Correctness of Cauchy-Type Problems for Abstract Differential Equations with Fractional Derivatives

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Abstract—We prove the uniform correctness of a Cauchy-type problem with two fractional derivatives and a bounded operator A. We propose a criterion for the uniform correctness of unbounded operator A.

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Let $M_k^{\alpha,\beta}$ be a differential operator in the form $M_k^{\alpha,\beta} = D^\alpha \left(t^k D^\beta\right)$ with left-hand fractional Riemann–Liouville derivatives of orders $\alpha \in (0,1)$ and $\beta \in (0,1)$ ([1], P. 84)

$$D^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t (t-s)^{-\alpha} u(s) ds, \;\; t \in (0,\infty);$$

here $\Gamma(\cdot)$ is the Euler gamma function.

In a Banach space X we consider the following Cauchy-type problem with a linear closed operator A:

$$M_k^{\alpha,\beta} u(t) = t^\gamma A u(t), \;\; t > 0, \tag{1}$$

$$\lim_{t\to 0} D^{\beta-1} u(t) = u_0, \;\; \lim_{t\to 0} D^{\alpha-1}\left(t^k D^\beta u(t)\right) = u_1, \tag{2}$$

where

$$D^{\beta-1} u(t) = I^{1-\beta} u(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} u(s) ds$$

is the left-hand fractional Riemann–Liouville integral of order $1-\beta$.

The definition of initial conditions (2) has the following specificity: In the first condition the total order of derivatives is less by 1 than that in Eq. (1), in the second condition the difference of orders increases by α.

Moreover, the peculiarity of the considered problem consists in the presence of two conditions in the form (2) even if $0 < \alpha + \beta < 1$. In the case of an ordinary differential equation of fractional order ($E = \mathbf{R}$, A is the operator of multiplication by a number) this peculiarity can be explained by the equality ([1], formula (2.68))

$$D^\alpha D^\beta u\,(t) = D^{\alpha+\beta} u\,(t) - \frac{D^{\beta-1} u\,(0)\, t^{-\alpha-1}}{\Gamma(-\alpha)}.$$

This equality allows one to reduce Eq. (1) with $k = \gamma = 0$ to the following inhomogeneous equation:

$$D^{\alpha+\beta} u\,(t) = Au(t) + \frac{D^{\beta-1} u\,(0)\, t^{-\alpha-1}}{\Gamma(-\alpha)}.$$

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For extracting a unique solution one should state two conditions: the first condition

$$\lim_{t\to 0} D^{\beta-1}u\,(t) = u_0$$

allows one to define the right-hand side of the equation, while the second condition enables one to obtain a Cauchy-type problem.

In particular, if $E = \mathbf{R}$, $k = \gamma = 0$, $A = 0$, then the general solution to Eq. (1) takes the form

$$u(t) = t^{\beta-1}c_1 + t^{\alpha+\beta-1}c_2$$

with arbitrary constants $c_1, c_2 \in E$; in order to find them one should impose two conditions in the form (2).

Seemingly, the reduction to an inhomogeneous equation is less useful, because this approach requires to consider each case $0 < \alpha + \beta \leq 1$ and $1 < \alpha + \beta < 2$ separately.

Note that papers [2–4] contain a review of the latter publications devoted to the resolvability of differential equations with fractional derivatives. Among the papers dedicated to the study of abstract differential equations with fractional derivative of order α let us point out the paper [5], where one investigates a weakened Cauchy problem with a regularized fractional derivative, and papers [6, 7] which contain a criterion for the uniform correctness of Cauchy-type problems.

In [8] one studies an inhomogeneous differential equation of order $1 + \alpha$ with a regularized fractional derivative and a positive operator A by the method of sums developed by Da Prato and Grisvard.

In [9] we investigate the uniform correctness of problem (1), (2) in the case when $u_0 \neq 0$, $u_1 = 0$. In this paper, where we continue the research of [9], we study the uniform correctness of problem (1), (2) in the case when $u_0 = 0$, $u_1 \neq 0$. We prove the uniform correctness of problem (1), (2) with a bounded operator and adduce a criterion for the uniform correctness of problems with an unbounded operator in the case when $k = \gamma = 0$. In conclusion, we establish connections between solutions of several Cauchy-type problems that contain fractional derivatives.

Definition 1. A solution to problem (1), (2) is a strongly continuous with $t > 0$ function $u(t)$ such that $I^{1-\beta}u(t)$ and $I^{1-\alpha}\left(t^k D^\beta u(t)\right)$ represent strongly continuously differential with $t > 0$ functions, the function $u(t)$ takes on values in $D(A)$ and satisfies conditions (1), (2).

Definition 2. Problem (1), (2) with $u_0 = 0$ is said to be uniformly correct if one can find an operator function $\Psi^{\alpha,\beta}_{\gamma,k}(t)$ defined on X that commutes with A and numbers $M > 0$ and $\omega \geq 0$ such that for any $u_1 \in D(A)$ the function $\Psi^{\alpha,\beta}_{\gamma,k}(t)u_1$ is its unique solution, and

$$\left\| \frac{\Gamma(\alpha)\Gamma(\alpha+\beta-k)}{\Gamma(\alpha-k)} t^{1+k-\alpha-\beta}\Psi^{\alpha,\beta}_{\gamma,k}(t)u_1 - u_1 \right\| = \mathcal{O}\left(t^{\alpha+\beta+\gamma-k}\right)\|Au_1\|, \;\; t\to 0, \tag{3}$$

$$\|\Psi^{\alpha,\beta}_{\gamma,k}(t)\| \leq Mt^{\alpha+\beta-k-1}e^{\omega t}. \tag{4}$$

In accordance with Definition 2 problem (1), (2) is uniformly correct if it has a unique solution that (as follows from (4)) continuously depends on the initial data uniformly with respect to t in any compact in $(0, \infty)$. Along with these usual requirements, Definition 2 contains the additional information about the order of convergence of a solution to the initial element (correlation (3)) and about its behavior with $t \to 0$ and $t \to \infty$ (inequality (4)). One can establish these additional properties of the solution in the cases considered below.

Note that if the operator A is bounded, then problem (1), (2) is always uniformly correct and the operator function $\Psi^{\alpha,\beta}_{\gamma,k}(t)$ in Theorem 1 is adduced in the form of a series. But if the operator A is unbounded and $k = \gamma = 0$, then the considered problem is uniformly correct only when the resolvent of the operator A satisfies certain estimates. In this case the operator function $\Psi^{\alpha,\beta}_{0,0}(t)$ is expressed in Theorem 3 in terms of the resolvent of the operator A with the help of the Mellin integral.

Let further $B(X)$ be the space of linear bounded operators from X to X.

Theorem 1. *Let $u_0 = 0$, $A \in B(X)$, and let parameters of problem* (1), (2) *satisfy inequalities* $\alpha - k > 0$, $\alpha + \beta + \gamma - k > 0$, *and* $\gamma \leq k$. *Then problem* (1), (2) *is uniformly correct and*

$$\Psi^{\alpha,\beta}_{\gamma,k}(t)u_1 = \frac{\Gamma(\alpha-k)t^{p-\gamma-1}}{\Gamma(\alpha)\Gamma(p-\gamma)}\left(u_1 + \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{\Gamma(\alpha-k+pj)\Gamma(pj)}{\Gamma(\alpha+pj)\Gamma(p-\gamma+pj)}\right)t^{pi}A^i u_1\right), \tag{5}$$

where $p = \alpha + \beta + \gamma - k > 0$.

Proof. Let us apply the fractional integration operator I^α of order α to both parts of Eq. (1). Taking into account the second condition in (2), we obtain

$$D^\beta u(t) = \frac{t^{\alpha-k-1}u_1}{\Gamma(\alpha)} + \frac{t^{-k}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^\gamma Au(s)ds,$$

and taking into account the first one, we have

$$u(t) = \frac{\Gamma(\alpha-k)t^{p-\gamma-1}u_1}{\Gamma(\alpha)\Gamma(p-\gamma)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\tau^{-k}d\tau\int_0^\tau (\tau-s)^{\alpha-1}s^\gamma Au(s)ds. \tag{6}$$

Further in the proof we use the method of successive approximations. Let

$$u_0(t) = \frac{\Gamma(\alpha-k)t^{p-\gamma-1}u_1}{\Gamma(\alpha)\Gamma(p-\gamma)},$$
$$u_m(t) = \frac{\Gamma(\alpha-k)t^{p-\gamma-1}u_1}{\Gamma(\alpha)\Gamma(p-\gamma)} + \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\tau^{-k}d\tau\int_0^\tau (\tau-s)^{\alpha-1}s^\gamma Au_{m-1}(s)ds.$$

Using integral 2.2.4.8 from [10], with $m = 1, 2, \ldots$ we find

$$u_1(t) = u_0(t) + \frac{\Gamma(\alpha-k)\Gamma(\alpha-k+p)t^{2p-\gamma-1}Au_1}{\Gamma(\alpha)\Gamma(\alpha+p)\Gamma(p-\gamma)\Gamma(2p-\gamma)};$$
$$u_2(t) = u_1(t) + \frac{\Gamma(\alpha-k)\Gamma(\alpha-k+p)\Gamma(\alpha-k+2p)\Gamma(p)\Gamma(2p)t^{3p+\gamma-1}A^2u_1}{\Gamma(\alpha)\Gamma(\alpha+p)\Gamma(\alpha+2p)\Gamma(p-\gamma)\Gamma(2p-\gamma)\Gamma(3p-\gamma)};\ldots,$$

which in a general case gives the formula

$$u_m(t) = u_{m-1}(t) + \prod_{j=1}^{m}\frac{\Gamma(\alpha-k+pj)\Gamma(pj)t^{(m+1)p-\gamma-1}A^m u_1}{\Gamma(\alpha+pj)\Gamma(p(j+1)-\gamma)}.$$

Note that in the above correlations for the convergence of integrals we need the fulfillment of conditions $\alpha - k > 0$, $p = \alpha + \beta + \gamma - k > 0$. Hence, proceeding to the limit as $m \to \infty$, we obtain the following representation of the desired solution to problem (1), (2):

$$\Psi^{\alpha,\beta}_{\gamma,k}(t)u_0 = \frac{\Gamma(\alpha-k)t^{p-\gamma-1}}{\Gamma(\alpha)\Gamma(p-\gamma)}\left(u_1 + \sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{\Gamma(\alpha-k+pj)\Gamma(pj)}{\Gamma(\alpha+pj)\Gamma(p-\gamma+pj)}\right)t^{pi}A^i u_1\right),$$

which can be established by direct checking.

Now let us prove the validity of correlations (3) and (4). Really, from (5) it follows that

$$\left\|\frac{\Gamma(\alpha)\Gamma(p-\gamma)}{\Gamma(\alpha-k)}t^{1+\gamma-p}\Psi^{\alpha,\beta}_{\gamma,k}(t)u_1 - u_1\right\|$$
$$= \left\|\sum_{i=1}^{\infty}\left(\prod_{j=1}^{i}\frac{\Gamma(\alpha-k+pj)\Gamma(pj)}{\Gamma(\alpha+pj)\Gamma(p-\gamma+pj)}\right)t^{pi}A^i u_1\right\| = \mathcal{O}\left(t^p\right)\|Au_1\|$$

as $t \to 0$. Therefore, equality (3) is proved. Let us prove estimate (4). From (5) we deduce

$$\left\|t^{1+\gamma-p}\Psi^{\alpha,\beta}_{\gamma,k}(t)\right\| \leq \frac{\Gamma(\alpha-k)}{\Gamma(\alpha)\Gamma(p-\gamma)}\left(1 + \sum_{i=1}^{\infty}\frac{\Gamma(\alpha-k+p)\Gamma(p)\cdots\Gamma(\alpha-k+pi)\Gamma(pi)t^{pi}\|A\|^i}{\Gamma(\alpha+p)\Gamma(2p-\gamma)\cdots\Gamma(\alpha+pi)\Gamma(p-\gamma+pi)}\right).$$

Let us show that for sufficiently great j,

$$\frac{\Gamma(\alpha-k+pj)\Gamma(pj)}{\Gamma(\alpha+pj)\Gamma(pj-\gamma)}<1. \tag{7}$$

Let $\alpha+pj=q>\alpha+\gamma$. Then we rewrite inequality (7) in the form

$$\frac{\Gamma(q-\alpha)\Gamma(q-k)}{\Gamma(q-\alpha-\gamma)\Gamma(q)}<1. \tag{8}$$

The validity of inequality (8) is proved in [9], therefore

$$\|t^{1+\gamma-p}\Psi_{\gamma,k}^{\alpha,\beta}(t)\|\leq M_1\sum_{i=0}^{\infty}\frac{t^{pi}\left\|A\right\|^i}{\Gamma(pi+p-\gamma)}=M_1E_{p,p-\gamma}(t^p\left\|A\right\|^{1/p})\leq Me^{\omega t},\ \ \omega>\left\|A\right\|^{1/p^2},$$

here we use the asymptotic equality ([11], P. 224, formula (22))

$$E_{\alpha,\beta}(z)=\frac{1}{\alpha}z^{(1-\beta)/\alpha}\exp(z^{1/\alpha})+\mathcal{O}\left(1/\left|z\right|\right),\ \ z\to\infty,\ \ |\arg z|\leq\frac{\alpha\pi}{2},\ \ 0<\alpha<2, \tag{9}$$

that is fulfilled for the Mittag–Lefiler-type function

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)}.$$

Since the uniqueness of a solution to problem (1), (2) is already established in [9], Theorem 1 is proved.

In particular, if $A\in B(X)$, then

$$\Psi_{0,0}^{\alpha,\beta}(t)u_1=t^{\alpha+\beta-1}E_{\alpha+\beta,\alpha+\beta}\left(t^{\alpha+\beta}A\right)u_1=I^{\alpha}\Phi_{0,0}^{\alpha,\beta}(t)u_1,$$

where $\Phi_{0,0}^{\alpha,\beta}(t)=t^{\beta-1}E_{\alpha+\beta,\beta}\left(t^{\alpha+\beta}A\right)$ is the resolving operator (see [9]) of problem (1), (2) with $k{=}\gamma{=}0$, $u_0\neq 0$, $u_1=0$.

Note also that if one of solutions to problem (1), (2) $\Phi_{\gamma,k}^{\alpha,\beta}(t)u_0$ always has a singularity of order $t^{\beta-1}$ at zero (see [9]), then the behavior of the second one $\Psi_{\gamma,k}^{\alpha,\beta}(t)u_1$ depends on the sign of the number $p-\gamma-1=\alpha-k+\beta-1$ $(\alpha-k>0,\beta-1<0)$.

Let us now study the problem

$$D^{\alpha}D^{\beta}u(t)=Au(t),\ \ t>0, \tag{10}$$

$$\lim_{t\to 0}D^{\beta-1}u(t)=0,\ \ \lim_{t\to 0}D^{\alpha-1}D^{\beta}u(t)=u_1, \tag{11}$$

which can be obtained from (1), (2) with $k=\gamma=0$, but in this case we assume that A is a linear closed operator in X with a dense definition domain $D(A)$, $u_1\in D(A)$.

In what follows we denote the operator function $\Psi_{0,0}^{\alpha,\beta}(t)$ by $\Psi(t)$. Let us first establish the necessary condition for the uniform correctness of problem (10), (11). Let $R(\lambda)=(\lambda I-A)^{-1}$ stand for the resolvent of the operator A.

Theorem 2. *Let problem* (10), (11) *be uniformly correct and* $\operatorname{Re}\lambda>\omega$, *then* $\lambda^{\alpha+\beta}$ *belongs to the resolvent set* $\rho(A)$ *of the operator* A, *and for any* $x\in X$ *the following representation is true*:

$$R\left(\lambda^{\alpha+\beta}\right)x=\int_0^{\infty}e^{-\lambda t}\Psi(t)x\,dt, \tag{12}$$

in addition, for all integer $n\geq 0$,

$$\left\|\frac{d^nR\left(\lambda^{\alpha+\beta}\right)}{d\lambda^n}\right\|\leq\frac{M\,\Gamma(n+\alpha+\beta)}{(\operatorname{Re}\lambda-\omega)^{n+\alpha+\beta}}. \tag{13}$$

Proof. Let us verify that the operator $R\left(\lambda^{\alpha+\beta}\right)$ defined by equality (12) satisfies the correlation

$$(\lambda^{\alpha+\beta}I - A)R(\lambda^{\alpha+\beta}) = R(\lambda^{\alpha+\beta})(\lambda^{\alpha+\beta}I - A) = I.$$

Let $x \in D(A)$ and $\operatorname{Re}\lambda > \omega$. Then, taking into account conditions (11), we obtain

$$(\lambda^{\alpha+\beta}I - A)R(\lambda^{\alpha+\beta})x = \lambda^{\alpha+\beta}\int_0^\infty e^{-\lambda t}\Psi(t)x dt - \int_0^\infty e^{-\lambda t}\, D^\alpha D^\beta \Psi(t)x\, dt$$

$$= \lambda^{\alpha+\beta}\int_0^\infty e^{-\lambda t}\Psi(t)x\, dt - \lambda^\alpha \int\limits_0^\infty e^{-\lambda t} D^\beta \Psi(t)x\, dt + x$$

$$= \lambda^{\alpha+\beta}\int_0^\infty e^{-\lambda t}\Psi(t)x\, dt - \lambda^{\alpha+\beta}\int_0^\infty e^{-\lambda t}\Psi(t)x\, dt + x = x.$$

Since operators A and $\Psi(t)$ commute, so do A and $R\left(\lambda^{\alpha+\beta}\right)$, therefore the following equality is fulfilled:

$$R(\lambda^{\alpha+\beta})(\lambda^{\alpha+\beta}I - A)x = x, \;\; x \in D(A).$$

Hence, it remains to prove the validity of estimate (13). Differentiating n times equality (12) with respect to λ, we obtain

$$\frac{d^n R\left(\lambda^{\alpha+\beta}\right)x}{d\lambda^n} = \int_0^\infty (-t)^n e^{-\lambda t}\Psi(t)x\, dt. \tag{14}$$

Inequality (13) easily follows from (14) after the application of inequality (4). □

As we show further, an estimate in the form (13) for the resolvent of the operator A provides the uniform correctness of problem (10), (11). Note also that estimate (13) in the case when $\alpha + \beta = 1$ and the Hille–Yosida theorem ([12], P. 373) imply that the operator A is a generator of the C_0-semigroup.

Further we assume that the Banach space X has the Radon–Nikodym property (see [13]), i.e., each absolutely continuous function $u : [a, b] \to X$ is differentiable almost anywhere on (a, b), $u'(t) \in L_1(a, b)$, and

$$u(t) - u(a) = \int_a^t u'(s)ds.$$

For example, reflexive Banach spaces have this property, while spaces $C[0, 1]$, $L_1(0, 1)$, and c_0 (the space of sequences converging to zero) do not have it. In what follows the denotation $X = X_{RN}$ means that a Banach space X has the Radon–Nikodym property. We will use this property for the inversion of the Laplace transform of abstract functions.

Theorem 3. *Let $X = X_{RN}$. If with $\operatorname{Re}\lambda > \omega$ the operator A has a resolvent $R\left(\lambda^{\alpha+\beta}\right)$ that satisfies inequality* (13), *and $u_1 \in D(A)$, then the function*

$$\Psi(t)u_1 = I^\beta D^{1-\alpha}\frac{1}{2\pi i}\text{ v. p. }\int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\lambda^{\alpha+\beta-1}R\left(\lambda^{\alpha+\beta}\right)u_1 d\lambda, \tag{15}$$

where $\sigma > \max(\omega, 0)$, is a solution to problem (10), (11).

Proof. Denoting $\widetilde{R}(\lambda) = \lambda^{\alpha+\beta-1}R\left(\lambda^{\alpha+\beta}\right)$, we rewrite equality (12) in the form

$$\widetilde{R}(\lambda)u_1 = \lambda^{\alpha+\beta-1}R(\lambda^{\alpha+\beta})u_1 = \lambda^{\alpha+\beta-1}\int_0^\infty e^{-\lambda t}\Psi(t)u_1 dt = \lambda^{\alpha-1}\int_0^\infty e^{-\lambda t}D^\beta\Psi(t)u_1 dt$$

$$= \int_0^\infty e^{-\lambda t}I^{1-\alpha}D^\beta\Psi(t)u_1 dt. \tag{16}$$

Due to lemma 1 in [9] the function $\widetilde{R}(\lambda)$ satisfies the inequality

$$\left\| \frac{d^n \widetilde{R}(\lambda)}{d\lambda^n} \right\| \leq \frac{M\Gamma(\alpha+\beta)n!}{(\operatorname{Re}\lambda - \omega)^{n+1}}$$

with any integer $n \geq 0$, therefore, as follows from theorem 1.4 in [14] (just here we need the assumption $X = X_{RN}$), we can invert equality (16), and we obtain the following formula:

$$I^{1-\alpha} D^{\beta} \Psi(t) u_1 = \frac{1}{2\pi i} \,\mathrm{v.p.} \int_{\sigma - i\infty}^{\sigma + i\infty} e^{\lambda t} \lambda^{\alpha+\beta-1} R(\lambda^{\alpha+\beta}) u_1 d\lambda. \tag{17}$$

Consequently, for the desired solution $\Psi(t)u_1$ from (17) we obtain representation (15).

Let us make sure that the function $\Psi(t)u_1$ defined by equality (15) is really a solution to problem (10), (11).

Using the evident correlation

$$\mu R(\mu) u_1 = R(\mu) A u_1 + u_1, \;\; u_1 \in D(A), \tag{18}$$

we rewrite equality (15) for $u_1 \in D\left(A^{n+1}\right)$ in the form

$$\Psi(t) u_1 = t^{\alpha+\beta-1} \sum_{j=0}^{n} \frac{t^{j(2\alpha+\beta)} A^j u_1}{\Gamma(j(2\alpha+\beta)+\alpha+\beta)} + \frac{t^{\alpha+\beta-1}}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{E_{1,\alpha+\beta}(\lambda t)}{\lambda^{1+n(2\alpha+\beta)}} R(\lambda^{\alpha+\beta}) A^{n+1} u_1 d\lambda. \tag{19}$$

Representation (19) with a proper choice of n (it suffices to take $n > 1/(2\alpha+\beta)$) implies the continuous differentiability of functions $I^{1-\beta}\Psi(t)u_1$ and $I^{1-\alpha}D^{\beta}\Psi(t)u_1$ with $t > 0$, and the validity of initial conditions (11) and correlation (3). In addition, the asymptotic equality (9) for the function $E_{1,\alpha+\beta}(\lambda t)$ allows one to conclude that $\|\Psi(t)u_1\| = \mathcal{O}\left(\exp(\sigma_0 t)\right)$ as $t \to \infty$. The latter remark is necessary for the application of the Post–Widder inversion formula ([15], P. 359).

Let $u_1 \in D\left(A^{n+1}\right)$. Applying in (12) the Post–Widder inversion formula, inequality (13), and the known asymptotic formula for the correlation of gamma functions ([16], P. 62, formula (4))

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + \frac{(a-b)(a+b-1)}{2z} + \mathcal{O}(z^{-2})\right), \tag{20}$$

we obtain

$$\begin{aligned}
\|\Psi(t)u_1\| &= \left\| \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} \frac{d^k R\left(\lambda^{\alpha+\beta}\right) u_1}{d\lambda^k} \bigg|_{\lambda = k/t} \right\| \\
&\leq M \;\|u_1\|\; t^{\alpha+\beta-1} \lim_{k\to\infty} \frac{\Gamma(k+\alpha+\beta) k^{1-\alpha-\beta}}{\Gamma(k+1)(1-\omega t/k)^{k+\alpha+\beta}} \\
&= M \;\|u_1\|\; t^{\alpha+\beta-1} \lim_{k\to\infty} (1-\omega t/k)^{-k-\alpha-\beta} = M \|u_1\| t^{\alpha+\beta-1} e^{\omega t}.
\end{aligned}$$

Thus, inequality (4) is also proved for $k = \gamma = 0$.

Since the definition domain of any degree of the operator A is dense in X ([9], lemma 2), the function $\Psi(t)u_1$ or, more precisely, its continuous continuation, also satisfies the mentioned correlations even for $u_1 \in D(A)$.

In order to make sure that the function $\Psi(t)u_1$ satisfies Eq. (10), we find the Laplace transformation of the left- and right-hand sides of this equality. Taking into account (16), (12), and (18), after evident transformations we obtain

$$\begin{aligned}
L(D^{\alpha} D^{\beta} \Psi(t) u_1) &= \lambda^{\alpha} L(D^{\beta}\Psi(t)u_1) - u_1 \\
&= \lambda^{\alpha+\beta} L(\Psi(t)u_1) - u_1 = \lambda^{\alpha+\beta} R(\lambda^{\alpha+\beta}) u_1 - u_1, \\
L(A\Psi(t)u_1) &= L(I^{\beta} D^{\beta} \Psi(t) A u_1) = \lambda^{-\beta} L(D^{\beta}\Psi(t) A u_1) \\
&= R(\lambda^{\alpha+\beta}) A u_1 = \lambda^{\alpha+\beta} R(\lambda^{\alpha+\beta}) u_1 - u_1.
\end{aligned}$$

Therefore, the function $\Psi(t)u_1$ satisfies Eq. (10).

Theorem 4. *Let $X = X_{RN}$. The Cauchy-type problem* (10), (11) *is uniformly correct in the class of functions that allow estimate* (4) *if and only if with* $\operatorname{Re}\lambda > \omega$ *the operator A has a resolvent* $R\left(\lambda^{\alpha+\beta}\right)$ *that satisfies inequality* (13).

Proof. The necessity is proved already in Theorem 2. Taking into account Theorem 3, in order to prove the sufficiency, it remains to make sure of the uniqueness of the solution defined by formula (15). Let $u_1(t)$ and $u_2(t)$ be two solutions of problem (10), (11).

As in the proof of theorem 23.7.1 in [12], we obtain the correlation

$$\lambda^{\alpha+\beta}L(u) = AL(u),$$

where $L(u)$ is the Laplace transform of the function $u(t) = u_1(t) - u_2(t)$ distinct from the identical zero. Hence we obtain a contradiction, because with $L(u) \neq 0$ all points $\lambda^{\alpha+\beta}$, for which $\operatorname{Re}\lambda > \omega$, belong to the point spectrum of the operator A. Therefore, $u_1(t) \equiv u_2(t)$. □

In [9] we establish that the following inequality being valid with all integer $n \geq 0$ is a criterion for the uniform correctness of problem (1), (2) with $k = \gamma = 0$, $u_1 = 0$:

$$\left\| \frac{d^n\left(\lambda^\alpha R(\lambda^{\alpha+\beta})\right)}{d\lambda^n} \right\| \leq \frac{M\,\Gamma(n+\beta)}{(\operatorname{Re}\lambda - \omega)^{n+\beta}} \tag{21}$$

and the resolvent operator takes the form

$$\Phi(t)u_0 = D^{1-\beta}\frac{1}{2\pi i}\ \text{v. p.} \int_{\sigma-i\infty}^{\sigma+i\infty} e^{\lambda t}\lambda^{\alpha+\beta-1}R\left(\lambda^{\alpha+\beta}\right)u_0 d\lambda. \tag{22}$$

In the following theorem we show that if the resolvent of the operator A satisfies inequality (21), then inequality (13) is valid, i.e., in view of Theorem 4 problem (10), (11) is also uniformly correct. In addition, we establish that the resolving operator $\Psi(t)$ of problem (10), (11) is the resolving operator $\Phi(t)$ integrated α times.

Theorem 5. *Let $X = X_{RN}$, $u_1 \in D(A)$. Assume that with* $\operatorname{Re}\lambda > \omega$ *the operator A has a resolvent* $R\left(\lambda^{\alpha+\beta}\right)$ *that satisfies inequality* (21). *Then problem* (10), (11) *is uniformly correct and* $\Psi(t)u_1 = I^\alpha\Phi(t)u_1$, *where $\Phi(t)$ is defined by equality* (22).

Proof. Using formula (21) and properties of the beta function $B(\cdot,\cdot)$, after simple transformations we obtain

$$\begin{aligned}
\left\| \frac{d^n R\left(\lambda^{\alpha+\beta}\right)}{d\lambda^n} \right\| &= \left\| \frac{d^n\left(\lambda^{-\alpha}\lambda^\alpha R\left(\lambda^{\alpha+\beta}\right)\right)}{d\lambda^n} \right\| = \left\| \sum_{j=0}^{n} C_n^j \frac{d^{n-j}\lambda^{-\alpha}}{d\lambda^{n-j}} \frac{d^j\left(\lambda^\alpha R\left(\lambda^{\alpha+\beta}\right)\right)}{d\lambda^j} \right\| \\
&\leq \sum_{j=0}^{n} C_n^j |(-\alpha)(-\alpha-1)(-\alpha-n+j+1)|\,|\lambda|^{-\alpha-n+j}\frac{M\Gamma(j+\beta)}{(\operatorname{Re}\lambda-\omega)^{j+\beta}} \\
&\leq \frac{M}{\Gamma(\alpha)\,(\operatorname{Re}\lambda-\omega)^{n+\alpha+\beta}} \sum_{j=0}^{n} C_n^j\Gamma(n+\alpha-j)\Gamma(j+\beta) \\
&= \frac{M\,\Gamma(n+\alpha+\beta)}{\Gamma(\alpha)\,(\operatorname{Re}\lambda-\omega)^{n+\alpha+\beta}} \sum_{j=0}^{n} C_n^j\, B(n+\alpha-j, j+\beta) \\
&= \frac{M\,\Gamma(n+\alpha+\beta)}{\Gamma(\alpha)\,(\operatorname{Re}\lambda-\omega)^{n+\alpha+\beta}} \sum_{j=0}^{n} C_n^j \int_0^1 s^{n+\alpha-j-1}(1-s)^{j+\beta-1}ds \\
&= \frac{M\,\Gamma(n+\alpha+\beta)}{\Gamma(\alpha)\,(\operatorname{Re}\lambda-\omega)^{n+\alpha+\beta}} \int_0^1 s^{n+\alpha-1}(1-s)^{\beta-1}\sum_{j=0}^{n} C_n^j\left(\frac{1-s}{s}\right)^j ds
\end{aligned}$$

$$= \frac{M\,\Gamma(n+\alpha+\beta)}{\Gamma(\alpha)\,(\mathrm{Re}\,\lambda-\omega)^{n+\alpha+\beta}} \int_0^1 s^{\alpha-1}(1-s)^{\beta-1}\,ds = \frac{M\Gamma(\beta)\Gamma(n+\alpha+\beta)}{\Gamma(\alpha+\beta)\,(\mathrm{Re}\,\lambda-\omega)^{n+\alpha+\beta}}.$$

Therefore, inequality (13) and then the uniform correctness of problem (10), (11) are proved.

In [9] we prove that the function $I^{\alpha}\Phi(t)u_1$ satisfies problem (10), (11), therefore in view of the uniqueness theorem we have $\Psi(t)u_1 = I^{\alpha}\Phi(t)u_1$, which completes the proof of the theorem. □

In the concluding theorem we establish the uniform correctness of the Cauchy-type problem

$$D^{\delta}v\,(t) = Av\,(t)\,,\ \ t>0, \tag{23}$$

$$\lim_{t\to 0} D^{\delta-1}v\,(t) = v_0 \in D\,(A)\,, \tag{24}$$

which contains the fractional derivative of order $\delta = (\alpha+\beta)/2$, under the assumption that with the operator A problem (10), (11) is uniformly correct. We adduce a formula that connects the resolving operators of the mentioned problems. Note that the uniform correctness of a problem in the form (23), (24), the expression of its resolving operator $T_{\delta}(t)$ in terms of the resolvent of the operator A, and some other questions about an equation with one fractional derivative were studied by us in [7, 17–20].

Theorem 6. *Let $X = X_{RN}$, let inequality (13) be fulfilled and $\delta = (\alpha+\beta)/2$. Then problem (23), (24) is uniformly correct and its solution takes the form*

$$v\,(t) = T_{\delta}\,(t)\,v_0 = \frac{2t^{\delta-1}}{\pi}\int_0^{\infty}\int_0^{\infty} s^{2\delta-1}E_{1,\delta}\left(-ts^2\right)\sin\left(\delta\pi - \tau s\right)ds\;\Psi\,(\tau)\,v_0\,d\tau, \tag{25}$$

where $E_{\alpha,\beta}(\cdot)$ is a function of the Mittag–Leffler type.

Proof. Let us make sure that the resolvent $R(\mu)$ of the operator A with $\mathrm{Re}\,\mu > \omega_1$ satisfies the inequality

$$\left\|\frac{d^n R\left(\mu^{\delta}\right)}{d\mu^n}\right\| \le \frac{M\,\Gamma\,(n+\beta)}{(\mathrm{Re}\,\mu-\omega_1)^{n+\delta}} \tag{26}$$

for all integer $n \ge 0$. Taking into account the results of paper [7] (see also [20]), inequality (26) implies the uniform correctness of problem (23), (24).

By setting $\lambda = \sqrt{\mu}$ in (12), we obtain

$$R\left(\mu^{\delta}\right)x = \int_0^{\infty} e^{-t\sqrt{\mu}}\,\Phi\,(t)\,x\,dt, \tag{27}$$

hence, denoting $z = t\sqrt{\mu}$ and expressing $e^{-t\sqrt{\mu}}$ in terms of the Macdonald function

$$K_{1/2}(z) = \sqrt{\frac{\pi}{2z}}e^{-z},$$

after the use of the known formula for differentiation of the Macdonald function $K_{\nu}(\cdot)$ ([21], P. 729)

$$\left(\frac{1}{z}\frac{d}{dz}\right)^j\left(z^{-v}K_{\nu}\,(z)\right) = (-1)^j\,z^{-v-j}K_{\nu+j}\,(z)\,,$$

we have

$$\frac{d^n R\left(\mu^{\delta}\right)x}{d\mu^n} = \int_0^{\infty}\frac{d^n}{d\mu^n}\left(e^{-\sqrt{\mu}t}\right)\Psi(t)x\,dt = \int_0^{\infty}\left(\frac{t^2}{2z}\frac{d}{dz}\right)^n\left(\sqrt{\frac{2z}{\pi}}K_{1/2}(z)\right)\Psi(t)x\,dt$$
$$= \frac{(-1)^n}{\sqrt{\pi}\,2^{n-1/2}\mu^{(2n-1)/4}}\int_0^{\infty} t^{n+1/2}K_{n-1/2}\left(t\sqrt{\mu}\right)\Psi(t)x\,dt. \tag{28}$$

Since ([22], P. 217) at the point $z = 0$ the function $K_{n-1/2}(z)$ has a singularity in the form $z^{1/2-n}$, and with great $|z|$ and $|\arg z| < \frac{\pi}{2} - \gamma \, (\gamma > 0)$,

$$K_{n-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \left(1 + \mathcal{O}\left(\frac{1}{z}\right)\right),$$

from (28) for sufficiently small $\varepsilon > 0$ we obtain

$$\left\| \frac{d^n R\left(\mu^\delta\right)}{d\mu^n} \right\| \le \frac{M_1}{2^n |\mu|^{n-1/2}} \int_0^{\varepsilon/\sqrt{|\mu|}} t \, \|\Psi(t)\| \, dt + \frac{M_1}{2^n |\mu|^{n/2}} \int_{\varepsilon/\sqrt{|\mu|}}^{\infty} t^n e^{-t \operatorname{Re}\sqrt{\mu}} \|\Psi(t)\| \, dt. \qquad (29)$$

Therefore, from (29) and inequality (4) with $k = \gamma = 0$ we deduce

$$\left\| \frac{d^n R\left(\mu^\delta\right)}{d\mu^n} \right\| \le \frac{M_2}{2^n |\mu|^{n+\delta}} + \frac{M_2 \Gamma(n+2\delta)}{2^n |\mu|^{n/2} \left(\operatorname{Re}\sqrt{\mu} - \omega\right)^{n+2\delta}} \le \frac{M_3 \Gamma(n+2\delta)}{2^n \left(\operatorname{Re}\mu - \omega_1\right)^{n+\delta}}, \qquad (30)$$

where $\operatorname{Re}\mu > \omega_1 > 0$.

Now, in order to deduce the proof of estimate (26), it remains to use the asymptotic equality (20) in inequality (30).

Further we establish a formula that connects solutions to problems (10), (11) and (23), (24). Taking into account the representation of a solution to problem (23), (24) in terms of the resolvent (see [7]) and equality (27), after elementary transformations we obtain

$$\begin{aligned} T_\delta(t) v_0 &= D^{1-\delta} \frac{1}{2\pi i} \, \text{v. p.} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu^{\delta-1} e^{\mu t} R\left(\mu^\delta\right) v_0 d\mu \\ &= D^{1-\delta} \frac{1}{2\pi i} \, \text{v. p.} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu^{\delta-1} e^{\mu t} d\mu \int_0^\infty e^{-\tau\sqrt{\mu}} \Psi(\tau) v_0 \, d\tau \\ &= \int_0^\infty D_t^{1-\delta} \frac{1}{2\pi i} \, \text{v. p.} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu^{\delta-1} e^{\mu t - \tau\sqrt{\mu}} \, d\mu \, \Psi(\tau) v_0 \, d\tau. \qquad (31) \end{aligned}$$

As in [23] (P. 236), we transform the integral

$$\frac{1}{2\pi i} \, \text{v. p.} \int_{\sigma - i\infty}^{\sigma + i\infty} \mu^{\delta-1} e^{\mu t - \tau\sqrt{\mu}} \, d\mu = \frac{2}{\pi} \int_0^\infty s^{2\delta-1} e^{-ts^2} \sin\left(\delta\pi - \tau s\right) ds, \qquad (32)$$

and, taking into account equality ([1], P. 140)

$$D^{1-\delta} e^{\lambda t} = t^{\delta-1} E_{1,\delta}(\lambda t),$$

from (31), (32) we obtain representation (25). □

For the harmonization of the obtained transformation (25) we note that if we formally set $\alpha = \beta = 1$, then we can calculate integral (32), and formula (25) is transformed into the known formula that connects the C_0-semigroup $T(t, A)$ with the sine operator function $S(t, A)$:

$$T(t, A) v_0 = \frac{1}{2t\sqrt{\pi t}} \int_0^\infty \tau \, e^{-\tau^2/(4t)} S(\tau, A) v_0 \, d\tau.$$

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