On a Problem of Karatsuba

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Abstract—We obtain lower bounds for the fractional moments of linear combinations of analogs of the Hardy function. In addition, we apply these estimates to the Karatsuba problem of finding a lower bound for the number of zeros of the linear combination of analogs of Hardy functions on the interval $(0, T]$.

Key words: *Hardy function, Dirichlet character, Euler function, fractional moments of the Hardy function, Stirling's formula, Cauchy's theorem, Dirichlet L-function.*

In blessed memory of my teacher Anatolii Alekseevich Karatsuba

1. INTRODUCTION

Suppose that $N_0(T)$ is the number of zeros of $\zeta(1/2+it)$ on the interval $(0, T]$.

In 1921, Hardy and Littlewood [1] established the estimate

$$N_0(T) \geq c_1 T, \qquad c_1 > 0 \text{ is an absolute constant.}$$

In 1942, Selberg [2] obtained the following order-exact estimate of $N_0(T)$:

$$N_0(T) \geq c_2 T \log T, \qquad c_2 > 0 \text{ is an absolute constant.}$$

For arithmetic Dirichlet series satisfying a functional equation of Riemannian type, but not having an Euler product, no order-infimums for the number of zeros on the intervals of the critical line $\operatorname{Re} s = 1/2$ have been obtained so far.

Voronin was the first to show that, on the critical line, there are abnormally many zeros of the arithmetic Dirichlet series without an Euler product. In 1980, he proved [3] the estimate

$$N_0(T, f) > c_3 T \exp\left(\frac{1}{20}\sqrt{\log\log\log\log T}\right), \tag{1}$$

where $N_0(T, f)$ is the number of zeros of the Davenport–Heilbronn ρ function $f(s)$ such that $\operatorname{Re}\rho = 1/2$, $0 < \operatorname{Im}\rho \leq T$, and $c_3 > 0$ is an absolute constant.

In 1989, Karatsuba [4] developed a new method for finding lower bounds for the numbers of zeros of certain Dirichlet series on the intervals of the critical line, with the help of which he proved the following inequality significantly strengthening Voronin's result (1):

$$N_0(T, f) \geq T\sqrt{\log T}\,(\log T)^{-\varepsilon}, \tag{2}$$

where ε is an arbitrarily small positive number, $T > T_0(\varepsilon) > 1$.

In 1994, Karatsuba [5] replaced the multiplier $(\log T)^{-\varepsilon}$ in inequality (2) by the more slowly decreasing multiplier $e^{-c_4\sqrt{\log\log T}}$, where $c_4 > 0$ is an absolute constant.

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At present, the inequality

$$N_0(T,f) \geq T\sqrt{\log T}\exp(-c_4\sqrt{\log\log T}), \qquad c_4 > 0,$$

is the sharpest known estimate for $N_0(T,f)$.

Suppose that

$$Z(t,\chi) = e^{i\theta(t,\chi)} L\left(\frac{1}{2} + it\right),$$

where the function $\theta(t,\chi)$ is chosen so that $Z(t,\chi)$ is real for real t. Then $Z(t,\chi)$ is an analog of the Hardy function (see, for example, [6, p. 86]).

Suppose that

$$G(t) = a_1 Z(t,\chi_1) + \cdots + a_N Z(t,\chi_N),$$

where $a_1,\ldots,a_N$ are arbitrary real numbers.

In 1991, Karatsuba [7] posed and solved (using his method of 1989) the problem of finding a lower bound for the number of zeros of $G(t)$ on the closed interval $(0,T]$.

This problem will be called the *Karatsuba problem*.

Let us cite the Karatsuba theorem from [7].

Theorem A. *Suppose that $N \geq 2$, and $k_1,\ldots,k_N$ are arbitrary natural numbers satisfying the condition $K = [k_1,\ldots,k_N] \geq 3$, and $\chi_1,\ldots,\chi_N$ are arbitrary primitive Dirichlet characters modulo $k_1,\ldots,k_N$, respectively. Further, suppose that, $a_1,\ldots,a_N$ are arbitrary real numbers.*

Then the number $N_0(T,G)$ of zeros of odd order of $G(t)$ on the interval $(0,T)$ satisfies the inequality

$$N_0(T,G) \geq T(\log T)^{\beta-\varepsilon}, \tag{3}$$

where $\varepsilon > 0$ is an arbitrary number, $T \geq T_0(\varepsilon) > 0$, K is the least common multiple of the numbers $k_1,\ldots,k_N$, $\beta\varphi(K) = 1$, and $\varphi(K)$ is the Euler function.

If the characters $\chi_1,\ldots,\chi_N$ have identical parity, then

$$N_0(T,G) \geq T(\log T)^{2\beta-\varepsilon}. \tag{4}$$

In 2002, Karatsuba [8] obtained an estimate for the number of zeros of the Riemann zeta function on a closed interval of the critical line, by using the knowledge of the true order of the fractional moments of the zeta function on the line $\operatorname{Re} s = 1/2$. This estimate turned out to be less sharp than the order-exact Selberg estimate, but significantly sharper than the Hardy–Littlewood estimate.

The Karatsuba method of 1989 is very complicated. Considerations dating back to Selberg are supplemented in it by a number of original ideas due to Karatsuba himself.

In contrast, the Karatsuba method of 2002 is comparatively simple. It was noted in [8] that, in addition to the techniques of Hardy and Littlewood, it incorporates the knowledge of the true order of the fractional moments of $\zeta(s)$ on the critical line.

In the present paper, we obtain order-sharp estimates of the fractional moments of linear combinations of analogs of the Hardy function. In addition, we show that, in some special cases, the Karatsuba method of 2002 can be applied to the Karatsuba problem and, moreover, it can produce a sharper result than the elaborate method of 1989.

In what follows, we shall use the following definitions.

Suppose that

$$\varepsilon(\chi) = \frac{i^\delta\sqrt{k}}{\tau(\chi)},$$

where k is the modulus of the character $\chi(n)$, and $\tau(\chi)$ is the corresponding Gauss sum, and δ is equal to, respectively, 0 or 1 depending on whether the character $\chi(n)$ is even or odd.

We introduce the following functions:

$$\rho(s,\chi)=\varepsilon(\overline{\chi})\left(\frac{\pi}{k}\right)^{s-1/2}\frac{\Gamma((1-s+\delta)/2)}{\Gamma((s+\delta)/2)},$$

$$\Phi(s)=a_1(\rho(s,\chi_1))^{-1/2}L(s,\chi_1)+a_2(\rho(s,\chi_2))^{-1/2}L(s,\chi_2). \tag{5}$$

Here and elsewhere, the moduli of the characters χ_1 and χ_2 are assumed constant (not increasing with the main parameter T). The constants in all subsequent estimates depend on these constants.

Suppose that

$$G(t)=\Phi\left(\frac{1}{2}+it\right).$$

It follows from Stirling's formula that, for $T/2\le t\le T$, the following formula holds:

$$\rho(\sigma+it,\chi_j)=\left(\frac{k_jt}{2\pi}\right)^{1/2-\sigma-it}e^{it}e^{-\pi i(2\delta_j-1)/4}(1+O(T^{-1})),\qquad j=1,2; \tag{6}$$

this formula will be used on numerous occasions in what follows.

The main results of the paper are stated in the following theorems.

Theorem 1. *Suppose that k_1 and k_2 are natural numbers, $m\ge 2$ is an arbitrary natural number, and $d_{1/m}(n)$ are the Dirichlet coefficients of the function $(\zeta(s))^{1/m}$ for $\operatorname{Re}s>1$. Suppose that χ_1 and χ_2 are primitive Dirichlet characters modulo k_1 and k_2, respectively.*

Suppose that

$$c_{k,1/m}=\frac{\sin(\pi/m^2)}{\pi}\left(\frac{\varphi(k)}{k}\right)^{1/m^2}\int_0^\infty r^{-1/m^2-1}(1-e^{-r})\,dr \times\prod_{p\nmid k}\left(1+\frac{1}{m^2p}+\frac{|d_{1/m}(p^2)|^2}{p^2}+\cdots\right)\left(1-\frac{1}{p}\right)^{1/m^2}.$$

Suppose that, in formula (5), *a_1 and a_2 are arbitrary complex numbers such that*

$$|a_1|^{2/m}c_{k_1,1/m}\ne|a_2|^{2/m}c_{k_2,1/m}.$$

Then, for $T\ge T_1>2$, the following estimates hold:

$$T(\log T)^{1/m^2}\ll\int_0^T|G(t)|^{2/m}\,dt\ll T(\log T)^{1/m^2}.$$

Theorem 2. *Suppose that k_1 and k_2 are natural numbers and χ_1 and χ_2 are primitive Dirichlet characters modulo k_1 and k_2, respectively. Suppose that, in formula* (5), *a_1 and a_2 are arbitrary real numbers such that*

$$|a_1|^{2/3}c_{k_1,1/3}\ne|a_2|^{2/3}c_{k_2,1/3}.$$

Then the following estimate holds:

$$N_0(T,G)\ge c_4T(\log T)^{1/6},\qquad c_4>0.$$

Note that if the constant β from (3) (or 2β in the case of the same parity of the characters χ_1 and χ_2) is less than $1/6$, then the estimates of Theorem 2 are sharper than (3) and (4). At the same time, the scope of applications of the Karatsuba theorems is wider than that of Theorem 2.

Theorem 3. *Suppose that* k_1 *and* k_2 *are natural numbers and* χ_1 *and* χ_2 *are primitive Dirichlet characters modulo* k_1 *and* k_2*, respectively. Suppose that* a_1 *and* a_2 *is are arbitrary complex numbers such that*

$$|a_1|^2\left(\frac{\varphi(k_1)}{k_1}\right)^{1/m} \neq |a_2|^2\left(\frac{\varphi(k_2)}{k_2}\right)^{1/m}.$$

Then, for any natural number $m \geq 2$*, the following estimates hold*:

$$T(\log T)^{1/m^2} \ll \int_0^T |a_1 L\left(\frac{1}{2}+it,\chi_1\right) + a_2 L\left(\frac{1}{2}+it,\chi_2\right)|^{2/m}\,dt \ll T(\log T)^{1/m^2}.$$

In order to prove the theorems, we shall need the following lemmas.

2. LEMMAS

Lemma 1. *Suppose that* $\chi(n)$ *is a Dirichlet character modulo* q*,* $0<k<1$*, and* $d_k(n)$ *are the Dirichlet coefficients of the function* $(\zeta(s))^k$*. Then the following asymptotic formula holds*:

$$\sum_{n=1}^{N}\frac{|d_k(n)|^2|\chi(n)|^2}{n} = c_{q,k}(\log N)^{k^2} + O((\log N)^{k^2-1}),$$

where

$$c_{q,k} = \frac{\sin \pi k^2}{\pi}\left(\frac{\varphi(q)}{q}\right)^{k^2}\int_0^\infty r^{-k^2-1}(1-e^{-r})\,dr \prod_{p\nmid q}\left(1+\frac{k^2}{p}+\frac{|d_k(p^2)|^2}{p^2}+\cdots\right)\left(1-\frac{1}{p}\right)^{k^2}.$$

Proof. Suppose that $b = 1/\log N$, $T = e^{\sqrt{\log N}}$. Perron's formula (see [9, p. 427 (Russian transl.)]) implies the equality

$$\sum_{n=1}^{N}\frac{|d_k(n)|^2|\chi(n)|^2}{n} = \frac{1}{2\pi i}\int_{b-iT}^{b+iT} F(1+s)\frac{N^s}{s}\,ds + O\left(\frac{\log N}{T}\right),$$

where

$$F(1+s) = \sum_{n=1}^{\infty}\frac{|d_k(n)|^2|\chi(n)|^2}{n^{1+s}}.$$

Suppose that Γ_1 is the rectangle with vertices

$$b-iT, \quad b+iT, \quad -c_1+iT, \quad -c_1-iT,$$

where $c_1 = c_1'/\sqrt{\log N}$ and c_1' is a small positive constant such that the closure Γ_1 does not contain zeros of $\zeta(s)$. The existence of such a constant is a consequence of the well-known Vallée–Poussin theorem on the boundary of the zeros of $\zeta(s)$ (see, for example, [6, p. 33]). Suppose that $c = c'/\log N$, where $0 < c' < 0.1c_1'$.

Further, note that, for $\operatorname{Re} s > 0$,

$$F(1+s) = (\zeta(1+s))^{k^2}U(1+s), \tag{7}$$

where

$$U(1+s) = \prod_{p\nmid q}\left(1+\frac{k^2}{p^{1+s}}+\frac{|d_k(p^2)|^2}{p^{2(1+s)}}+\cdots\right)\left(1-\frac{1}{p^{1+s}}\right)^{k^2}\prod_{p\mid q}\left(1-\frac{1}{p^{1+s}}\right)^{k^2}.$$

The function $U(1+s)$ is regular in the half-plane $\operatorname{Re} s > -1/2$.

Inside the rectangle Γ_1, let us make a cut along the interval $[-c_1, -c]$. Define a closed oriented contour Γ as follows. From the point $b-iT$, along the sides of Γ_1, we go upward, turn left, and go down to the point $-c_1$; we go along the upper edge of the cut to the point $-c$; we bypass the point $s = 0$ along a negatively oriented circle of radius c centered at the point $s = 0$; along the lower edge of the cut, we go to the point $-c_1$; along the the sides of Γ_1, we go down, turn left, and return at the point $b - iT$.

The upper edge of the cut, the circle, and the lower edge of the cut with the given orientations are denoted, respectively, by Γ_2, Γ_3, and Γ_4. By Cauchy's residue theorem, we have

$$\frac{1}{2\pi i}\int_{\Gamma} F(1+s)\frac{N^s}{s}\,ds = 0.$$

Combining this with the estimates

$$|\zeta(1-c_1+it)| \ll \frac{1}{c_1} + T^{c_1}\log T \ll \sqrt{\log N} \qquad \text{for} \quad |t| \le T,$$
$$|\zeta(1+\sigma \pm iT)| \ll T^{c_1}\log T \ll \sqrt{\log N} \qquad \text{for} \quad -c_1 \le \sigma \le b,$$

we find that

$$\frac{1}{2\pi i}\int_{b-iT}^{b+iT} F(1+s)\frac{N^s}{s}\,ds = -\frac{1}{2\pi i}\biggl(\int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}\biggr) F(1+s)\frac{N^s}{s}\,ds + O(e^{-c_1'\sqrt{\log N}/2}). \tag{8}$$

Using the relation

$$F(1+s) = s^{-k^2}(s\zeta(1+s))^{k^2}U(1+s) = U(1)s^{-k^2} + O(|s|^{1-k^2}),$$

we obtain the equality

$$\biggl(\int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}\biggr) F(1+s)\frac{N^s}{s}\,ds = U(1)\biggl(\int_{\Gamma_2} + \int_{\Gamma_3} + \int_{\Gamma_4}\biggr) s^{-k^2}\frac{N^s}{s}\,ds + O((\log N)^{k^2-1}). \tag{9}$$

Let us evaluate $\int_{\Gamma_2} s^{-k^2}(N^s/s)\,ds$. We have $s = re^{\pi i}$, $ds/s = dr/r$, $N^s = e^{-r\log N}$,

$$\int_{\Gamma_2} s^{-k^2}\frac{N^s}{s}\,ds = e^{-\pi i k^2}\int_{c_1}^{c} r^{-k^2-1}e^{-r\log N}\,dr$$
$$= -e^{-\pi i k^2}(\log N)^{k^2}\int_{c'}^{\infty} r^{-k^2-1}e^{-r}\,dr + O(e^{-c_1'\sqrt{\log N}/2}).$$

Similarly, evaluating $\int_{\Gamma_4} s^{-k^2}(N^s/s)\,ds$, we obtain

$$\biggl(\int_{\Gamma_2} + \int_{\Gamma_4}\biggr) s^{-k^2}\frac{N^s}{s}\,ds = (e^{\pi i k^2} - e^{-\pi i k^2})(\log N)^{k^2}\int_{c'}^{\infty} r^{-k^2-1}e^{-r}\,dr + O(e^{-c_1'\sqrt{\log N}/2}).$$

Let us evaluate $\int_{\Gamma_3} s^{-k^2}(N^s/s)\,ds$. We have

$$\int_{\Gamma_3} s^{-k^2}\frac{N^s}{s}\,ds = \int_{\Gamma_3} s^{-k^2}\frac{ds}{}s + i(\log N)^{k^2}\sum_{l=1}^{\infty}\frac{(c')^{(l-k^2)}}{l!}J_l,$$

where

$$J_l = \int_{\pi}^{-\pi} e^{i\varphi(l-k^2)}\,d\varphi.$$

Let us evaluate $\int_{\Gamma_3} s^{-k^2}(ds/s)$. We have $s = ce^{i\varphi}$, $ds/s = i\,d\varphi$,

$$\int_{\Gamma_3} s^{-k^2}\frac{ds}{s} = i\int_{\pi}^{-\pi} c^{-k^2}e^{-i\varphi k^2}\,d\varphi = -\frac{(c')^{-k^2}}{k^2}(\log N)^{k^2}(e^{\pi i k^2} - e^{-\pi i k^2}).$$

Since

$$\frac{(c')^{-k^2}}{k^2} = \int_{c'}^{\infty} r^{-k^2-1}\,dr,$$

we obtain the equality

$$\left(\int_{\Gamma_2}+\int_{\Gamma_3}+\int_{\Gamma_4}\right)s^{-k^2}\frac{N^s}{s}\,ds=-2i(\sin\pi k^2)(\log N)^{k^2}\int_{c'}^{\infty}r^{-k^2-1}(1-e^{-r})\,dr$$
$$+\,i(\log N)^{k^2}\sum_{l=1}^{\infty}\frac{(c')^{(l-k^2)}}{l!}J_l+O(e^{-c_1'\sqrt{\log N}/2});$$

substituting this equality into (9) and passing to the limit as $c'\to+0$, we finally obtain

$$\left(\int_{\Gamma_2}+\int_{\Gamma_3}+\int_{\Gamma_4}\right)F(1+s)\frac{N^s}{s}\,ds$$
$$=-2iU(1)(\sin\pi k^2)(\log N)^{k^2}\int_0^{\infty}r^{-k^2-1}(1-e^{-r})\,dr+O((\log N)^{k^2-1}).$$

Now the assertion of the lemma immediately follows from (8). □

Lemma 2. *Suppose that $f(s)$ is a function regular in the strip $\alpha<\operatorname{Re}s<\beta$ and continuous in the strip $\alpha\le\operatorname{Re}s\le\beta$. Suppose that $f(s)\to0$ as $|\operatorname{Im}s|\to\infty$ uniformly in $\alpha\le\operatorname{Re}s\le\beta$. Then, for $\alpha\le\gamma\le\beta$ and $q>0$, the following inequality holds*:

$$\int_{-\infty}^{\infty}|f(\gamma+it)|^q\,dt\le\left(\int_{-\infty}^{\infty}|f(\alpha+it)|^q\,dt\right)^{(\beta-\gamma)/(\beta-\alpha)}\left(\int_{-\infty}^{\infty}|f(\beta+it)|^q\,dt\right)^{(\gamma-\alpha)/(\beta-\alpha)}.$$

For the proof, see [7].

Lemma 3. *Suppose that* $1/2\le\sigma\le3/4$, $l>0$, $T\ge2$,

$$J(\sigma)=\int_{-\infty}^{\infty}|\Phi(\sigma+it)|^{2l}w(t)\,dt,\qquad w(t)=\int_{T/2}^{T}e^{-2l(t-\tau)^2}\,d\tau.$$

Then the following inequality holds:

$$J(\sigma)\ll T^{(l+1)(\sigma-1/2)}J(1/2)^{3/2-\sigma}.$$

Proof. Let us apply Lemma 2, setting in it

$$\alpha=\frac{1}{2},\quad \beta=\frac{3}{2},\quad \gamma=\sigma,\quad q=2l,\quad f(s)=\Phi(s)e^{(s-i\tau)^2},$$

where τ is a number from the interval $[T/2,T]$. We obtain the inequality

$$\int_{-\infty}^{\infty}|f(\sigma+it)|^{2l}\,dt\le\left(\int_{-\infty}^{\infty}\left|f\left(\frac{1}{2}+it\right)\right|^{2l}dt\right)^{3/2-\sigma}\left(\int_{-\infty}^{\infty}\left|f\left(\frac{3}{2}+it\right)\right|^{2l}dt\right)^{\sigma-1/2}.$$

Let us integrate it over τ from $T/2$ to T and apply Hölder's inequality:

$$J(\sigma)\ll J\left(\frac{1}{2}\right)^{3/2-\sigma}J\left(\frac{3}{2}\right)^{\sigma-1/2}.$$

It follows from formula (6) that

$$\left|\rho\left(\frac{3}{2}+it,\chi_j\right)\right|\ll|t|^{1/2},\qquad j=1,2.$$

Now the definition $\Phi(s)$ (see (5)) and the fact that the L-Dirichlet function is bounded in the half-plane $\operatorname{Re}s\ge3/2$ implies that

$$J\left(\frac{3}{2}\right)\ll\int_{T/3}^{2T}\left|\Phi\left(\frac{3}{2}+it\right)\right|^{2l}dt\ll T^l\int_{T/3}^{2T}\left(\left|L\left(\frac{3}{2}+it,\chi_1\right)\right|^{2l}+\left|L\left(\frac{3}{2}+it,\chi_2\right)\right|^{2l}\right)dt\ll T^{l+1}.$$

The lemma is proved.

3. PROOF OF THEOREM 1

The upper bound in Theorem 1 is well known (see, for example, [11], [12]). Let us prove the validity of the lower bound.

First, consider the integral

$$J_1(\sigma) = \int_{T/2}^{T} |\Phi(\sigma + it)|^{2/m}\, dt.$$

Suppose that, for $\operatorname{Re} s > 1$,

$$(\zeta(s))^{1/m} = \sum_{n=1}^{\infty} d_{1/m}(n) n^{-s}.$$

Further, suppose that

$$S_N(s, \chi_1) = \sum_{n=1}^{N} d_{1/m}(n)\chi_1(n) n^{-s}, \qquad S_N(s, \chi_2) = \sum_{n=1}^{N} d_{1/m}(n)\chi_2(n) n^{-s},$$

where $N = T^{3/4}$.

Suppose that, for example, $|a_1|^{2/m} c_{k_1,1/m} > |a_2|^{2/m} c_{k_2,1/m}$. Then

$$\begin{aligned} J_1(\sigma) = \int_{T/2}^{T} \Big| & a_1(\rho(\sigma + it, \chi_1))^{-1/2} S_N^m(\sigma + it, \chi_1) \\ & + a_2(\rho(\sigma + it, \chi_2))^{-1/2} S_N^m(\sigma + it, \chi_2) \\ & + a_1(\rho(\sigma + it, \chi_1))^{-1/2} (L(\sigma + it, \chi_1) - S_N^m(\sigma + it, \chi_1)) \\ & + a_2(\rho(\sigma + it, \chi_2))^{-1/2} (L(\sigma + it, \chi_2) - S_N^m(\sigma + it, \chi_2)) \Big|^{2/m} dt. \end{aligned}$$

Suppose that

$$\sigma = \frac{1}{2} + \frac{C}{\log T},$$

while $C > 1$ will be chosen later.

Using the inequalities

$$|z_1|^{2/m} - |z_2|^{2/m} \le |z_1 + z_2|^{2/m} \le |z_1|^{2/m} + |z_2|^{2/m}$$

valid for all complex numbers z_1 and z_2 and any natural number $m \ge 2$, we obtain

$$\begin{aligned} &(|a_1|^{2/m} k_1^{(\sigma-1/2)/m} L_1(\sigma) - |a_2|^{2/m} k_2^{(\sigma-1/2)/m} L_2(\sigma)) T^{(\sigma-1/2)/m} \\ &\qquad \ll J_1(\sigma) + (K_1(\sigma) + K_2(\sigma)) T^{(\sigma-1/2)/m}, \end{aligned}$$

where

$$L_j(\sigma) = \int_{T/2}^{T} |S_N(\sigma + it, \chi_j)|^2\, dt,$$

$$K_j(\sigma) = \int_{T/2}^{T} |H_j(\sigma + it)|^{2/m}\, dt, \quad H_j(s) = a_j L(s, \chi_j) - a_j S_N^m(s, \chi_j), \qquad j = 1, 2.$$

Let us show that

$$T\left(\sigma - \frac{1}{2}\right)^{-1/m^2} \ll |a_1|^{2/m} k_1^{(\sigma-1/2)/m} L_1(\sigma) - |a_2|^{2/m} k_2^{(\sigma-1/2)/m} L_2(\sigma). \qquad (10)$$

The following relation holds:

$$L_1(\sigma) = \left(\frac{T}{2} + O(N\log N)\right)\sum_{n=1}^{N}\frac{d^2_{1/m}(n)|\chi_1(n)|^2}{n^{2\sigma}}$$

(see, for example, [6, p. 123]).

Let us rewrite the sum

$$\sum_{n=1}^{N}\frac{d^2_{1/m}(n)|\chi_1(n)|^2}{n^{2\sigma}}$$

Suppose that $N_1 = \exp((\sigma - 1/2)^{-1/2})$. Then, by Lemma 1, we have

$$\begin{aligned}\sum_{n=1}^{N}\frac{d^2_{1/m}(n)|\chi_1(n)|^2}{n^{2\sigma}} &= \sum_{N_1<n\le N}\frac{d^2_{1/m}(n)|\chi_1(n)|^2}{n}n^{1-2\sigma} + O\bigg(\Big(\sigma - \frac{1}{2}\Big)^{-1/2m^2}\bigg)\\ &= -\int_{N_1}^{N}\mathbb{C}(u)\,du^{1-2\sigma} + \mathbb{C}(N)N^{1-2\sigma} + O\bigg(\Big(\sigma - \frac{1}{2}\Big)^{-1/2m^2}\bigg),\end{aligned}$$

where

$$\mathbb{C}(u) = \sum_{N_1<n\le u}\frac{d^2_{1/m}(n)|\chi_1(n)|^2}{n}.$$

Again, using Lemma 1 and integrating by parts, we obtain the equality

$$\begin{aligned}&\sum_{n=1}^{N}\frac{d^2_{1/m}(n)|\chi_1(n)|^2}{n^{2\sigma}}\\ &\quad= \frac{c_{k_1,1/m}}{m^2}(2\sigma-1)^{-1/m^2}\int_0^{(2\sigma-1)\log N} v^{1/m^2-1}e^{-v}\,dv + O\bigg(\Big(\sigma - \frac{1}{2}\Big)^{-1/2m^2}\bigg).\end{aligned}$$

Using the same arguments to $J_2(\sigma)$ and taking into account the fact that

$$|a_1|^{2/m}c_{k_1,1/m} > |a_2|^{2/m}c_{k_2,1/m}, \qquad k_j^{(\sigma-1/2)/m} = 1 + O\Big(\sigma - \frac{1}{2}\Big), \quad j = 1, 2,$$

$$(2\sigma - 1)\log N > \frac{3}{2},$$

we obtain (10).

Thus,

$$T\Big(\sigma - \frac{1}{2}\Big)^{-1/m^2}T^{(\sigma-1/2)/m} \ll J_1(\sigma) + (K_1(\sigma) + K_2(\sigma))T^{(\sigma-1/2)/m}.$$

Our immediate problem is to estimate from above the integrals $K_1(\sigma)$ and $K_2(\sigma)$ for values of σ close to $1/2$.

The integrals $K_1(\sigma)$ and $K_2(\sigma)$ are estimated in a similar way. Suppose that j is any one of the numbers $1, 2$.

Let us apply Lemma 2, setting in it

$$\alpha = \frac{1}{2}, \quad \beta = \frac{5}{4}, \quad q = \frac{2}{m}, \quad f(s) = H_j(s)e^{(s-i\tau)^2}, \quad \gamma = \sigma,$$

where $T/2 \le \tau \le T$. We obtain

$$\int_{-\infty}^{\infty}|f(\sigma+it)|^{2/m}\,dt \le \left(\int_{-\infty}^{\infty}|f(1/2+it)|^{2/m}\,dt\right)^{(5-4\sigma)/3}\left(\int_{-\infty}^{\infty}|f(5/4+it)|^{2/m}\,dt\right)^{(4\sigma-2)/3}.$$

Let us integrate this inequality over τ from $T/2$ to T and use Hölder's inequality. We see that

$$M_j(\sigma) \ll \left(M_j\left(\frac{1}{2}\right)\right)^{(5-4\sigma)/3}\left(M_j\left(\frac{5}{4}\right)\right)^{(4\sigma-2)/3}, \tag{11}$$

where

$$M_j(\sigma) = \int_{-\infty}^{\infty} |H_j(\sigma+it)|^{2/m} w(t)\,dt, \qquad w(t) = \int_{T/2}^{T} e^{-2(t-\tau)^2/m}\,d\tau.$$

Further, for $\operatorname{Re} s > 1$, the function $H_j(s)$ can be expressed as the series

$$H_j(s) = \sum_{n=N+1}^{\infty} b_j(n) n^{-s}, \qquad \text{where} \quad |b_j(n)| \ll n^{\varepsilon}, \quad \varepsilon > 0.$$

Therefore, using Hölder's inequalities and the Montgomery–Vaughan theorem, we obtain the estimates

$$\begin{aligned}\left(M_j\left(\frac{5}{4}\right)\right)^m &\ll T^{m-1}\int_{T/3}^{3T} |H_j(5/4+it)|^2\,dt \\ &\ll T^{m-1}\sum_{n=N+1}^{\infty}(T+n)n^{2\varepsilon-5/2} \ll T^m N^{2\varepsilon-3/2} \ll T^m N^{-3/4},\end{aligned}$$

$$\left(M\left(\frac{5}{4}\right)\right)^{(4\sigma-2)/3} \ll T^{(4\sigma-2)/3} N^{-(\sigma-1/2)/m}. \tag{12}$$

In the case $|M_j(1/2)| \le T$, by (11), we have

$$M_j(\sigma) \ll T N^{-(\sigma-1/2)/m}.$$

Now suppose that $|M_j(1/2)| > T$. Then, in view of (11) and (12), we see that

$$M_j(\sigma) \ll M_j\left(\frac{1}{2}\right) N^{-(\sigma-1/2)/m}.$$

Using the inequalities

$$M_j\left(\frac{1}{2}\right) \ll \int_{-\infty}^{\infty}\left|L\left(\frac{1}{2}+it,\chi_j\right)\right|^{2/m} w(t)\,dt + L_j\left(\frac{1}{2}\right) \ll T(\log T)^{1/m^2},$$

we obtain

$$M_j(\sigma) \ll T(\log T)^{1/m^2} N^{-(\sigma-1/2)/m}.$$

Hence, using $M_j(\gamma) \asymp K_j(\gamma)$, we find that

$$T(\sigma-1/2)^{-1/m^2} T^{(\sigma-1/2)/m} \ll J_1(\sigma) + T(\log T)^{1/m^2} T^{(\sigma-1/2)/m} N^{-(\sigma-1/2)/m}.$$

Recalling that $\sigma = 1/2 + C/\log T$, $N = T^{3/4}$, and noting that $N^{-(\sigma-1/2)/m}$ decreases with the growth of C faster than C^{-1/m^2}, and that the constant in the Vinogradov sign is independent of C, for a sufficiently large constant C, we obtain

$$J_1(\sigma) \gg T(\log T)^{1/m^2}.$$

Now Theorem 1 follows immediately from Lemma 3.

4. PROOF OF THEOREM 2

We shall follow the scheme of proof from [8] with slight modifications.

Suppose that $h > 0$,

$$j_1(t) = \int_0^h |G(t+u)|\,du, \qquad j_2(t) = |\int_0^h G(t+u)\,du|.$$

Define the set E:

$$E = \{t \in (0,T] \mid j_1(t) > j_2(t)\}.$$

In the same way as in [8], we obtain the inequalities

$$I_1 \le I_2 + I_3,$$

where

$$I_1 = \int_0^T j_1(t)^{2/3}\,dt, \qquad I_2 = \int_0^T j_2(t)^{2/3}\,dt, \qquad I_3 = \int_E j_1(t)^{2/3}\,dt.$$

Let us estimate I_1 from below. Using Hölder's inequality, we obtain

$$h^{-1/3}\int_0^h |G(t+u)|^{2/3}\,du \le j_1(t)^{2/3},$$

whence

$$I_1 \ge h^{2/3}\int_h^T |G(t)|^{2/3}\,dt.$$

Theorem 1 in which m is set equal to 3 implies that

$$I_1 \ge d_1 T h^{2/3}(\log T)^{1/9}, \qquad d_1 > 0.$$

Let us estimate I_2 from above. We have

$$I_2^3 \ll T^2\biggl(\int_0^T\biggl|\int_0^h Z(t+u,\chi_1)\,du\biggr|^2 dt + \int_0^T\biggl|\int_0^h Z(t+u,\chi_2)\,du\biggr|^2 dt\biggr).$$

The following estimate was obtained in [7] and [8]:

$$\int_0^T\biggl|\int_0^h Z(t+u,\chi)\,du\biggr|^2 dt \ll Th;$$

it implies that there exists a $d_2 > 0$ such that

$$I_2 \le d_2 T h^{1/3}.$$

Choose h so that

$$d_2 T h^{1/3} = 0.5 d_1 T h^{2/3}(\log T)^{1/9}, \qquad \text{i.e.,} \quad h = \left(\frac{2d_2}{d_1}\right)^3 (\log T)^{-1/3};$$

for such an h, we have

$$T h^{2/3}(\log T)^{1/9} \ll I_3. \tag{13}$$

Let us estimate I_3 from above. Using Hölder's inequality, we can write

$$I_3^{3/2} \le \mu(E)^{1/2}\int_0^T j_1(t)\,dt,$$

where $\mu(E)$ is the measure of the set E.

Using the upper bound from Theorem 1, we obtain

$$I_3^{3/2} \ll \mu(E)^{1/2} h T(\log T)^{1/4} \qquad \text{or} \qquad I_3 \ll \mu(E)^{1/3} h^{2/3} T^{2/3}(\log T)^{1/6}.$$

Combining this with (13), we see that

$$\mu(E) \gg T(\log T)^{-1/6}.$$

From this inequality, just as in [7], [8], we derive the inequality

$$N_0(T, G) \gg T(\log T)^{1/6}.$$

Theorem 2 is proved.

Theorem 3 is proved, essentially, in the same way as Theorem 1.

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