds\,\lambda^{-\alpha}\int_0^\infty e^{-\lambda\tau}\Phi(\tau)P\Psi_{n-1}(s)x\,d\tau\\ &= R(\lambda^{\alpha+\beta},A)P\int_0^\infty e^{-\lambda s}\Psi_{n-1}(s)x\,ds.\end{aligned}$$

Applying relation (29) n times and using (5) we find

$$\begin{aligned}\int_0^\infty e^{-\lambda t}\Psi_n(t)x\,dt &= R(\lambda^{\alpha+\beta},A)(PR(\lambda^{\alpha+\beta},A))^{n-1}P\int_0^\infty e^{-\lambda t}\Psi_0(t)x\,dt\\ &= R(\lambda^{\alpha+\beta},A)(PR(\lambda^{\alpha+\beta},A))^{n-1}P\lambda^{-\alpha}\int_0^\infty e^{-\lambda t}\Phi(t)x\,dt\\ &= R(\lambda^{\alpha+\beta},A)(PR(\lambda^{\alpha+\beta},A))^{n}\end{aligned}$$

and hence, by (21), we have

$$\int_0^\infty e^{-\lambda t}\widehat{\Psi}(t)x\,dt = R(\lambda^{\alpha+\beta},A+P)x. \tag{30}$$

Taking (28) into account and using (30), we obtain

$$R(\lambda^{\alpha+\beta},A+P)x = \int_0^\infty e^{-\lambda t}I^\alpha\widehat{\Phi}(t)x\,dt = \lambda^{-\alpha}\int_0^\infty e^{-\lambda t}\widehat{\Phi}(t)x\,dt$$

and, finally,

$$\lambda^\alpha R(\lambda^{\alpha+\beta},A+P)x = \int_0^\infty e^{-\lambda t}\widehat{\Phi}(t)x\,dt. \tag{31}$$

Relations (31) and (27) imply the inequality

$$\left\|\frac{d^n(\lambda^\alpha R(\lambda^{\alpha+\beta},A+P))}{d\lambda^n}\right\| \le \frac{M_0\Gamma(n+\beta)}{(\operatorname{Re}\lambda-\omega_5)^{n+\beta}}$$

for all integers $n\ge 0$ and $\operatorname{Re}\lambda>\omega_5$; by Theorem 4 from [2], this inequality is a sufficient condition for problem (20), (2) to be uniformly well posed.

Finally, in view of Theorem 2 from [2] and the uniqueness of the Laplace transform, it follows from (31) that

$$\Phi(t,A+P)x = \widehat{\Phi}(t)x = \sum_{n=0}^\infty \Phi_n(t)x.$$

The theorem is proved. □

Remark 2. To verify the correctness of the result obtained, we note that, by setting $\alpha=\beta=1$ and replacing the operator functions $\Phi_0(t)$ and $\Psi_0(t)$, respectively, by the cosine operator function $C(t,A)$ and the sine operator function

$$S(t,A) = \int_0^t C(\tau,A)\,d\tau,$$

the assertion of Theorem 2 becomes the corresponding result of [17] dealing with the perturbation of the generator of a cosine operator function by a bounded operator.

3. INHOMOGENEOUS EQUATION

Let us now establish a formula for solving a Cauchy-type problem for the inhomogeneous equation

$$D^{\alpha} D^{\beta} u(t) = Au(t) + f(t), \qquad t > 0. \tag{32}$$

Moreover, we shall use the operator function $\Phi(t)$ (see (4)) and also the operator function $\Psi_0(t) = I^{\alpha}\Phi(t)$ introduced in Theorem 2 and now renamed $\Psi(t)$.

Note that the function $\Psi(t)u_1$, $u_1 \in D(A)$, satisfies Eq. (1) and the initial conditions

$$\lim_{t\to 0} D^{\beta-1}\Psi(t)u_1 = 0, \qquad \lim_{t\to 0} D^{\alpha-1} D^{\beta}\Psi(t)u_1 = u_1. \tag{33}$$

This fact can easily be established using the corresponding properties of the function $\Phi(t)u_1$ and Theorem 2.5 [1] on the semigroup property of the fractional integrodifferentiation operation.

Theorem 3. *Suppose that problem* (1), (2) *is uniformly well posed, $u_0 \in D(A)$, and the function $f(t) \in C((0,\infty), X)$, absolutely integrable at zero, assumes values in $D(A)$, satisfies the inclusion $Af(t) \in (C(0,\infty), X)$, and is absolutely integrable at zero. Then problem* (32), (2) *has a unique solution given by the equality*

$$u(t) = \Phi(t)u_0 + \int_0^t \Psi(t-s)f(s)\,ds. \tag{34}$$

Proof. It suffices to prove that, under our assumptions, the function

$$w(t) = \int_0^t \Psi(t-s)f(s)\,ds$$

satisfies Eq. (32) and the zero initial conditions (2). For $t > 0$, after elementary transformations, we obtain

$$\begin{aligned} D^{\beta}w(t) &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t (t-\tau)^{-\beta}\,d\tau \int_0^{\tau} \Psi(\tau-s)f(s)\,ds \\ &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t ds \int_s^t (t-\tau)^{-\beta}\Psi(\tau-s)f(s)\,d\tau \\ &= \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t ds \int_0^{t-s} (t-s-\xi)^{-\beta}\Psi(\xi)f(s)\,d\xi \\ &= \frac{1}{\Gamma(1-\beta)}\lim_{s\to t}\int_0^{t-s} (t-s-\xi)^{-\beta}\Psi(\xi)f(s)\,d\xi \\ &\qquad + \frac{1}{\Gamma(1-\beta)}\int_0^t ds\,\frac{d}{dt}\int_0^{t-s} (t-s-\xi)^{-\beta}\Psi(\xi)f(s)\,d\xi \\ &= \lim_{t-s\to 0} D^{\beta-1}\Psi(t-s)f(s) + \int_0^t D^{\beta}\Psi(t-s)f(s)\,ds \\ &= \int_0^t D^{\beta}\Psi(t-s)f(s)\,ds. \end{aligned} \tag{35}$$

Further, by analogy with the foregoing, we have

$$\begin{aligned} D^{\alpha}D^{\beta}w(t) &= \frac{1}{\Gamma(1-\alpha)}\lim_{s\to t}\int_0^{t-s} (t-s-\xi)^{-\alpha} D^{\beta}\Psi(\xi)f(s)\,d\xi \\ &\qquad + \frac{1}{\Gamma(1-\alpha)}\int_0^t ds\,\frac{d}{dt}\int_0^{t-s} (t-s-\xi)^{-\alpha} D^{\beta}\Psi(\xi)f(s)\,d\xi \end{aligned}$$

$$= \lim_{t-s\to 0} D^{\alpha-1}D^{\beta}\Phi(t-s)f(s) + \int_0^t D^{\alpha}D^{\beta}\Psi(t-s)f(s)\,ds$$
$$= f(t) + A\int_0^t \Psi(t-s)f(s)\,ds = f(t) + Aw(t);$$

therefore, the function $w(t)$ satisfies Eq. (32).

We now verify that the function $w(t)$ satisfies the zero initial conditions (2). We have

$$\lim_{t\to 0} D^{\beta-1}w(t) = \frac{1}{\Gamma(1-\beta)}\lim_{t\to 0}\int_0^t (t-\tau)^{-\beta}\,d\tau \int_0^{\tau}\Psi(\tau-s)f(s)\,ds. \tag{36}$$

For $t \in [0,1]$, inequality (23) implies

$$\left\|\int_0^t (t-\tau)^{-\beta}\,d\tau\int_0^{\tau}\Psi(\tau-s)f(s)\,ds\right\|$$
$$\le M_1\int_0^t \|f(s)\|\,ds\int_s^t (t-\tau)^{-\beta}(\tau-s)^{\alpha+\beta-1}e^{\omega(\tau-s)}\,d\tau$$
$$\le M_4\int_0^t (t-s)^{\alpha}\|f(s)\|\,ds. \tag{37}$$

It follows from (36), (37) that the function $w(t)$ satisfies the condition

$$\lim_{t\to 0} D^{\beta-1}w(t) = 0.$$

To verify the validity of the second initial condition in (2), we use (35), obtaining

$$D^{\alpha-1}D^{\beta}w(t) = \frac{1}{\Gamma(1-\alpha)}\int_0^t (t-\tau)^{-\alpha}\,d\tau\int_0^{\tau} D^{\beta}\Psi(\tau-s)f(s)\,ds$$
$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t ds\int_s^t (t-\tau)^{-\alpha}D^{\beta}\Psi(\tau-s)f(s)\,d\tau$$
$$= \frac{1}{\Gamma(1-\alpha)}\int_0^t ds\int_0^{t-s}(t-s-\xi)^{-\alpha}D^{\beta}\Psi(\xi)f(s)\,d\xi$$
$$= \int_0^t D^{\alpha-1}D^{\beta}I^{\alpha}\Phi(t-s)f(s)\,ds = \int_0^t D^{\beta-1}\Phi(t-s)f(s)\,ds;$$

hence, by analogy with (37), for $t \in [0,1]$ we find

$$\|D^{\alpha-1}D^{\beta}w(t)\| \le M_5\int_0^t \|f(s)\|ds$$

and, therefore,

$$\lim_{t\to 0} D^{\alpha-1}D^{\beta}w(t) = 0$$

The theorem is proved. □

Remark 3. Setting $\alpha = \beta = 1$ and replacing the operator functions $\Phi(t)$ and $\Psi(t)$, respectively, by the cosine operator function $C(t, A)$ and the sine operator function $S(t, A)$, we find that formula (34) becomes the well-known formula for solving second-order inhomogeneous equations with the generator of a cosine operator function.

4. THE CAUCHY-TYPE PROBLEM (1), (2) WITH AN OPERATOR A WHICH IS A FRACTIONAL POWER OF THE GENERATOR OF A COSINE OPERATOR FUNCTION

Suppose that E is a complex Banach space and A is the generator of a uniformly bounded cosine operator function. Then we can find the positive fractional power of the operator $-A$ (see, for example, [18, p. 96 (Russian transl.)]):

$$-(-A)^\gamma f = \frac{\sin\gamma\pi}{\pi}\int_0^\infty \lambda^{\gamma-1}(\lambda I - A)^{-1}Af\,d\lambda, \tag{38}$$

where $\gamma\in(0,1)$, $f\in D(A)$.

Moreover, if $g\in E$, $\mu>0$, then the resolvent of the operator $-(-A)^\gamma$, denoted by A_γ in what follows, can be expressed as

$$(\mu I - A_\gamma)^{-1}g = \frac{\sin\gamma\pi}{\pi}\int_0^\infty \frac{\lambda^\gamma(\lambda I - A)^{-1}g\,d\lambda}{\mu^2 - 2\mu\lambda^\gamma\cos\gamma\pi + \lambda^{2\gamma}}. \tag{39}$$

Let us show that, with the operator A_γ, where $\gamma=(\alpha+\beta)/2$, the Cauchy-type problem

$$D^\alpha D^\beta u(t) = A_\gamma u(t), \qquad t>0, \tag{40}$$

$$\lim_{t\to0} D^{\beta-1}u(t) = u_0, \qquad \lim_{t\to0} D^{\alpha-1}D^\beta u(t) = 0 \tag{41}$$

is uniformly well posed.

Theorem 4. *Suppose that $\gamma=(\alpha+\beta)/2$, A is the generator of a uniformly bounded cosine operator function, and the operator A_γ is defined by relation* (38). *Then the Cauchy-type problem* (40), (41) *is uniformly well posed.*

Proof. As already noted in the Introduction, the criterion for the uniform well-posedness of problem (40), (41) is that, for $\mu>0$, the resolvent

$$(\mu I - A_\gamma)^{-1}$$

satisfies an inequality of type (3), which in our case, is of the form

$$\left\|\frac{d^n(\mu^\alpha(\mu^{\alpha+\beta}I - A_\gamma)^{-1})}{d\mu^n}\right\| \le \frac{M\Gamma(n+\beta)}{\mu^{n+\beta}}. \tag{42}$$

Using the representation (39), we can establish estimate (42). Setting

$$R(\lambda) = (\lambda I - A)^{-1}$$

and making the change of variable from (39), we obtain

$$\begin{aligned}\mu^\alpha(\mu^{\alpha+\beta}I - A_\gamma)^{-1}g &= \frac{\mu^{2-\beta}\sin\gamma\pi}{\gamma\pi}\int_0^\infty \frac{s^{1/\gamma}R(\mu^2 s^{1/\gamma})g\,ds}{s^2 - 2s\cos\gamma\pi + 1}\\ &= \frac{\sin\gamma\pi}{\gamma\pi}\int_0^\infty \frac{s^{\beta/(2\gamma)}p^{2-\beta}R(p^2)g\,ds}{s^2 - 2s\cos\gamma\pi + 1},\end{aligned}$$

where

$$p = \mu s^{1/(2\gamma)}$$

and, therefore,

$$\begin{aligned}&\frac{d^n(\mu^\alpha(\mu^{\alpha+\beta}I - A_\gamma)^{-1}g)}{d\mu^n}\\ &\quad= \frac{\sin\gamma\pi}{\gamma\pi}\int_0^\infty (s^{(\beta+n)/(2\gamma)}s^2 - 2s\cos\gamma\pi + 1)\frac{d^n}{dp^n}(p^{1-\beta}(pR(p^2)g))\,ds.\end{aligned} \tag{43}$$

Using the Leibniz formula and the inequality

$$\left\|\frac{d^n(pR(p^2))}{dp^n}\right\| \le \frac{M_1 n!}{p^{n+1}}, \qquad p>0,$$

which holds for the resolvent of the generator of a cosine operator function by Sova's theorem (see [7], [18]), we estimate

$$\begin{aligned}&\left\|\frac{d^n}{dp^n}(p^{1-\beta}(pR(p^2)g))\right\| \\ &\le \frac{M_1\|g\|}{p^{n+\beta}}\sum_{j=0}^{n}\binom{n}{j}(1-\beta)\beta\times\cdots\times|-\beta-n+j+2|j! \\ &= \frac{M_1\|g\|\Gamma(n+\beta-1)}{|\Gamma(\beta-1)|p^{n+\beta}}\sum_{j=0}^{n}\binom{n}{j}\binom{n+\beta-2}{j}^{-1} \\ &= \frac{M_1\|g\|\Gamma(n+\beta)}{\Gamma(\beta)p^{n+\beta}}\left(1-\binom{n}{n+1}\binom{n+\beta-1}{n+1}^{-1}\right) = \frac{M_1\Gamma(n+\beta)\|g\|}{\Gamma(\beta)p^{n+\beta}}; \qquad (44)\end{aligned}$$

here we have used formula [19, 4.2.8.1].

It follows from (43), (44) that

$$\left\|\frac{d^n}{d\mu^n}(\mu^\alpha(\mu^{\alpha+\beta}I-A_\gamma)^{-1})\right\| \le \frac{M_1\Gamma(n+\beta)}{\mu^{n+\beta}}\int_0^\infty \frac{ds}{s^2-2s\cos\gamma\pi+1} \le \frac{M\Gamma(n+\beta)}{\mu^{n+\beta}},$$

and thus we have proved inequality (42). The proof of the theorem is complete. □

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REFERENCES

1. S. G. Samko, A. A. Kilbas, and O. I. Marichev, *Integrals and Derivatives of Fractional Order and Some of Their Applications* (Nauka i Tekhnika, Minsk, 1987) [in Russian].
2. A. V. Glushak, "Cauchy-type problem for an abstract differential equation with fractional derivatives," Mat. Zametki **77** (1), 28–41 (2005) [Math. Notes **77** (1–2), 26–38 (2005)].
[J] 2001, No. 2, 74-77 (2001). ISSN 1609-0705
3. A. V. Glushak, "The problem of Cauchy-type for an abstract differential equation with fractional derivative," Vestn. Voronezh. Gos. Univ., Ser. Fiz. Mat., No. 2, 74–77 (2001).
4. A. V. Glushak, "On a problem of Cauchy-type for an inhomogeneous abstract differential equation with fractional derivative," Vestn. Voronezh. Gos. Univ., Ser. Fiz. Mat., No. 1, 121–123 (2002).
5. A. V. Glushak, "On the relationship between the solutions of abstract differential equations containing fractional derivatives" Vestn. Voronezh. Gos. Univ., Ser. Fiz. Mat., No. 2, 61–63 (2002).
6. A. V. Glushak, "On periodic solutions of abstract differential equations with fractional derivatives," Vestn. Voronezh. Gos. Univ., Ser. Fiz. Mat., No. 1, 96–98 (2003).
7. V. V. Vasil′ev, S. G. Krein, and S. I. Piskarev, "Semigroups of operators, cosine operator functions, and linear differential equations," in *Itogi Nauki Tekh., Ser. Mat. Anal.* (VINITI, Moscow, 1990), Vol. 28, pp. 87–202 [J. Sov. Math. **54** (4), 1042-1129 (1991)].
8. A. N. Kochubei, "A Cauchy problem for evolution equations of fractional order," Differentsial′nye Uravneniya **25** (8), 1359–1368 (1989) [Differential Equations **25** (8), 967–974 (1989)].
9. V. A. Kostin, "The Cauchy problem for an abstract differential equation with fractional derivatives," Dokl. Ross. Akad. Nauk **326** (4), 597–600 (1992) [Dokl. Math. **46** (2), 316–319 (1992)].
10. Ph. Clement, G. Gripenberg, and S.-O. Loden, "Regularity properties of solution of fractional evolution equation," in *Evolution Equations and Their Applications in Physical and Life Sciences*, Bad Herrenalb, 1998, *Lecture Notes in Pure and Appl. Math.* (Dekker, New York, 2001), Vol. 215, pp. 235–246.
11. H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 3: *Elliptic and Modular Functions, Lamé and Mathieu Functions* (McGraw–Hill, New York–Toronto–London, 1955; Nauka, Moscow, 1967).

12. S. A. Tersenov, *Introduction to the Theory of Equations Degenerate on the Boundary* (Novosibirsk Gos. Univ., Novosibirsk, 1973) [in Russian].
13. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Special Functions* (Nauka, Moscow, 1983) [in Russian].
14. A. G. Zemanyan, *Integral Transformations of Generalized Functions* (Nauka, Moscow, 1974) [in Russian].
15. H. Bateman and A. Erdélyi, *Higher Transcendental Functions*, Vol. 1: *The Hypergeometric Function, Legendre Functions* (McGraw–Hill, New York–Toronto–London, 1953; Nauka, Moscow, 1965).
16. A. G. Sveshnikov and A. N. Tihonov, *Theory of Functions of a Complex Variable*, in *Course in Higher Mathematics and Mathematical Physics* (Nauka, Moscow, 1967), Vol. 4 [in Russian].
17. C. C. Travis and G. F. Webb, "Perturbation of strongly continuous cosine family generators," Colloq. Math. **45** (2), 277–285 (1981).
18. J. Goldstein, *Semigroups of Linear Operators and Applications* (Oxford University Press, New York, 1985; Vyshcha Shkola, Kiev, 1989).
19. A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series. Elementary Functions* (Nauka, Moscow, 1981) [in Russian].