New Equation for Average Topological Characteristics of the Interphase Surface in Heterogeneous Media

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Abstract—On the basis of an equation for the distribution function of interphase specific area in terms of tilt angles of its normals, a new equation involving partial derivatives with respect to coordinates and time was derived that defines the functions and tensor components of a specific surface, which are used in an exact formula for the volumetric density vector of capillary forces.

INTRODUCTION

Powders, emulsions, suspensions, bubbles in liquid, droplets in gas, and other mixtures are examples of heterogeneous environments with developed interphases. Spatially averaged equations describing heterogeneous media [1] contain functions of coordinates and surface normals averaged over the interphase surface. However, no equation for determining these functions is known. We refer to those functions which depend on normal components or their derivatives as average topological characteristics of the surface. The most important average topological characteristic of the interphase surface is the specific surface tensor s^{ml}, where $s^{ll} = s_{12}$ is the interphase specific area. For example, the volumetric density of capillary forces in a heterogeneous medium linearly depends on the divergence of the specific surface tensor, namely,

$$P^m_\Sigma = 2\Sigma \frac{\partial s^{ml}}{\partial x^l},$$

where Σ is the surface tension coefficient [2].

An exact description of the variation of such time-dependent mean values, including average topological interphase characteristics, in a homogeneous medium and at each point of a heterogeneous medium appears to be an unsolvable problem. This is caused by the chaotic position of the interphase surface, its entangled configuration, the large number of particles, the multivariance of the interphase position at the beginning of the process, local interactions between phases on the interphase (e.g., those caused by Laplace forces), and the randomness of particle size and many other parameters. Sometimes, it is possible to determine fractal dependences between interphase average topological characteristics. The question arises: is it possible to accurately calculate the above quantities? Hypothetically, knowing the position of the interphase surface at the initial time and the velocity vector $\mathbf{V}(\mathbf{t}, \mathbf{x}')$ of interphase points at each time point t, we can determine its position in a heterogeneous medium at every moment of time. This would require high-precision calculations on high-speed computers with huge memory. A similar problem arises in calculating the coordinates and velocities of gas molecules in a container; it can be solved using the Boltzmann kinetic equation for the molecular velocity distribution function. In the case of a heterogeneous medium, because of the chaotic interphase configuration and random factors mentioned above, it is possible to describe interphase average topological characteristics by means of a kinetic equation for the distribution function $s(t, \mathbf{x}, \theta, \varphi)$, i.e., the dependence of the interphase specific area on the tilt angles of normals to the interphase [3]. Here, θ and φ are angles defining the position of the unit normal $\mathbf{n}$ with respect to interphase in local spherical coordinates at the point $\mathbf{x}'$. By definition, $s(t, \mathbf{x}, \theta, \varphi)d\Omega d\mathbf{x}$ is the area of the interphase, contained in the heterogeneous medium volume $d\mathbf{x}$ with normals concentrated in the solid angle $d\Omega = \sin\theta d\theta d\varphi$.

An equation for the distribution function s was obtained in [3] by the following simple argument. The area variation of an interphase with normals θ and φ in a fixed representative volume of a heterogeneous medium can be written in terms of s and set equal to the variation calculated directly provided that the velocity $\mathbf{V}' = \mathbf{V}(t, \mathbf{x}')$ of interphase points is known. The resulting equation is written as

$$\frac{\partial s}{\partial t} + \operatorname{div}(s\langle \mathbf{V}'\rangle_{12,\Omega}) - \frac{1}{\sin\theta} \times \frac{\partial}{\partial\theta}[\sin\theta s n^p n^q_\theta \langle \nabla'^q V'^p\rangle_{12,\Omega}] - \frac{1}{\sin^2\theta} \times \frac{\partial}{\partial\theta}[s n^p n^q_\theta \langle \nabla'^q V'^p\rangle_{12,\Omega}] = s(\delta^{pq} - n^p n^q)\langle \nabla'^q \nabla'^p\rangle_{12,\Omega}, \quad (1)$$

where $\nabla = \{\nabla^1, \nabla^2, \nabla^3\}$ and $\nabla^k = \dfrac{\partial}{\partial x^k}$. Here and in what follows, superscripts denote the numbers of vector and tensor components, the vector components themselves are braced, and primes refer to local interphase parameters. We use the notation

$$\langle \mathbf{V}' \rangle_{12\Omega} s dV d\Omega = \sum_{\chi} (\mathbf{V}')_{\chi} d'S_{\chi},$$
$$\langle \nabla'^{q} V'^{p} \rangle_{12\Omega} s dV d\Omega = \sum_{\chi} (\nabla'^{p} V'^{q})_{\chi} d'S_{\chi}, \quad (2)$$

where $\langle\,\rangle_{12\Omega}$ denotes averaging over all interphase area elements $d'S_{\chi}$ with numbers χ having equal tilt angles θ and φ, both contained in the solid angle $d\Omega$ in the heterogeneous volume $d\mathbf{x}$. The unit vectors $\mathbf{n}_{\theta}$, $\frac{\mathbf{n}_{\varphi}}{\sin\theta}$, and $\mathbf{n}$ form a right-oriented orthogonal basis; they are given by

$$\mathbf{n}_{\theta} = \frac{\partial \mathbf{n}}{\partial \theta} = \{\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta\},$$
$$\mathbf{n}_{\varphi} = \frac{\partial \mathbf{n}}{\partial \varphi} = \{-\sin\theta\sin\varphi, \sin\theta\cos\varphi, 0\}, \quad (3)$$
$$\mathbf{n} = \{\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta\},$$

Cartesian coordinates of vectors are braced. In Eq. (1), the function $s = s(t, \mathbf{x}, \theta, \varphi)$, the operator div, and the average quantities $\langle\,\rangle_{12\Omega}$ are evaluated at the point $\mathbf{x}$ (without prime) of a distinguished fixed representative volume of the heterogeneous medium. We note that Eq. (1) does not describe the birth of a surface, such as the formation of bubbles in a liquid. The birth of a surface can be accounted for by adding the rate of birth of the surface to the right-hand side of Eq. (1).

TOPOLOGICAL HYPOTHESIS

In what follows, in seeking a solution of Eq. (1) as a series of spherical functions and in deriving an equation for average topological interphase characteristics, we use integration over the solid angle $\Omega = 4\pi$ (with respect to the variables $0 \le \theta \le \pi$ and $0 \le \varphi \le 2\pi$). Physically, such integration is equivalent to averaging over the interphase surface, which is applied to obtain spatially averaged equations in the mechanics of heterogeneous media [1], which involve averaging over the interphase of the form

$$s_{12}\langle \Psi(t, \mathbf{x})\Phi(\mathbf{n}')\rangle_{12} = \frac{1}{d\mathbf{x}} \int_{dS_{12}} \Psi(t, \mathbf{x}')\Phi(\mathbf{n}')d'S, \quad (4)$$

where dS_{12} is the interphase contained in the heterogeneous medium volume element $d\mathbf{x}$, $s_{12} = \int_{\Omega = 4\pi} s(t, \mathbf{x}, \theta, \varphi) d\Omega$ is the interphase specific area, and Ψ and Φ are scalar functions. Let us write Eq. (4) in terms of the interphase area distribution function $s(t, \mathbf{x}, \theta, \varphi)$. For the local interphase element, the equality $d'S = s_{12} d'\mathbf{x}$ formally holds; therefore, considering all possible directions of the normal $\mathbf{n}$ at a local point $\mathbf{x}'$ of the surface in Eq. (2) and applying (4), we obtain

$$\int_{dS_{12}} \Psi(t, \mathbf{x}')\Phi(\mathbf{n}')d'S$$
$$= \int_{dS_{12}} \Psi(t, \mathbf{x}')\left[\frac{1}{s_{12}} \int_{\Omega = 4\pi} s(t, \mathbf{x}', \theta, \varphi)\Phi(\mathbf{n})d\Omega\right] d'S, \quad (5)$$

where the bracketed expression is, essentially, the mean value of the function $\Phi(\mathbf{n})$ in the local unit volume $d'\mathbf{x}$.

In [3], the function s is defined by using the notion of a representative volume $d\mathbf{x}$ with coordinate $\mathbf{x}$ of the heterogeneous medium. Thus, for an independent variable of s in (5), we use the coordinate $\mathbf{x}$ of the heterogeneous volume $d\mathbf{x}$ instead of the local coordinate $\mathbf{x}'$. In other words, we assume that the typical variation length of the function s is on the order of the linear size of the representative volume, which is much larger than the typical variation length of local parameters. This is the essence of our topological hypothesis. Thus, the bracketed expression in (5) depends on the coordinates $\mathbf{x}$ of the representative volume rather than on the local coordinates $\mathbf{x}'$. Hence, (5) can be written as

$$\langle \Psi(t, \mathbf{x}')\Phi(\mathbf{n}')\rangle_{12} = \langle \Psi(t, \mathbf{x}')\rangle_{12}\langle \Phi(\mathbf{n}')\rangle_{12}. \quad (6)$$

Equation (6) is a mathematical expression of the topological hypothesis. This hypothesis is akin to the requirement of the stability, regularity, and representativeness of spatially averaged equations of heterogeneous medium mechanics (see [1]).

A NEW EQUATION FOR THE AVERAGE TOPOLOGICAL CHARACTERISTIC OF THE INTERPHASE SURFACE

Let us multiply Eq. (1) by $\Phi(\mathbf{n})$ and integrate the result over all solid angles $d\Omega$. Using definition (5), assuming the boundedness of the function $\Phi(\mathbf{n})$, taking into account the equalities $\sin 0 = \sin\pi = 0$ and the periodicity of functions of the angle φ when integrating the left-hand side of (1), and applying the topological hypothesis, we obtain the following equation for the average quantity $\langle\Phi'\rangle_{12}$:

$$\frac{1}{s_{12}}\left[\frac{\partial}{\partial t}(s_{12}\langle\Phi'\rangle_{12}) + \nabla^{k}(s_{12}\upsilon^{k}\langle\Phi'\rangle_{12})\right]$$
$$= \left\langle -\frac{\partial\Phi'}{\partial\theta'} n'^{p} n'^{q}_{\theta} - \frac{1}{\sin^{2}\theta'}\frac{\partial\Phi'}{\partial\varphi'} n'^{p} n'^{q}_{\varphi} \right. \quad (7)$$
$$\left. + \Phi'(\delta^{pq} - n'^{p} n'^{q}) \right\rangle_{12} \langle \nabla'^{q} V'^{p}\rangle_{12},$$

where $\upsilon^{k} = \langle V'^{k}\rangle_{12}$.

An Example of the Application of the Equation for the Interphase Average Topological Characteristic

Setting $\Phi' = 1$, $n'^{m}n'^{l}$ in (7), we obtain, respectively, the following equations for the specific interphase s_{12}

and the components of the topological tensor $\nu^{ml} = \langle n'^m n'^l \rangle_{12}$ (the components of the tensor of the specific surface are linear with respect to those of the topological tensor and are $s^{ml} = \frac{1}{2}s_{12}(\delta^{ml} - \nu^{ml})$; see [2]):

$$\frac{\partial s_{12}}{\partial t} + \nabla^k(s_{12}\upsilon^k) = s^{pq}\langle \nabla'^q V'^p \rangle_{12}, \qquad (8)$$

$$\frac{\partial(s_{12}\nu^{ml})}{\partial t} + \nabla^k(s_{12}\nu^{ml}\upsilon^k) = -s_{12}\langle \nabla'^q V'^p \rangle_{12}$$
$$\times\left\langle (n'^m_\theta n'^l + n'^m n'^l_\theta)n'^p n'^q_\theta + \frac{(n'^m_\varphi n'^l + n'^m n'^l_\varphi)n'^p n'^q_\varphi}{\sin^2\theta'} - n'^m n'^l(\delta^{pq} - n'^p n'^q)\right\rangle_{12}. \qquad (9)$$

System (8)–(9) is not generally closed. Note that Eq. (8) is the sum of Eqs. (9) with $m = l$. Later on, we shall consider an example of a closed system (8)–(9) with interphase being the free surface of an incompressible Newtonian fluid.

Representation of the Distribution Function as a Series in Spherical Functions

We seek a solution of Eq. (1) for the function s in the form of an expansion in the spherical functions $Y_j^m(\theta, \varphi) = p_j^m(\cos\theta)\exp(im\varphi)$,

$$s = \sum_{j=0}^{\infty}\sum_{m=-j}^{j} s_{j,m}(t, \mathbf{x})p_j^{|m|}(\cos\theta)\exp(im\varphi), \qquad (10)$$

where

$$s_{j,-m} = \bar{s}_{j,m}, \quad s_{j,m} = \frac{1}{2}(\xi_{j,m} - i\eta_{j,m}). \qquad (11)$$

Here, $\xi_{j,m}$ and $\eta_{j,m}$ are real-valued functions of time t and coordinates, i is the imaginary unit, a horizontal bar over a symbol denotes conjugation, and $p_j^m(\cos\theta)$ is an associated Legendre polynomial. Successively substituting the spherical functions from (10) into (7) instead of $\Phi(\mathbf{n})$, we obtain an infinite system of equations for the coefficients $s_{j,m}$.

Let us show, using properties of associated Legendre polynomials, that for $j \le 2$, the functions $s_{j,m}$ depend linearly with constant coefficients on the components of the specific surface tensor s^{ml} and the gradient of the phase volume fraction $\nabla\alpha$. For the associated Legendre polynomials with $j \le 2$, we have (see [4])

$$p_1^0 = x; \quad p_2^0 = \frac{3x^2 - 1}{2}; \quad p_1^1 = \sqrt{1 - x^2};$$
$$p_2^1 = 3x\sqrt{1 - x^2}; \quad p_2^2 = 3(1 - x^2), \qquad (12)$$

where $x = \cos\theta$.

The definitions of normal (3) and of ν^{ml} easily imply

$$s_{12}\nu^{ml} = \begin{pmatrix} \frac{S_{0,0} - S_{2,0}}{3} + \frac{S_{2,-2} + S_{2,2}}{12} & i\frac{S_{2,2} - S_{2,-2}}{12} & \frac{S_{2,-1} + S_{2,1}}{6} \\ & \frac{S_{0,0} - S_{2,0}}{3} - \frac{S_{2,-2} + S_{2,2}}{12} & i\frac{S_{2,1} - S_{2,-1}}{6} \\ & & \frac{S_{0,0} + 2S_{2,0}}{3} \end{pmatrix},$$
$$S_{j,m} = \begin{pmatrix} 1 & 0 & 0 \\ -\nabla^3\alpha & -\nabla^1\alpha + i\nabla^2\alpha & 0 \\ s_{12}\left(\frac{3}{2}\nu^{33} - \frac{1}{2}\right) & s_{12}(3\nu^{13} - i3\nu^{23}) & s_{12}(3\nu^{11} + 3\nu^{22} - i6\nu^{12}) \end{pmatrix}, \qquad (13)$$

where $s_{j,m}\frac{2}{2j+1}\frac{(j+m)!}{(j-m)!}$ is denoted by $S_{j,m}$ for $j, m = 0, 1, 2$; we take into account the equality, $s_{12}\langle\mathbf{n}\rangle_{12} = -\nabla\alpha$ [1] (here, α is the phase volume fraction with respect to which $\mathbf{n}$ is an outer normal).

An Example of a Solution of the Equation for the Distribution Function with Interphase Being the Free Surface of an Incompressible Newtonian Fluid

First, we show that the following equalities are valid on the free surface of an incompressible Newtonian fluid,

$$n^p n_\theta^q \nabla^q V^p = \mathbf{n}(\mathbf{n}_\theta \boldsymbol{\nabla})\mathbf{V} = -\frac{\mathbf{n}_\varphi \boldsymbol{\Omega}}{2\sin\theta},$$
$$n^p n_\varphi^q \nabla^q V^p = \mathbf{n}(\mathbf{n}_\varphi \boldsymbol{\nabla})\mathbf{V} = \frac{\sin\theta \mathbf{n}_\theta \boldsymbol{\Omega}}{2}, \quad (14)$$

where $\boldsymbol{\Omega} = \text{rot}\mathbf{V}$.

On the free surface of an incompressible Newtonian fluid, the stress tangents vanish,

$$2\mu e^{pq} n^p t^q = 0 \text{ or } 2\mu(n^q e^{lq} - n^l n^p n^q e^{pq}) = 0. \quad (15)$$

Here, $e^{pq} = \frac{1}{2}(\nabla^p V^q + \nabla^q V^p)$ is the rate of deformation tensor, $\mathbf{t}$ is the unit tangent vector to the free surface, and μ is the constant coefficient of dynamic viscosity. On the interphase surface, we have

$$\frac{d\mathbf{n}}{dt} = \mathbf{n}[\mathbf{n}(\mathbf{n}\boldsymbol{\nabla})\mathbf{V}] - (\mathbf{n}\boldsymbol{\nabla})\mathbf{V} - \mathbf{n} \times \boldsymbol{\Omega}$$
$$= \mathbf{e}^l(n^l n^p n^q e^{pq} - n^q e^{lq}) - \frac{1}{2}\mathbf{n} \times \boldsymbol{\Omega}, \quad (16)$$

where the $\mathbf{e}^l$ are the orts of the Cartesian coordinate system. It follows from (16) and (15) that, on the free surface,

$$\frac{d\mathbf{n}}{dt} = -\frac{1}{2}\mathbf{n} \times \boldsymbol{\Omega},$$
$$\mathbf{n}[\mathbf{n}(\mathbf{n}\boldsymbol{\nabla})\mathbf{V}] - (\mathbf{n}\boldsymbol{\nabla})\mathbf{V} = \frac{1}{2}\mathbf{n} \times \boldsymbol{\Omega}. \quad (17)$$

Taking $\mathbf{n}_\theta$ and $\mathbf{n}_\varphi$ successively for the tangent vector $\mathbf{t}$ in (15), we obtain, respectively,

$$\mathbf{n}(\mathbf{n}_\theta \boldsymbol{\nabla})\mathbf{V} = -\mathbf{n}_\theta(\mathbf{n}\boldsymbol{\nabla})\mathbf{V},$$
$$\mathbf{n}(\mathbf{n}_\varphi \boldsymbol{\nabla})\mathbf{V} = -\mathbf{n}_\varphi(\mathbf{n}\boldsymbol{\nabla})\mathbf{V}. \quad (18)$$

Taking the inner product of the second equation from (17) by $\mathbf{n}_\theta$ and $\mathbf{n}_\varphi$, respectively, and taking into account the equalities $\mathbf{n} = \mathbf{n}_\theta \times \frac{\mathbf{n}_\varphi}{\sin\theta}$ and (18), we obtain the required equation (14).

Using (14), we can rewrite equation (7) for average topological characteristics as

$$\frac{1}{s_{12}}\left[\frac{\partial}{\partial t}(s_{12}\langle\Phi'\rangle_{12}) + \nabla^k(s_{12}\upsilon^k\langle\Phi'\rangle_{12})\right]$$
$$+ \frac{1}{2}\boldsymbol{\omega}\langle\mathbf{B}[\Phi]\rangle_{12} + \langle C[\Phi]\rangle_{12} = 0. \quad (19)$$

Here,

$$\mathbf{B}[\Phi] = \frac{\partial\Phi}{\partial\varphi}\frac{\mathbf{n}_\theta}{\sin\theta} - \frac{\partial\Phi}{\partial\theta}\frac{\mathbf{n}_\varphi}{\sin\theta}$$
$$= \left\{\frac{\partial\Phi}{\partial\varphi}\cot\theta\cos\varphi + \frac{\partial\Phi}{\partial\theta}\sin\varphi,\right.$$
$$\left.\frac{\partial\Phi}{\partial\varphi}\cot\theta\sin\varphi - \frac{\partial\Phi}{\partial\theta}\cos\varphi, -\frac{\partial\Phi}{\partial\varphi}\right\}, \quad (20)$$
$$C[\Phi] = \Phi n^p n^q e^{pq},$$
$$\boldsymbol{\omega} = \langle\boldsymbol{\Omega}'\rangle_{12}, \quad \varepsilon^{pq} = \langle e'^{pq}\rangle_{12}.$$

The functional dependence on the functions Φ is indicated by brackets. We seek a solution for the distribution function in the form of series (10). It follows from (12) and (20) that, for $j \le 2$ and $\Phi = 1$, p_1^0; p_2^0; $p_1^1 e^{i\varphi}$; $p_2^1 e^{-i\varphi}$; $p_2^2 e^{-i2\varphi}$,

$$\mathbf{B}[1] = \{0, 0, 0\},$$
$$\mathbf{B}[p_1^0] = \left\{ip_1^1\frac{e^{i\varphi} - e^{-i\varphi}}{2}, p_1^1\frac{e^{i\varphi} + e^{-i\varphi}}{2}, 0\right\},$$
$$\mathbf{B}[p_2^0] = \left\{ip_2^1\frac{e^{i\varphi} - e^{-i\varphi}}{2}, p_2^1\frac{e^{i\varphi} + e^{-i\varphi}}{2}, 0\right\},$$
$$\mathbf{B}\lfloor p_1^1 e^{-i\varphi}\rfloor = \{-ip_1, p_1, ip_1^1 e^{-i\varphi}\}, \quad (21)$$
$$\mathbf{B}[p_2^1 e^{-i\varphi}] = \left\{-3ip_2 - \frac{1}{2}ip_2^2 e^{-i2\varphi},\right.$$
$$\left.-3p_2 + \frac{1}{2}ip_2^2 e^{-i2\varphi}, ip_2^1 e^{-i\varphi}\right\},$$
$$\mathbf{B}\lfloor p_2^2 e^{-i2\varphi}\rfloor = \{-2ip_2^1 e^{-i\varphi}, -2p_2^1 e^{-i\varphi}, 2ip_2^2 e^{-i2\varphi}\}.$$

In calculating $C[\Phi]$, we use equalities that follow from (3) and (12):

$$p_2 = \frac{3}{2}(n^3)^2 - \frac{1}{2}, \quad p_1^1 e^{-i\varphi} = 3n^1 n^3 - 3in^2 n^3,$$
$$p_2^2 e^{-i2\varphi} = 3[(n^1)^2 - (n^2)^2] + 6in^1 n^2.$$

We have

$$C[1] = n^p n^q e^{pq}, \quad C[p_1] = n^p e^{p3},$$
$$C[p_2] = \frac{3}{2}n^p n^3 e^{p3} - \frac{1}{2}n^p n^q e^{pq},$$
$$C\lfloor p_1^1 e^{-i\varphi}\rfloor = n^p e^{p1} - in^p e^{p2},$$
$$C\lfloor p_2^1 e^{-i\varphi}\rfloor = 3(n^p n^1 e^{p1} + n^p n^3 e^{p3})$$
$$- 3i(n^p n^2 e^{p2} + n^p n^3 e^{p3}), \quad (22)$$
$$C\lfloor p_2^2 e^{-i2\varphi}\rfloor = 3(n^p n^1 e^{p1} - n^p n^2 e^{p2})$$
$$+ 6i(n^p n^1 e^{p2} + n^p n^2 e^{p1}).$$

Substituting (21)–(22) into (19), we obtain the following system of equations for the components of the topological tensor

$$\frac{\partial S_{0,0}}{\partial t} + \mathrm{div}(S_{0,0}\mathbf{\upsilon}) + s_{12}\nu^{pq}\varepsilon^{pq} = 0,$$

$$\frac{\partial S_{1,0}}{\partial t} + \mathrm{div}(S_{1,0}\mathbf{\upsilon})$$

$$+ \frac{\boldsymbol{\omega}}{2}\left\{\frac{i}{2}(S_{2,-1} - S_{2,1}), \frac{1}{2}(S_{2,-1} + S_{2,1}), 0\right\}$$

$$+ \frac{3}{2}s_{12}\nu^{p3}\varepsilon^{p3} - \frac{1}{2}s_{12}\nu^{pq}\varepsilon^{pq} = 0,$$

$$\frac{\partial S_{1,1}}{\partial t} + \mathrm{div}(S_{1,1}\mathbf{\upsilon}) + \frac{\boldsymbol{\omega}}{2}\{-iS_{1,0}, -S_{1,0}, iS_{1,1}\}$$

$$-\varepsilon^{p1}\nabla^p\alpha + i\varepsilon^{p2}\nabla^p\alpha = 0, \qquad (23)$$

$$\frac{\partial S_{2,1}}{\partial t} + \mathrm{div}(S_{2,1}\mathbf{\upsilon})$$

$$+ \frac{\boldsymbol{\omega}}{2}\left\{-3iS_{2,0} - \frac{1}{2}iS_{2,2}, -3S_{2,0} + \frac{1}{2}S_{2,2}, iS_{2,1}\right\}$$

$$+ 3s_{12}\nu^{p1}\varepsilon^{p1} + 3s_{12}\nu^{p3}\varepsilon^{p3}$$

$$-3is_{12}\nu^{p2}\varepsilon^{p2} - 3is_{12}\nu^{p3}\varepsilon^{p3} = 0,$$

$$\frac{\partial S_{2,2}}{\partial t} + \mathrm{div}(S_{2,2}\mathbf{\upsilon}) + \frac{\boldsymbol{\omega}}{2}\{-2iS_{2,1}, -2S_{2,1}, 2iS_{2,2}\}$$

$$+ 3s_{12}\nu^{p1}\varepsilon^{p1} - 3s_{12}\nu^{p3}\varepsilon^{p2} + 6is_{12}\nu^{p1}\varepsilon^{p2}$$

$$+ 6is_{12}\nu^{p2}\varepsilon^{p1} = 0,$$

where the quantities $S_{j,m}$ are expressed in terms of the components of the topological tensor from (13). Thus, it follows from (23) and (13) that if the interphase is the free surface of an incompressible Newtonian fluid, then the system of equations for the components of the topological tensor is closed.

CONCLUSIONS

A new equation (7) for average topological characteristics of the interphase surface in a heterogeneous media is obtained. Interestingly, despite the complex interphase configuration in a heterogeneous medium, it turns out to be possible to determine average topological characteristics of the surface on the basis of the exact equation (7). Substituting the product $n^p n^q$ of the components of the normal for Φ in Eq. (7) yields system (8)–(9) for the components of the topological tensor ν^{pq}, which occur in the following formulas for calculating specific area and the volumetric density vector of the capillary forces: $s_{12} = s^{ll}$ and $P^m_\Sigma = 2\Sigma\frac{\partial s^{ml}}{\partial x^l}$. Up to the second harmonic, the coefficients in the spherical harmonic expansion of the distribution function of the interphase specific area in terms of the tilt angles of interphase normals can be linearly expressed in terms of the components of the specific surface tensor and the gradient of the phase volume fraction (see (13)). The dependence of the expansion coefficients on the components of the topological tensor can be considered as the physical meaning of the coefficients. The remarkable fact that the system of equations (23) for the components of the topological tensor is closed for an interphase being the free surface of an incompressible Newtonian fluid is discovered.

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