STOCHASTIC FRACTALS WITH MARKOVIAN REFINEMENTS

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We consider the random point fields with Markovian refinements we previously introduced. For this class of disordered structures possessing scaling and spatial homogeneity, we give the complete proof of the self-averageability theorem for the fractal dimension.

1. Introduction

In this work, we investigate stochastic fractals that can be constructed using cell refinements of the immersion space in the method proposed in [1]. This method is a development of the method of iterated functions [2] applied to the problem of constructing realistic models of media whose disorderedness is characterized by a broad spectrum of scales with the dimension of length. In this case, the statistical characteristics of the media in question averaged over domains of the corresponding sizes vary self-similarly in the transition from one scale to another. This work must be considered a continuation of [1], and we therefore keep all the notation and terminology used there. In Sec. 2, we only briefly describe the main construction of *random point fields with Markovian refinements* and set up the self-averageability problem for the fractal dimension of stochastic fractals and the nonrandomness problem for the fractal measure type defined on these fractals. In Sec. 3, we construct an example of stochastic fractals with a random fractal dimension. In Sec. 4, we give the complete proof of the nonrandomness theorem for the fractal dimension of class-F$[q]$ fields. In [1], this proof was given under the restriction that for each cell with the size L/N^{md} (where L and N are parameters of the model and d is the space dimension) that contains points from the fractal, the probability distribution π_l, $l = 1, , 2, \ldots, N^d$, of its "decay" into l cells with the sizes $L/N^{(m+1)d}$ is such that $\pi_1 = 0$. This restriction is not physical and must be removed.

2. Refinements and stochastic fractals

By a cell refinement of a cube $\Lambda = [0, L)^D$, the immersion space of the fractal, we mean a sequence of partitions into nonintersecting right-semiopen d-dimensional cubes (cells)

$$\mathfrak{K}_m = \{A^{(m)}_{\mathbf{n}_1,\ldots,\mathbf{n}_m};\ \mathbf{n} \in \mathbb{Z}^d,\ \mathbf{n}_j \in \mathfrak{I}^d_N,\ j = 1, 2, \ldots, m\},$$

where $\mathfrak{I}_N = \{0, \ldots, N-1\}$ and $m = 1, 2, \ldots$. The cells $A^{(m)}_{\mathbf{n}_1,\ldots,\mathbf{n}_m}$ of an mth-order partition are such that their edge is of the length L/N^m, where $N > 1$ is the refinement degree. The set $\mathfrak{K}_m$ of all cells of the mth-order partition is described by all the possible ordered sets $\boldsymbol{\xi} = [\mathbf{n}, \mathbf{n}_1, \ldots, \mathbf{n}_m]$. At the same time, each random realization $\mathfrak{X}$ is determined by a sequence of cell coverings $\mathfrak{X}^{(m)}$, $m = 1, 2, \ldots$, where the sets $\mathfrak{X}^{(m)}$ are constructed from the cells of the mth-order partitions that have a nonempty intersection with $\mathfrak{X}$. We then have $\mathfrak{X}^{(m+1)} \subset \mathfrak{X}^{(m)}$. It is useful to introduce the projection operation $\mathsf{K}_m(\mathfrak{X}) = \mathfrak{X}^{(m)}$.

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For each random realization $\mathfrak{X}$ of the fractal (at each of its points), we have the fractal dimension

$$D = \sup\{\alpha\colon \lim_{m\to\infty} N^{-\alpha m}|\mathfrak{X}^{(m)}| = 0\}, \tag{1}$$

where $|\cdot|$ is the number of cells entering the mth-order covering. The dimension D is therefore a random variable in general. Similarly, for each realization $\mathfrak{X}$ and a domain $G \subset \Lambda$, there is the Hausdorff–Carathéodory measure

$$\mu(X \cap G; s) = \lim_{m\to\infty} \sum_{\xi\colon A_\xi^{(m)} \in \text{covering } \mathfrak{X}\cap G} s\left(\frac{L}{N^m}\right), \tag{2}$$

where $s(\cdot)$ is a positive increasing function on $[0,\infty)$ that determines the measure type (see [3]). In our case, this implies that for a fixed realization $\mathfrak{X}$ and a fixed domain G, we find a decreasing sequence $0 < s_m = s(L/N^m)$ such that the limit in Eq. (2) is nonvanishing and finite. This sequence is unambiguously defined up to the asymptotic equivalence ($s'_m \sim s_m$ if $\lim(s'_m/s_m) = \text{const}$). The corresponding equivalence class $\{s_m\}$ is a characteristic of the measure. With this characteristic defined, Eq. (2) specifies a random variable determined by the chosen realization $\mathfrak{X}$. However, the characteristic $\{s_m\}$ in general depends on $\mathfrak{X}$ and on the choice of the domain G. Stochastic fractals for which the class $\{s_m\}$ is independent of $\mathfrak{X}$ and of G with probability 1 were previously [1] called fractals with a homogeneous fractal stochastic measure of the *nonrandom type* (stochastic because it takes random values). Structures of precisely this type are primarily important in describing fractally disordered physical media and in constructing the statistical mechanics of fluctuating physical fields on such media.

The probabilistic description of random sets $\{\mathfrak{X}\}$ in the cube Λ (the stochastic fractals) is provided by a sequence of probability distributions $\{P_m(\cdot); m = 1, 2, \dots\}$ for random realizations of the projections $\mathfrak{X}^{(m)}$ [1], where

$$P_m(H) = \mathbf{P}\{\mathfrak{X}\colon \mathsf{K}_m(\mathfrak{X}) = H\}, \quad H \subset \mathfrak{K}_m, \quad m = 1, 2, \dots. \tag{3}$$

The functions $P_m(\cdot)$ are normalized,

$$\sum_{H\subset\mathfrak{K}_m} P_m(H) = 1, \quad m = 1, 2, \dots,$$

and satisfy the compatibility condition [1]

$$\sum_{\{\mathsf{S}_m(\boldsymbol{\xi},G);\, \boldsymbol{\xi}\in\mathsf{K}_m(G)\}} P_{m+1}(G) = P_m(\mathsf{K}_m(G)),$$

where $\mathsf{S}_l(\boldsymbol{\xi}, H) = \{\boldsymbol{\eta} \in H\colon \mathsf{K}_l(\boldsymbol{\eta}) = \boldsymbol{\xi}\}$.

In [1], we introduced the class $\mathrm{F}[q]$ of random point sets for which the terms in the sequence $\{\{P_m(H); H \subset \mathfrak{K}_m\}; m = 1, 2, \dots\}$ are related by

$$P_{m+1}(H) = Q_m(H \mid \mathsf{K}_m(H))P_m(\mathsf{K}_m(H)), \tag{4}$$

where the conditional probability $\mathbf{P}\{\mathfrak{X}\colon \mathsf{K}_{m+1}(\mathfrak{X}) = H \mid \mathsf{K}_m(\mathfrak{X}) = \mathsf{K}_m(H)\} = Q_m(H \mid \mathsf{K}_m(H))$ can be represented as

$$Q_m(H \mid \mathsf{K}_m(H)) = \prod_{\boldsymbol{\xi}\in\mathsf{K}_m(H)} q(\mathsf{T}_m(\boldsymbol{\xi}; H)), \quad H \in \mathfrak{K}_{m+1}. \tag{5}$$

Here, the function $q(\cdot)$ is a probability distribution on the set $2^{\mathfrak{I}_N^d}$ with $q_m(\varnothing, \boldsymbol{\xi}) = 0$,

$$\sum_{\varnothing\neq\sigma\subset\mathfrak{I}_N^d} q_m(\sigma, \boldsymbol{\xi}) = 1,$$

and $\mathsf{T}_m(\boldsymbol{\xi}; H) = \{\mathbf{n}_{m+1}\colon [\boldsymbol{\xi}, \mathbf{n}_{m+1}] \in \mathsf{S}_m(\boldsymbol{\xi}, H)\}$ (see [1]). These random sets were called random point fields with Markovian refinements.

3. Random-dimension fractals

We now construct an example of stochastic fractals whose dimension is a random variable. We use the results in [1]. The fractal dimension of a random set with Markovian refinements is defined as $D = \log a/\log N$, where $a = \Pi'(1)$ and

$$\Pi(x) = \sum_{\sigma\subset\mathfrak{I}_N^d} q(\sigma)x^{|\sigma|} \equiv \sum_{l=1}^{N^d} x^l \pi_l \tag{6}$$

is the generating function of the probability distribution

$$\pi_l = \sum_{\sigma:\ \sigma\subset\mathfrak{I}_N^d, |\sigma|=l} q(\sigma)$$

of the random variable $\tilde{l}$ with the set of values $\{l = |\sigma|; \sigma \subset \mathfrak{I}_N^d\}$.

We introduce the random process $\{\tilde{q}_m(\,\cdot\,); m = 1, 2, \dots\}$ with values in distributions on $2^{\mathfrak{I}_N^d}$. The exact structure of the probabilistic space corresponding to this process is inessential (physically, the process can be interpreted as a stochastic dynamic system forming a fractal), and we do not specify it explicitly. We use the tilde to indicate random variables pertaining to this probabilistic space. We now construct the conditional transition probabilities $\widetilde{Q}_m(\cdot \mid \cdot)$ using a formula similar to (5),

$$\widetilde{Q}_m(H \mid \mathsf{K}_m(H)) = \prod_{\boldsymbol{\xi}\in\mathsf{K}_m(H)} \tilde{q}_m(\mathsf{T}_m(\boldsymbol{\xi}; H)), \quad H \in \mathfrak{K}_{m+1},$$

and use these in Eq. (4) to represent the distributions $\widetilde{P}_m(\,\cdot\,)$ that are similar to (3) as

$$\widetilde{P}_{m+1}(H) = \widetilde{Q}_m(H \mid \mathsf{K}_m(H))\widetilde{P}_m(\mathsf{K}_m(H)).$$

Each set of these distributions determines a probabilistic measure for the stochastic fractal. This measure is conditional and can be obtained by fixing a realization of the process $\{\tilde{q}_m(\,\cdot\,)\}$. The "true" probability distribution of the stochastic fractal under construction is then defined by the collection of unconditional distributions $\{P_m(\,\cdot\,) = \langle\widetilde{P}_m(\,\cdot\,)\rangle; m = 1, 2, \dots\}$ of the average values with respect to the distribution of the process $\{\tilde{q}_m(\,\cdot\,)\}$.

We assume the process $\{\tilde{q}_m(\,\cdot\,)\}$ to be such that each trajectory converges with probability 1 to a final state $\tilde{q}(\,\cdot\,)$, which is random for processes of this type in the general case. Then $\tilde{a}_m \to \tilde{a}$, where $\tilde{a}_m = \widetilde{\Pi}'_m(1)$ and, similarly to (6),

$$\widetilde{\Pi}_m(x) = \sum_{\sigma\subset\mathfrak{I}_N^d} \tilde{q}(\sigma)x^{|\sigma|}.$$

Therefore, $\tilde{a}$ is not a certain variable in general. Hence, the dimension of the fractal, which is given by $\log\tilde{a}/\log N$ because the variable $\tilde{a}$ is independent of m as $m \to \infty$, is not certain either and has a specific probability distribution. A specific realization of this construction can be built, e.g., using the *ruin scheme* [4] involving two possible final states realized with probabilities that are different from 0 and 1.

We note that in modeling the process of the stochastic fractal formation based on the random processes $\{\tilde{q}_m(\,\cdot\,)\}$ with random final states, the stochastic self-similarity condition (understood physically) for the structure under formation is violated in general. We can therefore conclude that the above class of stochastic fractals does not admit a physical interpretation. However, because the stochastic self-similarity combined with the spatial homogeneity is a rather subtle notion from the mathematical standpoint, the problem of selecting the broadest family of random point sets possessing a nonrandom fractal dimension is pertinent.

4. Self-averageability theorem for the fractal dimension

As in [1], the main tool in proving the self-averageability of the fractal dimension is the *ramified Markov process* $\{\mathfrak{k}_m = |\mathfrak{X}^{(m)}|; m = 0, 1, 2, \dots\}$ given by the number of cells in the covering of the fractal by the partition cells at the mth step. The probability distribution $\{P_m(\,\cdot\,); m = 1, 2, \dots\}$ determines the probability distribution for this process [1], namely, there is the relation

$$\mathbf{P}\{\mathfrak{k}_{m+1} = i\} = \sum_{j=0}^{N^{md}} \mathbf{P}\{\mathfrak{k}_{m+1} = i \mid \mathfrak{k}_m = j\}\mathbf{P}\{\mathfrak{k}_m = j\}.$$

In accordance with (5), the conditional transition probability then becomes

$$\mathbf{P}\{\mathfrak{k}_{m+1} = i \mid \mathfrak{k}_m = j\} = \sum_{\substack{l_1,\dots,l_j \geq 1 \\ l_1+\dots+l_j = i}}^{N^d} \prod_{k=1}^{j} \pi_{l_k}. \tag{7}$$

Associated with the above random process is the random sequence $\{\mathfrak{c}_m; m = 0, 1, 2, \dots\}$, $\mathfrak{c}_m = \mathfrak{k}_m / a^m$. The following theorem was proved in [1].

Theorem 1. *With probability* 1, *the sequence* $\{\mathfrak{c}_m; m = 0, 1, 2, \dots\}$ *has a limit, a random variable* $\mathfrak{c}$.

Moreover, it was proved that under the restriction $\pi_1 = 0$ on the distribution π_l, it follows that $\mathfrak{c} \neq 0$, and as a corollary of these two facts, the fractal dimension becomes nonrandom.

In what follows, we establish the self-averageability of the fractal dimension for any random set with a Markovian refinement. The main point is the proof that for any probability distribution $\pi_l \neq \delta_{1l}$, the inequality $\mathfrak{c} \neq 0$ is satisfied with probability 1. Before proving that, we formulate and prove several auxiliary statements.

We introduce an auxiliary random process by letting the random sequence $\{\mathfrak{i}_m\}$ be the constancy number in the chain $\{\mathfrak{k}_m\}$. It is defined by

$$\mathfrak{i}_m = \sum_{l=1}^{m} \delta(\mathfrak{k}_l - \mathfrak{k}_{l-1}), \qquad \delta(k) = \begin{cases} 1, & k = 0, \\ 0, & k \neq 0. \end{cases} \tag{8}$$

The sequence of pairs $\{(\mathfrak{k}_m, \mathfrak{i}_m); m = 0, 1, 2, \dots\}$ is a Markov chain because such is the random sequence $\{\mathfrak{k}_m\}$. Indeed,

$$\mathbf{P}\{\mathfrak{k}_{m+1} = k_{m+1} \mid \mathfrak{E}_m = \{\mathfrak{k}_m = k_m, \dots, \mathfrak{k}_0 = 1\} = \mathbf{P}\{\mathfrak{k}_{m+1} = k_{m+1} \mid \mathfrak{k}_m = k_m\}.$$

Together with (8), this in turn implies the relation

$$\begin{aligned}
&\mathbf{P}\{\mathfrak{k}_{m+1} = k_{m+1}, \mathfrak{i}_{m+1} = i_{m+1} \mid \mathfrak{E}_m\} = \\
&\qquad = \mathbf{P}\{\mathfrak{k}_{m+1} = k_{m+1}, i_m + \delta(k_{m+1} - k_m) = i_{m+1} \mid \mathfrak{E}_m\} = \\
&\qquad = \mathbf{P}\{\mathfrak{k}_{m+1} = k_{m+1}, i_m + \delta(k_{m+1} - k_m) = i_{m+1} \mid (\mathfrak{k}_m = k_m, \mathfrak{i}_m = i_m)\},
\end{aligned}$$

where $\mathfrak{E}_m = \{(\mathfrak{k}_m = k_m, \mathfrak{i}_m = i_m), \dots, (\mathfrak{k}_0 = 1, \mathfrak{i}_0 = 0)\}$, which indicates that the chain $\{(\mathfrak{k}_m, \mathfrak{i}_m); m = 0, 1, 2, \dots\}$ is Markovian.

We introduce the function

$$\mu(k \mid s) = \begin{cases} \dfrac{k}{s}, & k = 0 \pmod{s}, \\ \left[\dfrac{k}{s}\right] + 1, & k \neq 0 \pmod{s}. \end{cases}$$

The equation for the Markov chain $\{(\mathfrak{k}_m, \mathfrak{i}_m); m = 0, 1, 2, \dots\}$ then becomes

$$\mathbf{P}\{\mathfrak{i}_{m+1} = i+1, \mathfrak{k}_{m+1} = k\} = \pi_1^k \mathbf{P}\{\mathfrak{i}_m = i, \mathfrak{k}_m = k\} +$$
$$+ \sum_{k'=\mu(k|N^d)}^{k-1} \mathbf{P}\{\mathfrak{k}_{m+1} = k \mid \mathfrak{k}_m = k'\}\mathbf{P}\{\mathfrak{k}_m = k', \mathfrak{i}_m = i+1\}(1-\delta_{k1}). \tag{9}$$

We next define the generating function $\{\Psi_m(x \mid z)\}$ of the one-point probability distribution of the chain $\{(\mathfrak{k}_m, \mathfrak{i}_m); m = 0, 1, 2, \dots\}$ by the relation

$$\Psi_m(x \mid z) = \sum_{k=1}^{N^{dm}} \sum_{i=0}^{m} x^k z^i \mathbf{P}\{\mathfrak{k}_m = k, \mathfrak{i}_m = i\}. \tag{10}$$

It is easy to see that $\Psi_m(x \mid 1) = \Psi_m(x)$, where $\Psi_m(\,\cdot\,)$ is the generating function of the one-point probability distribution of the process $\{\mathfrak{k}_m\}$,

$$\Psi_m(x) = \sum_{k=1}^{N^{dm}} x^k \mathbf{P}\{\mathfrak{k}_m = k\}.$$

Theorem 2. *The generating function $\Psi_m(x \mid z)$ satisfies the difference equation*

$$\Psi_{m+1}(x \mid z) - \Psi_{m+1}(x) = [z-1]\Psi_m(\pi_1 x \mid z) + \Psi_m(\Pi(x) \mid z) - (\Psi_m(\Pi(x) \mid 0) - \Psi_m(\pi_1 x \mid 0)). \tag{11}$$

Proof. Using definition (10) and Eq. (9), we obtain

$$\Psi_{m+1}(x \mid z) - \Psi_{m+1}(x \mid 0) = \sum_{k=1}^{N^{d(m+1)}} \sum_{i=1}^{m+1} z^i x^k \mathbf{P}\{\mathfrak{i}_{m+1} = i, \mathfrak{k}_{m+1} = k\} =$$
$$= \sum_{k=1}^{N^{d(m+1)}} \sum_{i=0}^{m} z^{i+1} x^k \mathbf{P}\{\mathfrak{i}_{m+1} = i+1, \mathfrak{k}_{m+1} = k\}.$$

It then follows from Eq. (11) that

$$\Psi_{m+1}(x \mid z) - \Psi_{m+1}(x \mid 0) =$$
$$= z\Psi_m(\pi_1 x \mid z) + \sum_{k=2}^{N^{d(m+1)}} x^k \sum_{i=0}^{m-1} z^{i+1} \sum_{l=\mu(k|N^d)}^{k-1} \mathbf{P}\{\mathfrak{k}_{m+1} = k \mid \mathfrak{k}_m = l\}\mathbf{P}\{\mathfrak{k}_m = l, \mathfrak{i}_m = i+1\} \tag{12}$$

because for $i = m$, we have $\mathbf{P}\{\mathfrak{k}_m = l, \mathfrak{i}_{m+1} = m+1\} = \delta_{l1}$. Using the explicit representation in Eq. (7), we write

$$\sum_{k=2}^{N^{d(m+1)}} x^k \sum_{i=0}^{m-1} z^{i+1} \sum_{l=\mu(k|N^d)}^{k-1} \mathbf{P}\{\mathfrak{k}_m = l, \mathfrak{i}_m = i+1\} \sum_{\substack{s_1+s_2+\dots+s_l=k \\ s_j \leq 1}} \prod_{j=1}^{l} \pi_{l_j} =$$
$$= \sum_{i=0}^{m} z^{i+1} \sum_{l=1}^{N^{dm}} \mathbf{P}\{\mathfrak{k}_m = l, \mathfrak{i}_m = i+1\} \sum_{k=l+1}^{lN^{dm}} x^k \sum_{\substack{s_1+s_2+\dots+s_l=k \\ s_j \leq 1}} \prod_{j=1}^{l} \pi_{s_j}. \tag{13}$$

Transforming the inner sum in the right-hand side of this formula as

$$\sum_{k=l+1}^{lN^{dm}} x^k \sum_{\substack{s_1+s_2+\dots+s_l=k\\ s_j\ge 1}} \prod_{j=1}^{l} \pi_{s_j} = \sum_{k=l}^{lN^{dm}} x^k \sum_{\substack{s_1+s_2+\dots+s_l=k\\ s_j\ge 1}} \prod_{j=1}^{l} \pi_{s_j} - (x\pi_1)^l,$$

$$\sum_{\substack{s_1+s_2+\dots+s_l\le lN^{dm}\\ s_j\ge 1}} \prod_{j=1}^{l} x^{s_j}\pi_{s_j} = \prod_{j=1}^{l}\sum_{s_j=1}^{N^d} \pi_{s_j}x^{s_j} = (\Pi(x))^l \tag{14}$$

(where we use $\pi_l = 0$ for $l > N^d$) and taking Eqs. (12)–(14) into account, we finally obtain

$$\Psi_{m+1}(x \mid z) - \Psi_{m+1}(x \mid 0) = z\Psi_m(\pi_1 x \mid z) + \sum_{i=0}^{m} z^{i+1} \sum_{l=1}^{N^{dm}} \mathbf{P}\{\mathfrak{k}_m = l, \mathfrak{i}_m = i+1\}((\Pi(x))^l - (\pi_1 x)^l) =$$

$$= z\Psi_m(\pi_1 x \mid z) + \Psi_m(\Pi(x) \mid z) -$$

$$- \Psi_m(\pi_1 x \mid z) - (\Psi_m(\Pi(x) \mid 0) - \Psi_m(\pi_1 x \mid 0)),$$

as was to be shown.

Directly from Eq. (11), we obtain an expression for the average constancy number $\langle \mathfrak{i}_m \rangle$.

Lemma 1. *The formula*

$$\langle \mathfrak{i}_m \rangle = \sum_{l=0}^{m-1} \Psi_l(\pi_1)$$

holds.

Proof. We note that

$$\langle \mathfrak{i}_m \rangle = \sum_{i=1}^{m} i\mathbf{P}\{\mathfrak{i}_m = i\} = \left(\frac{\partial \Psi_m(1 \mid z)}{\partial z}\right)\Bigg|_{z=1}.$$

Because $\Pi(1) = 1$, it follows from Eq. (11) that

$$\langle \mathfrak{i}_{m+1} \rangle = \Psi_m(\pi_1) + \left(\frac{\partial \Psi_m(1 \mid z)}{\partial z}\right)\Bigg|_{z=1} = \Psi_m(\pi_1) + \langle \mathfrak{i}_m \rangle.$$

Under the condition $\langle \mathfrak{i}_0 \rangle = 0$, the solution of the resulting difference equation is given by

$$\langle \mathfrak{i}_m \rangle = \sum_{l=0}^{m-1} \Psi_l(\pi_1). \tag{15}$$

The lemma is proved.

Corollary. *The average value of the limiting random variable* $\mathfrak{i}_\infty = \lim_{m\to\infty} \mathfrak{i}_m$ *is given by*

$$\langle \mathfrak{i}_\infty \rangle = \sum_{l=0}^{\infty} \Psi_l(\pi_1). \tag{16}$$

Proof. The sequence $\mathfrak{i}_m$ is monotonically nondecreasing. Therefore, there exists $\mathfrak{i}_\infty$ (with the infinite value allowed). Taking the limit in (15), we obtain Eq. (16), as was to be proved.

We next establish a property of the random sequence $\{\mathfrak{i}_m; m = 0, 1, 2, \dots\}$ that is used in proving the main theorem.

Theorem 3. *The random variable* $\mathfrak{i}_\infty$ *has a finite first moment as* $\Pi(x) \not\equiv x$,

$$\langle \mathfrak{i}_\infty \rangle < \infty.$$

Therefore, in particular, the sequence is finite with probability 1.

Proof. We must prove that the series $\sum_{m=0}^{\infty} \Psi_m(\pi_1)$ converges. Because of the inequality $x\Pi'(x) > \Pi(x)$, we have $(\Pi(x)/x)' > 0$. For $y < x$, we then have the inequality $\Pi(y)/y < \Pi(x)/x$, which we use to obtain

$$\frac{\Pi^{(k)}(\pi_1)}{\Pi^{(k-1)}(\pi_1)} = \frac{\Pi(\Pi^{(k-1)}(\pi_1))}{\Pi^{(k-1)}(\pi_1)} \leq \frac{\Pi(\Pi^{(k-2)}(\pi_1))}{\Pi^{(k-2)}(\pi_1)} \leq \cdots \leq \frac{\Pi(\pi_1)}{\pi_1}.$$

This implies that

$$\Pi^{(l)}(\pi_1) = \pi_1 \prod_{k=1}^{l} \frac{\Pi^{(k)}(\pi_1)}{\Pi^{(k-1)}(\pi_1)} < \pi_1 \left(\frac{\Pi(\pi_1)}{\pi_1}\right)^l.$$

Because $\Pi(x) \not\equiv x$, the estimate derived above and the property $\Pi(x) \leq x$ of the polynomial $\Pi(x)$ (which is obvious for $x \leq 1$) imply that

$$\sum_{m=0}^{\infty} \Psi_m(\pi_1) = \sum_{m=0}^{\infty} \Pi^{(m)}(\pi_1) \leq \pi_1 \sum_{m=0}^{\infty} \left(\frac{\Pi(\pi_1)}{\pi_1}\right)^m = \frac{\pi_1}{1 - \Pi(\pi_1)/\pi_1} < \infty.$$

The theorem is proved.

Remark. The case where $\Pi(x) = x$ occurs for $\pi_l = 0$, $l > 1$, and $\pi_1 = 1$; the fractal dimension is then $D = 0$.

The statement proved above implies, in particular, that with probability 1, there exists only a finite set of constancy numbers of the random sequence $\{\mathfrak{k}_m\}$.

We introduce the numerical sequences $\{r_{l,m}; m = 1, 2, \ldots\}$, $l = 1, 2, \ldots$,

$$r_{l,m} = a^m \int_0^1 dx_1 \int_0^{x_1} dx_2 \cdots \int_0^{x_{l-1}} \Psi_m(x_l) dx_l,$$

where $a = \Pi'(1)$. Then $r_{l,m}/a^m = \langle [(\mathfrak{k}_m + 1)(\mathfrak{k}_m + 2) \cdots (\mathfrak{k}_m + l)]^{-1} \rangle$.

Lemma 2. *For the sequences* $\{r_{l,m};\ m = 1, 2, \ldots\}$, $l = 1, 2, \ldots$, *the recursive estimate*

$$r_{l,m+1} \leq a^{1-l} r_{l,m} + C_l r_{l+1,m} \tag{17}$$

holds, where

$$C_l = \frac{a}{(l-1)!} \left(r_{l,l}(1) + \int_0^1 |r_{l+1,l}(x)| dx \right).$$

Estimate (17) is derived in Theorem 2 in [1].

Lemma 2 and Theorem 3 now imply the following corollary.

Corollary. *The estimate* $\langle \mathfrak{k}_m^{-1} \rangle < \mathrm{const} \cdot a^{-m}$ *holds.*

Proof. Setting $l = 1$ in (17), we have

$$\langle (\mathfrak{k}_{m+1} + 1)^{-1} \rangle = a^{-(m+1)} r_{l,m+1} \leq a^{-1}(a^{-m} r_{1,m}) + a^{-1} C_1 a^{-m} r_{2,m},$$

where

$$C_1 = a\left(r_{1,1}(1) + \int_0^1 |r_{2,1}(x)|dx\right), \qquad r_{1,1}(y) = \frac{d}{dy}\Pi^{(-1)}(y) > 0.$$

It then follows that

$$\langle(\mathfrak{k}_{m+1}+1)^{-1}\rangle < a^{-1}\langle(\mathfrak{k}_m+1)^{-1}\rangle + a^{-1}C_1\langle[(\mathfrak{k}_m+1)(\mathfrak{k}_m+2)]^{-1}\rangle. \tag{18}$$

In view of Theorem 3, the series $\sum_{m=0}^{\infty}\langle[(\mathfrak{k}_m+1)(\mathfrak{k}_m+2)]^{-1}\rangle$ converges. Indeed,

$$\sum_{m=0}^{M}\langle[(\mathfrak{k}_m+1)(\mathfrak{k}_m+2)]^{-1}\rangle < \left\langle \sum_{\substack{m=0,\\ \mathfrak{k}_m\neq\mathfrak{k}_{m+1}}}^{M} [(\mathfrak{k}_m+1)(\mathfrak{k}_m+2)]^{-1} + \mathfrak{i}_M \right\rangle <$$

$$< \langle \mathfrak{i}_M\rangle + \sum_{m=0}^{M}[(m+1)(m+2)]^{-1}$$

because $\mathfrak{k}_m \geq m$. Taking the limit as $M\to\infty$, we obtain the sought convergence of the series.

Applying inequality (18) derived above, we find

$$\langle\mathfrak{k}_m^{-1}\rangle < a^{-m}\left(\langle\mathfrak{k}_0^{-1}\rangle + a^{-1}C_1\sum_{n=0}^{m-1}\langle[(\mathfrak{k}_m+1)(\mathfrak{k}_m+2)]^{-1}\rangle\right),$$

which leads to the sought estimate with the constant

$$\text{const} = \langle\mathfrak{k}_0^{-1}\rangle + a^{-1}C_1\sum_{n=0}^{m-1}\langle[(\mathfrak{k}_m+1)(\mathfrak{k}_m+2)]^{-1}\rangle,$$

as was to be shown.

Among all sequences $\{r_{l,m};\ m=0,1,2,\dots\}$, the one with $l=2$ is particularly important.

Lemma 3. *If the series* $S_2=\sum_{m=0}^{\infty} r_{2,m}$ *converges, then* $\mathfrak{c}^{-1}<\infty$ *with probability* 1.

Proof. If the series S_2 converges, we consecutively use inequalities (17) with $l=1$,

$$r_{1,m+1} < r_{1,m} + C_1 r_{2,m},$$

for $m=0,1,2,\dots,n$ to obtain

$$r_{1,n+1} < r_{1,0} + C_1\sum_{m=0}^{n} r_{2,m}$$

by induction on n. Taking the limit as $n\to\infty$, we obtain $\sup_n r_{1,n} < r_{1,0}+C_1S_2$. Therefore, there exists a finite upper bound $\sup_m a^m\langle\mathfrak{k}_m^{-1}\rangle = \sup_m r_{1,m}$. Then

$$\langle\mathfrak{c}^{-1}\rangle = \lim_{m\to\infty}\langle\mathfrak{c}_m^{-1}\rangle \leq \sup_m a^m\langle\mathfrak{k}_m^{-1}\rangle.$$

Because the averaged value in the right-hand side is finite, the lemma is proved.

To establish the main result, it therefore suffices to prove that the series S_2 converges. For an arbitrary distribution π_l (except the trivial one, in which case $\pi_l=\delta_{1l}$), this proof is rather cumbersome, and we divide it into several lemmas.

Lemma 4. *Let α_m be a monotonically decreasing sequence and $\mathfrak{f}_n$ be a random sequence for which with probability 1, there exists a random number $\mathfrak{m}$ such that the condition $\mathfrak{f}_m < \alpha_m$ is satisfied for $m > \mathfrak{m}$. Starting with a certain nonrandom number n, the sequence $\langle \mathfrak{f}_m \rangle$ then also satisfies the inequality*

$$\langle \mathfrak{f}_m \rangle < \alpha_m. \tag{19}$$

Proof. We suppose the converse, i.e., that an infinite set of numbers $m_1, m_2, \dots$ exists such that the inequalities $\langle \mathfrak{f}_{m_i} \rangle \geq \alpha_{m_i}$ are satisfied. For each m_i, $i = 1, 2, \dots,$ the probability that the inequality is satisfied is then nonzero, which contradicts Eq. (19). The lemma is proved.

We introduce the random sequence

$$\frac{\mathfrak{k}_{m+1}}{\mathfrak{k}_m} = \mathfrak{b}_m. \tag{20}$$

Lemma 5. *The formula*

$$\langle |\mathfrak{b}_m - a|^2 \rangle = (a + a' - a^2)\langle \mathfrak{k}_m^{-1} \rangle \tag{21}$$

holds.

Proof. We represent the left-hand side of (21) as

$$\langle |\mathfrak{b}_m - a|^2 \rangle = \langle \mathfrak{b}_m^2 \rangle - 2a\langle \mathfrak{b}_m \rangle + a^2. \tag{22}$$

We recall (see [1]) that the two-point generating function $\Psi_{m,m+1}(x,y)$ defined by

$$\Psi_{m,m+1}(x,y) = \sum_{k=1}^{N^{md}} \sum_{l=k}^{kN^d} x^k y^l \mathbf{P}\{\mathfrak{k}_m = k, \mathfrak{k}_{m+1} = l\} \tag{23}$$

is related to the one-point function by

$$\Psi_{m,m+1}(x,y) = \Psi_m(x\Pi(y)).$$

Because the average $\langle \mathfrak{b}_m \rangle$ is defined by the formula

$$\langle \mathfrak{b}_m \rangle = \sum_{k=1}^{N^{md}} \sum_{l=k}^{kN^d} \frac{l}{k} \mathbf{P}\{\mathfrak{k}_m = k, \mathfrak{k}_{m+1} = l\} = \left(\frac{d}{dy} \int_0^1 \frac{\Psi_{m,m+1}(x,y)}{x}\, dx \right)\Bigg|_{y=1},$$

it follows from (23) that

$$\langle \mathfrak{b}_m \rangle = \left(\frac{d}{dy} \int_0^1 \Psi_m(x\Pi(y)) \frac{dx}{x} \right)\Bigg|_{y=1}. \tag{24}$$

Because

$$\langle \mathfrak{k}_m^{-1} \rangle = \int_0^1 \Psi_m(x) \frac{dx}{x}$$

and $\Pi'(1) = a$, Eq. (24) implies

$$\langle \mathfrak{b}_m \rangle = a \int_0^1 \Psi_m'(x)\, dx = a.$$

It is slightly more difficult to evaluate $\langle \mathfrak{b}_m^2 \rangle$. From the definition

$$\langle \mathfrak{b}_m^2 \rangle = \sum_{k=1}^{N^{md}} \sum_{l=k}^{kN^d} \frac{l^2}{k^2} \mathbf{P}\{\mathfrak{k}_m = k, \mathfrak{k}_{m+1} = l\} =$$

$$= \left(\frac{d}{dy} y \frac{d}{dy} \int_0^1 \frac{dx}{x} \int_0^x \Psi_{m,m+1}(x', y) \, \frac{dx'}{x'} \right) \bigg|_{y=1},$$

we obtain

$$\langle \mathfrak{b}_m^2 \rangle = \left(\frac{d}{dy} y \frac{d}{dy} \int_0^1 \frac{dx}{x} \int_0^x \Psi_m(x'\Pi(y)) \frac{dx'}{x'} \right) \bigg|_{y=1} =$$

$$= \left[\frac{d}{dy} (y\Pi'(y)) \int_0^1 \frac{dx}{x} \int_0^x \Psi_m'(x'\Pi(y)) \, dx' \right] \bigg|_{y=1} =$$

$$= \left[\frac{d}{dy} \left(\frac{y\Pi'(y)}{\Pi(y)} \right) \int_0^1 \Psi_m(x\Pi(y)) \, \frac{dx}{x} \right] \bigg|_{y=1} =$$

$$= \left[\frac{y(\Pi'(y))^2}{\Pi(y)} \right] \bigg|_{y=1} + \left[\frac{d}{dy} \left(\frac{y(\Pi'(y))^2}{\Pi(y)} \right) \right] \bigg|_{y=1} \langle \mathfrak{k}_m^{-1} \rangle.$$

Using that $\Pi'(1) = a$ and $\Pi''(1) = a'$ and evaluating the derivatives, we obtain

$$\langle \mathfrak{b}_m^2 \rangle = a^2 + (a' + a - a^2) \langle \mathfrak{k}_m^{-1} \rangle.$$

Substituting this expression in (22), we obtain (21). The lemma is proved.

We now prove the key statement used in deriving the main result.

Theorem 4. *With probability* 1, *the random variable* $\mathfrak{c}$ *is nonzero.*

Proof. The proof consists of several steps.

1. We show that under the conditions of Lemma 4, the series S_2 converges. Indeed, Lemma 4 allows concluding that the series S_{l+1} converges for sufficiently large l. We then apply Lemma 3 consecutively and establish that the series S_j with $j = l, l-1, \ldots, 2$ also converge. This follows from the recursive inequalities

$$r_{j,m+1} \le a^{1-j} r_{j,m} + C_j r_{j+1,m}$$

with $m = 0, 1, 2, \ldots$. Summing these inequalities over m, we obtain the estimate

$$(1 - a^{1-j}) S_j \le r_{j,0} + C_j S_{j+1}$$

for $j > 1$, which implies the sought statement.

2. We show that with probability 1, there exists a random number $\mathfrak{n}$ such that the inequality

$$|\mathfrak{b}_n - a| \le \sigma, \quad a - 1 > \sigma > 0, \tag{25}$$

is satisfied for $n > \mathfrak{n}$. Lemma 5 and the Chebyshev inequality imply that

$$\mathbf{P}\{|\mathfrak{b}_m - a| > \sigma\} \le \varepsilon_m, \qquad \varepsilon_m = \frac{(a + a' - a)}{\sigma^2} \langle \mathfrak{k}_m^{-1} \rangle.$$

On the other hand, using the corollary of Lemma 2, we have $\sum_m \varepsilon_m < \infty$. Finally, invoking the theorem given in the appendix, we find that the sequence $\mathfrak{b}_m$ converges to a with probability 1, which implies the required statement.

3. We let the number σ be such that $0 < \sigma < a - 1$. It follows from step 2 that with probability 1, a random number $\mathfrak{m}$ exists such that inequality (25) is satisfied for $n > \mathfrak{m}$, and therefore (see (20))

$$\mathfrak{k}_n^{-1} = \mathfrak{k}_m^{-1}(\mathfrak{b}_m \mathfrak{b}_{m+1} \cdots \mathfrak{b}_{n-1}) < \mathfrak{k}_{\mathfrak{m}}{}^{-1}(a - \sigma)^{-(n-\mathfrak{m})}.$$

By increasing $\mathfrak{m}$, we can always ensure the inequality $\mathfrak{k}_n^{-1} < (a - \sigma)^{-n}$. It then follows that the inequality $a^n/\mathfrak{k}_n^l < (a^{1/l}/(a - \sigma))^{ln}$ is satisfied for these values of n. We next choose l such that $a^{1/l}/(a - \sigma) < 1$ and set $[a/(a - \sigma)^l]^n = \alpha_n$. With probability 1, we then have the inequality $a^n/\mathfrak{k}_n^l < \alpha_n$ for all sufficiently large n. Therefore, Lemma 6 is applicable, and hence $r_{l,n} < \alpha_n$ starting with a certain number.

4. It follows from step 3 that there exists a number l for which the series $S_l < \sum_{n=0}^{\infty} \alpha_n$ converges. In step 1, it was shown that S_2 converges. Together with Lemma 3, this in turn implies that $\mathfrak{c}^{-1} < \infty$ with probability 1. The theorem is proved.

Remark. Obviously, we could restrict ourselves to the variable $l = 2$ because we can ensure the inequality $a^{1/2}/(a - \sigma) < 1$ for $a > 1$ and sufficiently small σ.

Main Theorem. *Random sets with Markovian refinements have a nonrandom fractal dimension D that is evaluated as*

$$D = \frac{\log a}{\log N}.$$

On these sets, a stochastic D-measure exists with probability 1.

Proof. The proof is equivalent to the concluding argument in the proof of Theorem 2 in [1]. From the existence of a nonzero limit $\mathfrak{c}$ (Theorems 1 and 4), it first follows that the lower bound

$$D = \inf\left\{\alpha\colon \lim_{m\to\infty} N^{-\alpha m}|\mathsf{K}_m(\mathfrak{X})| = 0\right\}$$

(see (1)) is determined by the equation $N^D = a$ and is therefore nonrandom. Second, choosing this value of D, we find that the limit

$$\lim_{m\to\infty} N^{-mD}|\mathsf{K}_m(\mathfrak{X})| = \mathfrak{c} \neq 0$$

exists with probability 1. This implies the existence of a stochastic D-measure (see (2)) on realizations of the random set with probability 1.

5. Conclusions

The nonrandomness theorem for the fractal dimension and the fractal measure type on fractals with Markovian refinements allows introducing a stochastic integral, which in turn opens the possibility of introducing phenomenological Hamiltonians of fluctuating fields distributed on these fractals. On the other hand, we must note that the class of stochastic fractals considered in this work is insufficient for modeling an arbitrary fractally disordered medium. This is related to the anisotropy of the corresponding random point fields. At the same time, it would be desirable to have models of stochastic isotropic homogeneous fractals possessing a nonrandom dimension and a certain fractal measure type. It appears that the main theorem proved here must admit a generalization to a broader class of random point sets that are stochastically homogeneous and allow introducing the notion of stochastic self-similarity.

Appendix: Convergence of almost certainly random sequences

The statement proved here is a version of the "0 and 1 law" and is a reformulation of the corresponding statement in the abstract Lebesgue integral theory [5].

Theorem. *Let $\{c^{(n)}(\mathfrak{X});\ n = 1, 2, \dots\}$ be a sequence of random variables such that the probabilities satisfy the estimates*

$$\mathbf{P}\{\mathfrak{X}\colon \exists(k > 0)(|c^{(n)}(\mathfrak{X}) - c^{(n+k)}(\mathfrak{X})| \geq \sigma_n)\} < \varepsilon_n,$$

where the sequence $\{\varepsilon_n > 0\colon n = 1, 2, \dots\}$ is summable, i.e., $\sum_n \varepsilon_n < \infty$, and the sequence $\{\sigma_n > 0\colon n = 1, 2, \dots\}$ tends to zero. Then the sequence of random variables $\{c^{(n)}(\mathfrak{X}); n = 1, 2, \dots\}$ converges to $c(\mathfrak{X})$ with probability 1.

Proof. 1. Let the event $\mathfrak{C}_n$, $n = 1, 2, \dots$, be such that $\mathbf{P}\{\mathfrak{C}_n\} < \varepsilon_n$ and

$$\mathfrak{A} = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \mathfrak{C}_n. \tag{A.1}$$

The events $\mathfrak{B}_m = \bigcup_{n=m} \mathfrak{C}_n$ possess the property $\mathfrak{B}_{m+1} \subset \mathfrak{B}_m$, and because the probability is continuous, it therefore follows that $\lim_{m\to\infty} \mathbf{P}\{\mathfrak{B}_m\} = \mathbf{P}(\mathfrak{A})$. On the other hand,

$$\mathbf{P}\{\mathfrak{B}_m\} \leq \sum_{n=m}^{\infty} \mathbf{P}\{\mathfrak{C}_n\} < \sum_{n=m}^{\infty} \varepsilon_n.$$

Therefore, $\mathbf{P}\{\mathfrak{B}_m\} \to 0$ as $m \to \infty$, and hence $\mathbf{P}\{\mathfrak{A}\} = 0$.

2. We set $\mathfrak{C}_n = \{\mathfrak{X}\colon \mathfrak{J}^{(n)}(\mathfrak{X})\}$, where $\mathfrak{J}^{(n)}(\mathfrak{X})$ are propositional forms depending on the random realization. We consider the corresponding event $\mathfrak{A}$. In accordance with item 1, the event $\overline{\mathfrak{A}}$ has the probability 1. By the duality law,

$$\overline{\mathfrak{A}} = \bigcup_{m=1}^{\infty} \bigcap_{n=m}^{\infty} \overline{\mathfrak{C}}_n.$$

The event $\bigcap_{n=m}^{\infty} \overline{\mathfrak{C}}_n$ has an equivalent description $\{\mathfrak{X}\colon \forall(n \geq m)(\rceil\mathfrak{J}^{(n)})(\mathfrak{X})\}$. Then $\overline{\mathfrak{A}}$ has an equivalent description $\{\mathfrak{X}\colon \exists(m \geq 1)(\rceil\mathfrak{J}^{(n)}(\mathfrak{X}), n \geq m)\}$.

3. We define a sequence of propositional forms

$$\mathfrak{J}^{(n)}(\mathfrak{X}) = \exists(k > 0)(|c^{(n)}(\mathfrak{X}) - c^{(n+k)}(\mathfrak{X})| \geq \sigma_n),$$

whose negation $\rceil\,\mathfrak{J}^{(n)}(\mathfrak{X})$ can be formulated as

$$\forall(k > 0)(|c^{(n)}(\mathfrak{X}) - c^{(n+k)}(\mathfrak{X})| < \sigma_n). \tag{A.2}$$

Using these forms, we define the events $\mathfrak{C}_n$ as indicated in item 2. Therefore, the event

$$\{\mathfrak{X}; \exists(m \geq 0)(\overline{\mathfrak{J}}^{(n)}(\mathfrak{X}), n \geq m)\}$$

has the probability 1. If the realization $\mathfrak{X}_0$ belongs to this event, this implies that a number m exists such that for $n \geq m$, all the inequalities in (A.2) are simultaneously satisfied for all k. The sequence of quantities $\{c^{(n)}(\mathfrak{X}_0); n = 1, 2, \dots\}$ is then fundamental and therefore converges to a certain quantity $c(\mathfrak{X}_0)$.

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