*Siberian Mathematical Journal, vol.* 59, *no.* 5, *pp.* 909–921, 2018
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# HOMOGENIZATION OF THE EQUATIONS OF FILTRATION OF A VISCOUS FLUID IN TWO POROUS MEDIA

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**Abstract:** A homogenized model of filtration of a viscous fluid in two domains with common boundary is deduced on the basis of the method of two-scale convergence. The domains represent an elastic medium with perforated pores. The fluid, filling the pores, is the same in both domains, and the properties of the solid skeleton are distinct.

**DOI:** 10.1134/S0037446618050154

**Keywords:** Lamé equations, Stokes equations, homogenization of periodic structures, two-scale convergence

## 1. Introduction and Statement of the Problem

In this article, we derive the homogenized equations that describe the process of fluid filtration in two adjoining poroelastic domains. For describing heterogeneous media, we apply an approach that is based on the construction of an exact mathematical model and its further simplification by the methods of mathematical analysis. As a rule, the differential equations of an exact mathematical model contain a small parameter. Therefore, the main tools for simplifying exact mathematical models are the methods of linearization and homogenization as the small parameter vanishes. In this paper, we adhere to the methods proposed in the works of Sánchez-Palencia [1], Burridge and Keller [2], and Levy [3]. The theorems on homogenization can be proved depending on a particular problem in various ways, and the monographs [4, 5] (as well as, for example, the articles [6–10], etc.) are devoted to all possible aspects of homogenization theory.

This article uses the method of two-scale convergence which was proposed in 1989 by Nguetseng in [11]. Two-scale convergence is used to homogenize filtration and acoustic problems in the papers [12–14] by Meirmanov. In [15], Zhikov and Iosifyan laid the foundations of the theory of two-scale convergence and demonstrated applications of this theory to homogenization problems. In [16], we gave basic information about two-scale convergence in a form convenient for us.

In [17–19], some results were obtained for a special geometry of the porous space (a disconnected solid skeleton) for a domain in $\mathbb{R}^2$. In [20], the boundary value problem was considered that describes the stationary motion of a two-component mixture of viscous compressible heat-conducting fluids in a bounded domain. Filtration from a water reservoir into a porous soil was investigated in [21].

This article deals with the filtration of a viscous fluid from a domain $Q = \Omega^0 \cup S^0 \cup \Omega$ that is the union of two different poroelastic media $\Omega^0$ and $\Omega$ respectively. We assume that $\Omega^0$ and $\Omega$ have the common boundary $S^0$. The fluid, filling the pores, is one and the same, and the properties of the solid skeleton are distinct in $\Omega^0$ and $\Omega$.

Here we use the particular case of a viscous fluid in an elastic skeleton

$$0 < \mu_0, \lambda_0, \lambda_0^0 < \infty.$$

A weakly viscous fluid in an absolutely solid skeleton and a weakly viscous fluid in a solid skeleton will be considered separately.

Belgorod; Moscow. Translated from *Sibirskii Matematicheskii Zhurnal*, vol. 59, no. 5, pp. 1145–1158, September–October, 2018; DOI: 10.17377/smzh.2018.59.515. Original article submitted August 23, 2017.

 

In the domain $\Omega^0$ for $t > 0$, the motion of the continuous medium is described by the equations

$$\nabla \cdot \mathbf{w} = 0, \tag{1}$$

$$\nabla \cdot \mathbb{P}^0 + \varrho^{0,\varepsilon}\mathbf{F} = 0, \tag{2}$$

where

$$\mathbb{P}^0 = \chi_0^\varepsilon \alpha_\mu \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + \big(1 - \chi_0^\varepsilon\big)\lambda_0^0 \mathbb{D}(x, \mathbf{w}) - p\,\mathbb{I}.$$

The motion in $\Omega$ for $t > 0$ is described by the continuity equation and the momentum balance equation

$$\nabla \cdot \mathbb{P} + \varrho^\varepsilon \mathbf{F} = 0, \tag{3}$$

where

$$\mathbb{P} = \chi^\varepsilon \alpha_\mu \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + (1 - \chi^\varepsilon)\lambda_0 \mathbb{D}(x, \mathbf{w}) - p\,\mathbb{I}.$$

On the common boundary $S^0 = \partial\Omega \cap \partial\Omega^0$ for $t > 0$, the continuity conditions

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^0}} \mathbf{w}(\mathbf{x}, t) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \mathbf{w}(\mathbf{x}, t), \tag{4}$$

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega^0}} \mathbb{P}^0(\mathbf{x}, t) \cdot \boldsymbol{n}(\mathbf{x}^0) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0\\ \mathbf{x}\in\Omega}} \mathbb{P}(\mathbf{x}, t) \cdot \boldsymbol{n}(\mathbf{x}^0) \tag{5}$$

for the displacements and the normal stresses hold. Here $n(\mathbf{x}^0)$ is the normal vector to the boundary $S^0$ at a point $\mathbf{x}^0 \in S^0$.

The problem is closed by the Dirichlet boundary condition

$$\mathbf{w}(\mathbf{x}, t) = 0 \tag{6}$$

on the exterior boundary $S = \partial Q$ for $t > 0$, the initial condition

$$\hat{\chi}^\varepsilon(\mathbf{x})\mathbf{w}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in Q, \tag{7}$$

and the normalization condition

$$\int_Q p^\varepsilon(\mathbf{x}, t)\, dx = 0. \tag{8}$$

In (1)–(8), $\mathbf{F}$ is the given density of the distributed mass forces.

Introduce the small parameter $\varepsilon$ as the ratio of the average size of the pores to the size of the domain under consideration: $\varepsilon = \frac{l}{L}$, and assume the dependence of the dimensionless quantity $\alpha_\mu$ on $\varepsilon$ and the existence of the limit

$$\mu_0 = \lim_{\varepsilon \searrow 0} \alpha_\mu(\varepsilon), \quad \alpha_\mu = \frac{2\mu}{\tau L g\, \rho^0},$$

where $L$ is the typical size of the physical domain under consideration, $\tau$ is the time of the physical process, $\rho^0$ is the water density, $g$ is the acceleration of gravity, and $\mu$ is the dynamical viscosity coefficient.

For using homogenization theory and the results on two-scale convergence, we also need additional assumptions on the geometry of the porous space and solid skeleton.

**Assumption 1.** 1. *Let $\chi(\mathbf{y})$ be a 1-periodic function, let $Y_s = \{\mathbf{y} \in Y : \chi(\mathbf{y}) = 0\}$ be the solid part of the unit cube $Y = (0,1)^3 \subset \mathbb{R}^3$, and assume that the fluid part $Y_f = \{\mathbf{y} \in Y : \chi(\mathbf{y}) = 1\}$ is the open complement to the solid part. Put $\gamma = \partial Y_f \cap \partial Y_s$; assume that $\gamma$ is a Lipschitz continuous surface.*

2. *The domain $E_f^\varepsilon$ is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y_f^\varepsilon = \varepsilon Y_f$, and the domain $E_s^\varepsilon$ is a periodic repetition in $\mathbb{R}^3$ of the elementary cell $Y_s^\varepsilon = \varepsilon Y_s$.*

3. *The porous set $\Omega_f^\varepsilon \subset \Omega = \Omega \cap E_f^\varepsilon$ is a periodic repetition in $\Omega$ of the elementary cell $\varepsilon Y_f$, and the solid skeleton $\Omega_s^\varepsilon \subset \Omega = \Omega \cap E_s^\varepsilon$ is a periodic repetition in $\Omega$ of the elementary cell $\varepsilon Y_s$. The Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition in $\Omega$ of the boundary $\varepsilon\gamma$.*

4. *$Y_s$ and $Y_f$ are connected sets.*

**Assumption 2.** *The solid skeleton $\Omega_s^\varepsilon$ is a connected domain.*

**Assumption 3.** *The porous space $\Omega_f^\varepsilon$ is a connected domain.*

Under these assumptions, $\chi^\varepsilon(\mathbf{x})$ is the characteristic function of the fluid domain $\Omega_f^\varepsilon$, $\chi^\varepsilon(\mathbf{x}) = (1 - \zeta(\mathbf{x}))\chi\big(\frac{\mathbf{x}}{\varepsilon}\big)$, and $\chi_0^\varepsilon(\mathbf{x})$ is the characteristic function of the fluid domain $\Omega_f^{0,\varepsilon}$, $\chi_0^\varepsilon(\mathbf{x}) = \zeta(\mathbf{x})\chi_0\big(\frac{\mathbf{x}}{\varepsilon}\big)$, and $\zeta(\mathbf{x})$ is the characteristic function of $\Omega^0$ in $Q$, while $\chi(\mathbf{y})$ is the characteristic function of $Y_f$ in the unit cube $Y$, and $\chi_0(\mathbf{y})$ is the characteristic function of $Y_f^0$ in $Y$.

We describe the general fluid and solid domains $Q_f^\varepsilon$ and $Q_s^\varepsilon$ as follows:

$$Q_f^\varepsilon = \Omega_f^{0,\varepsilon} \cup \Omega_f^\varepsilon, \quad Q_s^\varepsilon = \Omega_s^{0,\varepsilon} \cup \Omega_s^\varepsilon,$$

where $\Omega_f^{0,\varepsilon}$, $\Omega_s^{0,\varepsilon}$, $\Omega_f^\varepsilon$, and $\Omega_s^\varepsilon$ are the fluid and solid domains in $\Omega^0$ and $\Omega$ respectively.

Henceforth, $\hat{\chi}^\varepsilon(\mathbf{x}) = \zeta(\mathbf{x})\chi_0^\varepsilon(\mathbf{x}) + (1-\zeta(\mathbf{x}))\chi^\varepsilon(\mathbf{x})$ is the characteristic function of the fluid domain $Q_f^\varepsilon$,

$$\hat{m} = \zeta(\mathbf{x})m_0 + (1 - \zeta(\mathbf{x}))m, \quad m = \int\limits_Y \chi(\mathbf{y})\,dy, \quad m_0 = \int\limits_Y \chi_0(\mathbf{y})\,dy,$$

$$\varrho^{0,\varepsilon} = \varrho_f \chi_0^\varepsilon + \varrho_s^0\big(1 - \chi_0^\varepsilon\big), \quad \varrho^\varepsilon = \varrho_f \chi^\varepsilon + \varrho_s(1 - \chi^\varepsilon),$$

$$\hat{\varrho}^\varepsilon(\mathbf{x}) = \zeta(\mathbf{x})\varrho^{0,\varepsilon}(\mathbf{x}) + (1 - \zeta(\mathbf{x}))\varrho^\varepsilon(\mathbf{x}),$$

$\varrho_s^0$ and $\varrho_s$ are the dimensionless densities of the solid components in $\Omega^0$ and $\Omega$ respectively, $\lambda_0^0$ and $\lambda_0$ are the dimensionless Lamé constants of the solid components in $\Omega^0$ and $\Omega$ respectively, $\mathbb{I}$ is the identity matrix, $\nabla\mathbf{u}$ and $\nabla\cdot\mathbf{w}$ are the operation of computing the gradient and divergence which are performed with respect to the variable $\mathbf{x}$ throughout, the article except the cases when these operations are endowed with the subscript $y$. In these cases, derivation is carried out with respect to $\mathbf{y}$. Further,

$$\mathbb{D}(x, \mathbf{u}) = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T).$$

Also $B : C = \operatorname{tr}(B\cdot C^T)$, where $B$ and $C$ are second-rank tensors, $\mathbf{a}\otimes\mathbf{b}$ is a dyad, $(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$ for all vectors $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, $\mathbb{J}^{ij} = 1/2(e_i\otimes e_j + e_j\otimes e_i)$, $A\otimes B$ is a fourth-rank tensor, and $(A\otimes B) : C = A(B : C)$ for every second-rank tensor $C$.

Let us explain the meaning of (1)–(8) in the solid part $Q_s^\varepsilon = \Omega_s^{0,\varepsilon} \cup \Omega_s^\varepsilon$. The solid component is an elastic medium. In the classical continuum mechanics, an elastic medium is a continuous medium in which the stress tensor $\mathbb{P}$ depends on the strain tensor $\mathbb{E}$ and this dependence is expressed by a corresponding connection equation. In the linear theory of elasticity, the connection equation contains two coefficients, one at the strain tensor and one at the identity tensor $\mathbb{I}$:

$$\mathbb{P}(\boldsymbol{w}) = 2\,\mu\,\mathbb{E} + \lambda\,J(\mathbb{E})\mathbb{I}, \quad \mathbb{E}(\boldsymbol{w}) = \frac{1}{2}(\nabla\boldsymbol{w} + \nabla^*\boldsymbol{w}). \tag{9}$$

Since the scalar coefficient at the identity tensor depends linearly on the displacement vector $\boldsymbol{w}$, the only possible form of this coefficient is the product of the divergence vector and some constant:

$$\lambda J(\mathbb{E}) = \nabla\cdot\boldsymbol{w}. \tag{10}$$

In this case, the Lamé equations

$$\nabla\cdot\mathbb{P} + \varrho\boldsymbol{f} = 0 \tag{11}$$

take the form

$$\mu\,\Delta\boldsymbol{w} + (\mu + \lambda)\nabla(\nabla\cdot\boldsymbol{w}) + \varrho\boldsymbol{f} = 0. \tag{12}$$

By analogy with a viscous fluid (for simplicity and due to the coincidence of the Lamé equations with the Stokes equations for a compressible fluid), we can refer as the *pressure* $p$ to the expression

$$p = -(\mu + \lambda)\, \nabla \cdot \boldsymbol{w}. \tag{13}$$

The corresponding equation (which defines the pressure) is called the *continuity equation*. The analogy becomes complete if the equation

$$\mu \Delta \boldsymbol{w} - \nabla\, p + \varrho \boldsymbol{f} = 0 \tag{14}$$

is assumed to be the impulse conservation law.

For $\lambda > 0$, the system of equations (13), (14) in the domain $\Omega$ supplemented by the boundary condition

$$\boldsymbol{w}_\lambda = 0 \tag{15}$$

on the boundary $S$ of $\Omega$ has a unique (weak) solution

$$\boldsymbol{w}_\lambda \in \mathbb{W}^1_2(\Omega), \quad p_\lambda \in \mathbb{L}_2(\Omega)$$

and

$$\int_\Omega \left(\mu |\nabla \boldsymbol{w}_\lambda|^2 + p_\lambda^2 + (\mu + \lambda)|\nabla \cdot \boldsymbol{w}_\lambda|^2\right) dx < C^2,$$

where $C$ does not depend on $\lambda$.

By the well-known theorems of analysis, we can choose the convergent subsequences

$$\boldsymbol{w}_\lambda \rightharpoonup \boldsymbol{W} \text{ (weakly in } \mathbb{L}_2(\Omega)), \quad \nabla \boldsymbol{w}_\lambda \rightharpoonup \nabla \boldsymbol{W}, \quad p_\lambda \rightharpoonup P \quad \text{as } \lambda \to \infty,$$

and perform passage to the limit in equation (13) written down in the form

$$\nabla \cdot \boldsymbol{w}_\lambda = -\frac{p_\lambda}{(\mu + \lambda)},$$

and in the Lamé equations (14), written in the form of the corresponding integral identities:

$$\nabla \cdot \boldsymbol{W} = 0, \tag{16}$$

$$\mu \Delta \boldsymbol{W} - \nabla P + \varrho \boldsymbol{f} = 0. \tag{17}$$

The system of equations (16) and (17) is a well-known approximation of the general Lamé equations for the sufficiently large values of the parameter $\lambda > 0$ (as in the case of the Stokes equations for an incompressible viscous fluid).

Observe finally that, in real physical processes, all media are compressible but, for some values of the parameters of the problem, we can use the "incompressible fluid" approximation and substantially simplify the problem.

This approach is widely used in hydrodynamics, and its use in the theory of elasticity is equally justified.

## 2. The Main Results

The prime interest in this problem is the conditions of continuity on the common boundary $S^0$. These conditions depend on the structures of the corresponding porous spaces, i.e. on the functions $\chi_0(\mathbf{y})$ and $\chi(\mathbf{y})$. We consider the two types of structures.

1. For the first structure of a general porous space, the elementary fluid domains $Y^0_f$ and $Y_f$ have nonempty intersection in $Y$:

$$Y^0_f \cap Y_f \neq \varnothing. \tag{18}$$

2. For the second structure, the elementary fluid domains $Y_f^0$ and $Y_f$ are disjoint in $Y$:

$$Y_f^0 \cap Y_f = \varnothing. \tag{19}$$

For both structures of a general porous space, the elementary solid domains $Y_s^0$ and $Y_s$ have nonempty intersection in $Y$:

$$Y_s^0 \cap Y_s \neq \varnothing. \tag{20}$$

Suppose that Assumptions 1–3 are fulfilled for porous spaces defined by the characteristic functions $\chi(\mathbf{y})$ and $\chi_0(\mathbf{y})$.

Definition 1. Refer to a pair of functions $\{\mathbf{w}^\varepsilon, p^\varepsilon\}$ such that

$$\mathbf{w}^\varepsilon \in \overset{\circ}{\mathbf{W}}{}_2^{1,0}(G_T), \quad p^\varepsilon \in L_2(G_T), \quad G_T = Q \times (0,T),$$

a *weak solution to* (1)–(8) if they satisfy the continuity equation (1) a.e. in $G_T$, the normalization condition (8), and the integral identity

$$\int\limits_{G_T} \bigg( -\alpha_\mu \hat{\chi}^\varepsilon \mathbb{D}(x, \mathbf{w}^\varepsilon) : \mathbb{D}\bigg(x, \frac{\partial \boldsymbol{\varphi}}{\partial t}\bigg) + (1-\hat{\chi}^\varepsilon)\lambda(\mathbf{x})\mathbb{D}(x,\mathbf{w}^\varepsilon) : \mathbb{D}(x,\boldsymbol{\varphi})\bigg)\, dxdt$$
$$= \int\limits_{G_T} (p^\varepsilon(\nabla \cdot \boldsymbol{\varphi}) + \hat{\varrho}^\varepsilon \mathbf{F} \cdot \boldsymbol{\varphi})\, dxdt \tag{21}$$

for all functions $\boldsymbol{\varphi}$ equal to zero at $t = T$ and such that $\boldsymbol{\varphi}, \frac{\partial \boldsymbol{\varphi}}{\partial t} \in \overset{\circ}{\mathbf{W}}{}_2^{1,0}(G_T)$.

In (21), $\lambda(\mathbf{x}) = \lambda_0^0 \zeta(\mathbf{x}) + \lambda_0(1 - \zeta(\mathbf{x}))$.

**Theorem 1.** *Suppose that*

$$\max_{0<t<T} \int\limits_Q |\mathbf{F}(\mathbf{x},t)|^2\, dx = \mathfrak{P}^2 < \infty, \quad \int\limits_{G_T} (1-\hat{\chi}^\varepsilon) \bigg|\frac{\partial \mathbf{F}}{\partial t}\bigg|^2 dxdt = \mathfrak{P}_1^2 < \infty.$$

*Then for all* $\varepsilon > 0$ *and every time interval* $[0,T]$ *there exists a unique weak solution to* (1)–(8) *and*

$$\max_{0<t<T} \int\limits_Q \hat{\chi}^\varepsilon \left(\alpha_\mu |\mathbb{D}(x, \mathbf{w}^\varepsilon(\mathbf{x},t))|^2 + \frac{\alpha_\mu}{\varepsilon^2}\big|\mathbf{w}^\varepsilon(\mathbf{x},t) - \mathbf{w}_s^\varepsilon(\mathbf{x},t)\big|^2\right) dx$$
$$+ \int\limits_{G_T} (|\pi^\varepsilon|^2 + \lambda(\mathbf{x})|\mathbb{D}(x,\mathbf{w}^\varepsilon)|^2)\, dxdt \leqslant C_0 \mathfrak{P}^2, \tag{22}$$

$$\int\limits_{G_T} (|\mathbf{w}^\varepsilon|^2 + |\mathbb{D}(x,\mathbf{v}^\varepsilon)|^2)\, dxdt + \max_{0<t<T} \int\limits_G (|\mathbb{D}(x,\mathbf{w}^\varepsilon)|^2 + |p^\varepsilon|^2)\, dx \leqslant C_0\big(\mathfrak{P}^2 + \mathfrak{P}_1^2\big), \tag{23}$$

*where* $\mathbf{v}^\varepsilon = \mathbb{E}_{Q_f^\varepsilon}\big(\frac{\partial \mathbf{w}^\varepsilon}{\partial t}\big)$ *is an extension of* $\frac{\partial \mathbf{w}^\varepsilon}{\partial t}$ *from the fluid domain* $Q_f^\varepsilon$ *to* $Q$ *and the constant* $C_0$ *does not depend on* $\varepsilon$, $\lambda_0^0$, *and* $\lambda_0$ *for* $\lambda_0 > 1$, $\lambda_0^0 > 1$, *and*

$$\pi^\varepsilon(\mathbf{x},t) = \int\limits_0^t p^\varepsilon(\mathbf{x},\tau)\, d\tau.$$

**Theorem 2.** *Suppose that* $\alpha_\mu = \mu_0$, $0 < \mu_0, \lambda_0, \lambda_0^0 < \infty$ *and let* $\{\mathbf{w}^\varepsilon, p^\varepsilon\}$ *be the weak solution to* (1)–(8).

*Then, up to subsequences,* $\{p^\varepsilon\}$ *converges weakly in* $L_2(G_T)$ *as* $\varepsilon \to 0$ *to a function* $p$ *and* $\{\mathbf{w}^\varepsilon\}$ *converges weakly in* $\mathbf{W}_2^{1,0}(G_T)$ *as* $\varepsilon \to 0$ *to a function* $\mathbf{w}$.

*The limit functions are solutions to the homogenized system consisting of the continuity equation*

$$\nabla \cdot \mathbf{w} = 0 \tag{24}$$

*and the momentum balance equation*

$$\nabla \cdot \widehat{\mathbb{P}}^0 + \hat{\varrho}\mathbf{F} = 0, \tag{25}$$

$$\widehat{\mathbb{P}}^0 = -p\,\mathbb{I} + \mathfrak{N}_1^0 : \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + \mathfrak{N}_2^0 : \mathbb{D}(x, \mathbf{w}) + \int_0^t \mathfrak{N}_3^0(t-\tau) : \mathbb{D}(x, \mathbf{w}(\mathbf{x}, \tau))\, d\tau$$

*in* $\Omega^0$ *for* $t > 0$, *the continuity equation* (24), *and the homogenized momentum balance equation*

$$\nabla \cdot \widehat{\mathbb{P}} + \hat{\varrho}\mathbf{F} = 0, \tag{26}$$

$$\widehat{\mathbb{P}} = -p\,\mathbb{I} + \mathfrak{N}_1 : \mathbb{D}\Big(x, \frac{\partial \mathbf{w}}{\partial t}\Big) + \mathfrak{N}_2 : \mathbb{D}(x, \mathbf{w}) + \int_0^t \mathfrak{N}_3(t-\tau) : \mathbb{D}(x, \mathbf{w}(\mathbf{x}, \tau))\, d\tau$$

*in* $\Omega$ *for* $t > 0$.

*The problem is closed by the normalization condition*

$$\int_Q p(\mathbf{x}, t)\, dx = 0, \tag{27}$$

*the continuity condition for the normal stresses*

$$\lim_{\substack{\mathbf{x}\to\mathbf{x}^0 \\ \mathbf{x}\in\Omega^0}} \widehat{\mathbb{P}}^0(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}^0) = \lim_{\substack{\mathbf{x}\to\mathbf{x}^0 \\ \mathbf{x}\in\Omega}} \widehat{\mathbb{P}}(\mathbf{x}, t) \cdot \mathbf{n}(\mathbf{x}^0) \tag{28}$$

*on the common boundary* $S^0$, *the Dirichlet condition*

$$\mathbf{w}(\mathbf{x}, t) = 0 \tag{29}$$

*on the exterior boundary* $S$, *and the initial condition*

$$\mathbf{w}(\mathbf{x}, 0) = 0, \quad \mathbf{x} \in Q. \tag{30}$$

*The fourth-rank tensors* $\mathfrak{N}_1^0$, $\mathfrak{N}_2^0$, *and* $\mathfrak{N}_3^0(t)$ *are defined by* (47) *for* $\mu_0$ *and* $\lambda_0^0$ *and the porous space with characteristic function* $\chi_0(\mathbf{y})$, *the fourth-rank tensors* $\mathfrak{N}_1$, $\mathfrak{N}_2$, *and* $\mathfrak{N}_3(t)$ *are defined by* (47) *for* $\mu_0$, $\lambda_0$, *and the porous space with characteristic function* $\chi(\mathbf{y})$. *The symmetric tensors* $\mathfrak{N}_1^0$ *and* $\mathfrak{N}_1$ *are positive-definite.*

The extension is constructed on the basis of the extension lemma in [22].

## 3. Proof of Theorem 1

Choose a test function in (21) as follows: $\varphi(\mathbf{x},\tau) = h(\tau)\mathbf{w}^\varepsilon(\mathbf{x},\tau)$, where $h(\tau) = 1$ for $0 < \tau < t$ and $h(\tau) = 0$ for $t < \tau < T$. We obtain the estimates

$$\alpha_\mu \int\limits_{Q^\varepsilon_f} |\mathbb{D}(x, \mathbf{w}^\varepsilon(\mathbf{x}, t))|^2\, dx + \min\left(\lambda^0_0, \lambda_0\right) \int\limits_0^t \int\limits_{Q^\varepsilon_s} |\mathbb{D}(x, \mathbf{w}^\varepsilon(\mathbf{x}, \tau))|^2\, dx d\tau \leqslant C_0 \mathfrak{P}^2, \tag{31}$$

$$\int\limits_0^T \int\limits_Q \left|\mathbf{w}^\varepsilon(\mathbf{x}, t) - \mathbf{w}^\varepsilon_s(\mathbf{x}, t)\right|^2 dx dt \leqslant \frac{C_0}{\mu_1} \alpha_\mu \int\limits_0^T \int\limits_{Q^\varepsilon_f} \left|\mathbb{D}(x, \mathbf{w}^\varepsilon(\mathbf{x}, t)) - \mathbb{D}\big(x, \boldsymbol{w}^\varepsilon_s(\mathbf{x}, t)\big)\right|^2 dx, \tag{32}$$

where $\mathbf{w}^\varepsilon_s$ is an extension of $\mathbf{w}^\varepsilon$ from the solid part $Q^\varepsilon_s$ to the fluid part $Q^\varepsilon_f$:

$$\int\limits_Q \left|\mathbf{w}^\varepsilon_s(\mathbf{x}, t)\right|^2 dx \leqslant C_0 \int\limits_{Q^\varepsilon_s} |\mathbf{w}^\varepsilon(\mathbf{x}, t)|^2\, dx, \tag{33}$$

$$\int\limits_Q \left|\mathbf{w}^\varepsilon_s(\mathbf{x}, t)\right|^2 dx \leqslant C_0 \int\limits_Q \left|\mathbb{D}\big(x, \mathbf{w}^\varepsilon_s(\mathbf{x}, t)\big)\right|^2 dx \leqslant C_0 \int\limits_{Q^\varepsilon_s} |\mathbb{D}(x, \boldsymbol{w}^\varepsilon(\mathbf{x}, t))|^2\, dx, \tag{34}$$

$$Q^\varepsilon_s = \{\mathbf{x} \in Q : \hat{\chi}^\varepsilon(\mathbf{x}) = 0\}, \quad Q^\varepsilon_f = \{\mathbf{x} \in Q : \hat{\chi}^\varepsilon(\mathbf{x}) = 1\}.$$

In (31) and (32), $C_0$ depends only of the domain $Q$ and the geometry of the porous spaces $\Omega^0$ and $\Omega$ and does not depend on $\varepsilon$. In (31), $C_0$ additionally depends on $\min\left\{\lambda^0_0, \lambda_0, 1\right\}$.

For proving the estimate for the pressure $p^\varepsilon$, write (21) as

$$\int\limits_{G_T} \pi^\varepsilon\, \nabla \cdot \varphi\, dx dt = \int\limits_{G_T} \left(\mathbb{F} : \mathbb{D}(x, \varphi) + \hat{\varrho}^\varepsilon \int\limits_0^t \mathbf{F}(\mathbf{x}, \tau)\, d\tau \cdot \varphi\right) dx dt,$$

where

$$\frac{\partial \mathbb{F}}{\partial t} = \hat{\chi}^\varepsilon \alpha_\mu \mathbb{D}\left(x, \frac{\partial \mathbf{w}^\varepsilon}{\partial t}\right) + (1 - \hat{\chi}^\varepsilon)\lambda(x)\mathbb{D}(x, \mathbf{w}^\varepsilon), \quad \mathbb{F}, \frac{\partial \mathbb{F}}{\partial t} \in L_2(\Omega_T).$$

Then

$$\left|\int\limits_{G_T} \pi^\varepsilon \nabla \cdot \varphi\, dx dt\right| \leqslant C_0 F \left(\int\limits_{G_T} |\nabla \varphi|^2\, dx dt\right)^{\frac{1}{2}}. \tag{35}$$

Next, choose a test function $\varphi$ so that

$$\nabla \cdot \varphi = \pi^\varepsilon, \quad \int\limits_{G_T} |\nabla \varphi|^2\, dx dt \leqslant C_0 \int\limits_{G_T} |\pi^\varepsilon|^2\, dx dt.$$

To this end, represent $\varphi$ as the sum of two functions $\varphi_0$ and $\nabla\psi$ such that

$$\Delta \psi = \pi^\varepsilon, \quad \mathbf{x} \in \Omega, \quad \psi|_S = 0, \tag{36}$$

$$\nabla \cdot \varphi_0 = 0, \quad \mathbf{x} \in \Omega, \quad (\varphi_0 + \nabla \psi)_S = 0. \tag{37}$$

Basing on the results by Ladyzhenskaya [23, 24] and the equality

$$\int\limits_Q \pi^\varepsilon(\mathbf{x}, t)\, dx = 0,$$

which stems from the normalization condition, we infer that problem (36) has a unique solution $\psi \in L_2\big((0,T); W_2^2(Q)\big)$,

$$\int_0^T \big(\|\psi\|_2^{(2)}(t)\big)^2\,dt \leqslant C_0 \int_{G_T} |\pi^\varepsilon|^2\,dxdt,$$

and (37) has at least one solution $\varphi_0 \in \mathbf{W}_2^{1,0}(G_T)$,

$$\int_0^T \big(\|\varphi_0\|_2^{(1)}(t)\big)^2\,dt \leqslant C_0 \int_0^T \big(\|\psi\|_2^{(2)}(t)\big)^2\,dt.$$

The last two relations and (35) give an estimate for $\pi^\varepsilon$.

The assumption of the theorem that

$$\int_{G_T} (1-\hat{\chi}^\varepsilon)\left|\frac{\partial \mathbf{F}}{\partial t}\right|^2 dxdt = \mathfrak{P}_1^2 < \infty$$

enables us to similarly obtain (23).

## 4. Proof of Theorem 2

**4.1. Passage to the limit.** The a priori estimates of the previous theorem show that as $\varepsilon \to 0$,

$$p^\varepsilon \rightharpoonup p \quad \text{weakly in } L_2(G_T),$$

$$\mathbf{w}^\varepsilon \to \mathbf{w}(\mathbf{x},t) \quad \text{weakly and two-scalely in } \mathbf{L}_2(G_T),$$

$$\mathbf{v}^\varepsilon \to \mathbf{v}(\mathbf{x},t) = \frac{\partial \mathbf{w}}{\partial t} \quad \text{weakly in } \overset{\circ}{\mathbf{W}}{}_2^{1,0}(G_T) \text{ and two-scalely in } \mathbf{L}_2(G_T),$$

$$\mathbb{D}(x,\mathbf{w}^\varepsilon) \to \mathbb{D}(x,\mathbf{w}) + \mathbb{D}(y,\widehat{\mathbf{W}}) \quad \text{two-scalely in } \mathbf{L}_2(G_T),$$

$$\mathbb{D}(x,\mathbf{v}^\varepsilon) \to \mathbb{D}(x,\mathbf{v}) + \mathbb{D}(y,\widehat{\mathbf{V}}) = \mathbb{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big) + \mathbb{D}\Big(y,\frac{\partial \widehat{\mathbf{W}}}{\partial t}\Big) \quad \text{two-scalely in } \mathbf{L}_2(G_T),$$

$$\widehat{\mathbf{W}} = \zeta \mathbf{W}^0 + (1-\zeta)\mathbf{W},$$

where the functions $\mathbf{W}^0$ and $\mathbf{W}$ are determined separately in $\Omega_T^0 \times Y$ and $\Omega_T \times Y$. In the next section, we deduce formulas for $\Omega_T \times Y$.

Passing to the two-scale limit in the continuity equation as $\varepsilon \to 0$, we obtain the microscopic continuity equation:

$$\nabla_y \cdot \mathbf{W} = 0, \quad \mathbf{y} \in Y. \tag{38}$$

Now, pass to the limit as $\varepsilon \to 0$ in the integral identity (21) with two different types of test functions. First take the test functions $\varphi = \varphi(\mathbf{x},t)$ and then the test functions $\varphi = \varepsilon h(\mathbf{x},t)\varphi_0\big(\frac{\mathbf{x}}{\varepsilon}\big)$. We obtain the microscopic momentum balance equation in $\Omega$:

$$\nabla \cdot \widehat{\mathbb{P}} + \hat{\varrho}\mathbf{F} = 0, \tag{39}$$

$$\widehat{\mathbb{P}} = \mu_0\bigg(m\mathbb{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big) + \Big\langle \mathbb{D}\Big(y,\frac{\partial \mathbf{W}}{\partial t}\Big)\Big\rangle_{Y_f}\bigg) + \lambda_0((1-m)\mathbb{D}(x,\mathbf{w}) + \langle \mathbb{D}(y,\mathbf{W})\rangle_{Y_s}) - p\,\mathbb{I}, \tag{40}$$

and the microscopic momentum balance equation

$$\nabla_y \cdot \bigg(\mu_0\chi\Big(\mathbb{D}\Big(x,\frac{\partial \mathbf{w}}{\partial t}\Big) + \mathbb{D}\Big(y,\frac{\partial \mathbf{W}}{\partial t}\Big)\Big) + \lambda_0(1-\chi)(\mathbb{D}(x,\mathbf{w}) + \mathbb{D}(y,\mathbf{W})) - P\,\mathbb{I}\bigg) = 0.$$

Rewrite the last equation as

$$\nabla_y \cdot \left(\chi\left(\mu_0 \mathbb{D}\left(y, \frac{\partial \mathbf{W}}{\partial t}\right) + \mathbb{Z}\right) + \lambda_0(1-\chi)\mathbb{D}(y, \mathbf{W}) - P\,\mathbb{I}\right) = 0, \tag{41}$$

where

$$\mathbb{Z}(\mathbf{x}, t) = \mu_0 \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}\right) - \lambda_0 \mathbb{D}(x, \mathbf{w}) = \sum_{i,j=1}^{3} Z_{ij}(\mathbf{x}, t)\mathbb{J}^{(ij)}.$$

**4.2. Deduction of the problems on the periodicity cell.** For finding the tensors $\mathfrak{N}_1$, $\mathfrak{N}_2$, and $\mathfrak{N}_3(t)$, we must solve (38) and (41), find $\mathbb{D}\big(y, \frac{\partial \mathbf{W}}{\partial t}\big)$ and $\mathbb{D}(y, \mathbf{W})$ as operators of $\mathbb{D}\big(x, \frac{\partial \mathbf{w}}{\partial t}\big)$ and $\mathbb{D}(x, \mathbf{w})$ respectively and then insert these expressions in (40).

Let $\{\mathbf{W}^{(ij)}(\mathbf{y}, t), P^{(ij)}(\mathbf{y}, t)\}$ and $\big\{\mathbf{W}_0^{(ij)}(\mathbf{y}), P_0^{(ij)}(\mathbf{y})\big\}$, $i, j = 1, 2, 3$, be the solutions to the periodic problems

$$\begin{cases} \nabla_y \cdot \big(\chi\mu_0 \mathbb{D}\big(y, \frac{\partial \mathbf{W}^{(ij)}}{\partial t}\big) + \lambda_0(1-\chi)\mathbb{D}(y, \mathbf{W}^{(ij)}) - P^{(ij)}\,\mathbb{I}\big) = 0, \\ \nabla_y \cdot \mathbf{W}^{(ij)} = 0, \\ \chi(\mathbf{y})\mathbf{W}^{(ij)}(\mathbf{y}, 0) = \mathbf{W}_0^{(ij)}(\mathbf{y}), \end{cases} \tag{42}$$

$$\begin{cases} \nabla_y \cdot \big(\chi\big(\mu_0 \mathbb{D}\big(y, \mathbf{W}_0^{(ij)}\big) + \mathbb{J}^{(ij)} - P_0^{(ij)}\,\mathbb{I}\big)\big) = 0, \\ \nabla_y \cdot \mathbf{W}_0^{(ij)} = 0, \quad \int_Y \chi(\mathbf{y})\mathbf{W}_0^{(ij)}(\mathbf{y})\,dy = 0, \end{cases} \tag{43}$$

in the domain $Y$.

Then

$$\mathbf{W}(\mathbf{x}, t, \mathbf{y}) = \sum_{i,j=1}^{3} \int_0^t \mathbf{W}^{(ij)}(\mathbf{y}, t-\tau) Z_{ij}(\mathbf{x}, \tau)\,d\tau,$$

$$P(\mathbf{x}, t, \mathbf{y}) = \chi(\mathbf{y}) \sum_{i,j=1}^{3} P_0^{(ij)}(\mathbf{y}) Z_{ij}(\mathbf{x}, t) + \sum_{i,j=1}^{3} \int_0^t P^{(ij)}(\mathbf{y}, t-\tau) Z_{ij}(\mathbf{x}, \tau)\,d\tau,$$

$$\mathbb{D}(y, \mathbf{W}) = \sum_{i,j=1}^{3} \int_0^t \mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y}, t-\tau)) Z_{ij}(\mathbf{x}, \tau)\,d\tau$$

$$= \sum_{i,j=1}^{3} \int_0^t (\mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y}, t-\tau)) \otimes \mathbb{J}^{(ij)}) : \mathbb{Z}(\mathbf{x}, \tau)\,d\tau$$

$$= \left(\mu_0 \sum_{i,j=1}^{3} \mathbb{D}\big(y, \mathbf{W}_0^{(ij)}\big) \otimes \mathbb{J}^{(ij)}\right) : \mathbb{D}(x, \mathbf{w}) - \sum_{i,j=1}^{3} \int_0^t \left(\left(\lambda_0 \mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y}, t-\tau))\right.\right.$$

$$\left.\left. + \mu_0 \mathbb{D}\left(y, \frac{\partial \mathbf{W}^{(ij)}}{\partial \tau}(\mathbf{y}, t-\tau)\right)\right) \otimes \mathbb{J}^{(ij)}\right) : \mathbb{D}(x, \mathbf{w}(\mathbf{x}, \tau))\,d\tau.$$

Using the obvious relation

$$\frac{\partial \mathbf{W}^{(ij)}}{\partial \tau}(\mathbf{y}, t-\tau) = -\frac{\partial \mathbf{W}^{(ij)}}{\partial t}(\mathbf{y}, t-\tau),$$

we get

$$\mathbb{D}(y, \mathbf{W}) = \mathfrak{A}_0(\mathbf{y}) : \mathbb{D}(x, \mathbf{w}) + \int_0^t \mathfrak{A}_1(\mathbf{y}, t-\tau) : \mathbb{D}(x, \mathbf{w}(\mathbf{x}, \tau))\,d\tau, \tag{44}$$

where

$$\mathfrak{A}_0(\mathbf{y}) = \mu_0 \sum_{i,j=1}^{3} \mathbb{D}\big(y, \mathbf{W}_0^{(ij)}(\mathbf{y})\big) \otimes \mathbb{J}^{(ij)}, \tag{45}$$

$$\mathfrak{A}_1(\mathbf{y},t) = \sum_{i,j=1}^{3} \left(\mu_0 \mathbb{D}\left(y, \frac{\partial \mathbf{W}^{(ij)}}{\partial t}(\mathbf{y},t)\right) - \lambda_0 \mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y},t))\right) \otimes \mathbb{J}^{(ij)}. \tag{46}$$

Equations (44) and (45) lead to the relation

$$\mathbb{D}\left(y, \frac{\partial \mathbf{W}}{\partial t}\right) = \mathfrak{A}_0(\mathbf{y}) : \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}(\mathbf{x},t)\right) + \mathfrak{A}_1(\mathbf{y},0) : \mathbb{D}(x, \mathbf{w}(\mathbf{x},t))$$
$$+ \int_0^t \frac{\partial \mathfrak{A}_1}{\partial t}(\mathbf{y}, t-\tau) : \mathbb{D}(x, \mathbf{w}(\mathbf{x},\tau))\, d\tau.$$

Therefore,

$$\begin{cases} \mathfrak{N}_1 = \mu_0 m \sum\limits_{i,j=1}^{3} \mathbb{J}^{(ij)} \otimes \mathbb{J}^{(ij)} + \mu_0 \langle \mathfrak{A}_0 \rangle_{Y_f}, \\ \mathfrak{N}_2 = \lambda_0 (1-m) \sum\limits_{i,j=1}^{3} \mathbb{J}^{(ij)} \otimes \mathbb{J}^{(ij)} + \lambda_0 \langle \mathfrak{A}_0 \rangle_{Y_s} + \mu_0 \langle \mathfrak{A}_1(\mathbf{y},0) \rangle_{Y_f}, \\ \mathfrak{N}_3(t) = \mu_0 \left\langle \frac{\partial \mathfrak{A}_1}{\partial t}(\mathbf{y},t) \right\rangle_{Y_f} + \lambda_0 \langle \mathfrak{A}_1(\mathbf{y},t) \rangle_{Y_s}. \end{cases} \tag{47}$$

Uniting the obtained results, we infer that the limit functions satisfy (24)–(26), the boundary condition (29), and the initial condition (30). The fulfillment of (27) is a consequence of the weak convergence of $\{p^\varepsilon\}$ and (8).

**4.3. Proof of the solvability of the initial-boundary value problems on the cell.** Prove the solvability of (42) and (43). We have the chain of energy identities

$$\frac{1}{2} \int_Y \chi \mu_0 |\mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y},t))|^2\, dy + \int_0^t \int_Y (1-\chi) \lambda_0 |\mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y},\tau))|^2\, dy d\tau$$
$$= \frac{1}{2} \int_Y \chi \mu_0 \big|\mathbb{D}\big(y, \mathbf{W}_0^{(ij)}(\mathbf{y})\big)\big|^2\, dy,$$

$$\int_Y \chi \mu_0 \big|\mathbb{D}\big(y, \mathbf{W}_0^{(ij)}(\mathbf{y})\big)\big|^2\, dy + \int_Y \chi \mathbb{D}\big(y, \mathbf{W}_0^{(ij)}(\mathbf{y})\big) : \mathbb{J}^{(ij)}\, dy = 0, \tag{48}$$

which are obtained by multiplying the first equation in (42) by $\mathbf{W}^{(ij)}$ and integrating by parts in $Y \times (0,t)$ together with multiplying the first equation in (43) by $\mathbf{W}_0^{(ij)}$ and integrating by parts in $Y$. These identities guarantee the estimates

$$\max_{0<t<T} \int_{Y_f} |\mathbb{D}(y, \mathbf{W}^{(ij)}(\mathbf{y},t))|^2\, dy + \int_0^T \int_{Y_s} |\mathbb{D}(y, \mathbf{W}^{(ij)})|^2\, dy dt \leqslant C_0.$$

Considering $(1-\chi(\mathbf{y}))\mathbf{W}^{(ij)}(\mathbf{y},0)$ as a solution in $Y_s$ to the periodic Stokes system

$$\nabla_y \cdot (\lambda_0\,\mathbb{D}(y,\mathbf{W}^{(ij)}) - P^{(ij)}\,\mathbb{I}) = 0, \quad \nabla_y \cdot \mathbf{W}^{(ij)} = 0,$$

coinciding on the boundary $\gamma$ with the function

$$\chi(\mathbf{y})\mathbf{W}^{(ij)}(\mathbf{y},0) \in \mathbf{W}_2^1(Y_s),$$

we obtain (by [23])

$$\int\limits_{Y_s} |\mathbb{D}(y,\mathbf{W}^{(ij)}(\mathbf{y},0))|^2\,dy \leqslant C_0.$$

For $t=0$, this estimate and (42) imply

$$\int\limits_{Y_f} \left|\mathbb{D}\left(y, \frac{\partial \mathbf{W}^{(ij)}}{\partial t}(\mathbf{y},0)\right)\right|^2 dy \leqslant C_0.$$

The obtained estimates imply the unique solvability of (42) and (43).

**4.4. Deduction of the continuity condition on the common boundary $S^0$.** Before passing to our case, return to (16) and (17) in the Introduction. For simplicity, consider the second boundary value problem

$$\nabla \cdot \boldsymbol{W} = 0,$$

$$\mu\Delta\boldsymbol{W} - \nabla P + \varrho\boldsymbol{f} = 0, \tag{49}$$

$$\mathbb{P}(\boldsymbol{w})\cdot\boldsymbol{n} = 0, \quad \boldsymbol{w}\in S, \tag{50}$$

supplemented by the normalization condition

$$\int\limits_\Omega P\,dx = 0,$$

where $\boldsymbol{n}$ is the normal vector to $S$ and the expression $\mathbb{P}(\boldsymbol{w})\cdot\boldsymbol{n}$ is the stress vector of a viscous fluid on the boundary $S$.

How can one understand the boundary condition (50) if it contains the pressure $P$ which is just a square-integrable function in $\mathbb{L}_2(\Omega)$? Equation (49) must simply be written as the integral identity

$$\int\limits_\Omega \mathbb{P}(\boldsymbol{w}) : \mathbb{E}(\boldsymbol{\varphi})\,dx = \int\limits_\Omega \varrho\boldsymbol{f}\cdot\boldsymbol{\varphi}\,dx \tag{51}$$

with an arbitrary smooth function $\boldsymbol{\varphi}$. As we can see, all terms in (51) are well defined, and the normal stress (the trace on the boundary) is absent therein. But if it turns out that $\nabla P \in \mathbb{L}_2(\Omega)$ (which is so in our simple case of $\boldsymbol{f}\in\mathbb{L}_2(\Omega)$) then the boundary condition is understood in the usual sense. In the case of general position, it is simply included in (51) in the above sense (see, for example, the papers by Ladyzhenskaya [23, 24]).

All this is carried over to the case of more difficult problems by the same scheme.

Pass to the two-scale limit as $\varepsilon \to 0$ with test functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\mathbf{x}, t)$ in the integral identity (21). We infer

$$\int\limits_{\Omega_T^0} \left( \mu_0 \left( m_0 \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}\right) + \left\langle \mathbb{D}\left(y, \frac{\partial \mathbf{W}^0}{\partial t}\right) \right\rangle_{Y_f^0} \right) + \lambda_0((1-m_0)\mathbb{D}(x,\mathbf{w}) \right.$$

$$\left. + \langle \mathbb{D}(y, \mathbf{W}^0)\rangle_{Y_s^0}) - p\,\mathbb{I} \right) : \mathbb{D}(x, \boldsymbol{\varphi})\, dxdt - \int\limits_{\Omega_T^0} \hat{\varrho}^0 \mathbf{F} \cdot \boldsymbol{\varphi}\, dxdt$$

$$+ \int\limits_{\Omega_T} \left( \mu_0 \left( m \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}\right) + \left\langle \mathbb{D}\left(y, \frac{\partial \mathbf{W}}{\partial t}\right) \right\rangle_{Y_f} \right) + \lambda_0((1-m)\mathbb{D}(x,\mathbf{w}) \right.$$

$$\left. + \langle \mathbb{D}(y, \mathbf{W})\rangle_{Y_s}) - p\,\mathbb{I} \right) : \mathbb{D}(x, \boldsymbol{\varphi})\, dxdt - \int\limits_{\Omega_T} \hat{\varrho} \mathbf{F} \cdot \boldsymbol{\varphi}\, dxdt = 0. \tag{52}$$

Since

$$\widehat{\mathbb{P}}^0 = -p\,\mathbb{I} + \mu_0 \left( m_0 \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}\right) + \left\langle \mathbb{D}\left(y, \frac{\partial \mathbf{W}^0}{\partial t}\right) \right\rangle_{Y_f^0} \right)$$
$$+ \lambda_0((1-m_0)\mathbb{D}(x,\mathbf{w}) + \langle \mathbb{D}(y, \mathbf{W}^0)\rangle_{Y_s^0}),$$

$$\widehat{\mathbb{P}} = -p\,\mathbb{I} + \mu_0 \left( m \mathbb{D}\left(x, \frac{\partial \mathbf{w}}{\partial t}\right) + \left\langle \mathbb{D}\left(y, \frac{\partial \mathbf{W}}{\partial t}\right) \right\rangle_{Y_f} \right)$$
$$+ \lambda_0((1-m)\mathbb{D}(x,\mathbf{w}) + \langle \mathbb{D}(y, \mathbf{W})\rangle_{Y_s});$$

therefore, (52) takes the form

$$\int\limits_{G_T} (\zeta \widehat{\mathbb{P}}^0 + (1-\zeta)\widehat{\mathbb{P}}) : \mathbb{D}(x, \boldsymbol{\varphi})\, dxdt = \int\limits_{G_T} (\zeta \hat{\varrho}^0 + (1-\zeta)\hat{\varrho}) \mathbf{F} \cdot \boldsymbol{\varphi}\, dxdt,$$

which obviously implies the boundary condition (28).

**4.5. Properties of the tensors.** Let $\zeta = (\zeta_{ij})$ and $\eta = (\eta_{ij})$ be arbitrary symmetric matrices and

$$\boldsymbol{Y}_\zeta = \sum_{i,j=1}^{3} \mathbf{W}_0^{(ij)} \zeta_{ij}, \quad \boldsymbol{Y}_\eta = \sum_{i,j=1}^{3} \mathbf{W}_0^{(ij)} \eta_{ij}.$$

By definition,

$$(\mathfrak{N}_1 : \zeta) : \eta = \mu_0 m \zeta : \eta + \mu_0^2 \langle \mathbb{D}(y, \boldsymbol{Y}_\zeta)\rangle_{Y_f} : \eta.$$

Now, use the equalities

$$\int\limits_{Y} \chi \mu_0 \mathbb{D}\big(y, \mathbf{W}_0^{(ij)}\big) : \mathbb{D}\big(y, \mathbf{W}_0^{(kl)}\big)\, dy + \int\limits_{Y} \chi \mathbb{D}\big(y, \mathbf{W}_0^{(kl)}\big) : \mathbb{J}^{(ij)}\, dy = 0$$

for $i, j = 1, 2, 3$, which are simply consequences of (48), and come to the expression

$$\mu_0 \langle \mathbb{D}(y, \boldsymbol{Y}_\eta)\rangle_{Y_f} : \zeta + \mu_0^2 \langle \mathbb{D}(y, \boldsymbol{Y}_\zeta) : \mathbb{D}(y, \boldsymbol{Y}_\eta)\rangle_{Y_f} = 0. \tag{53}$$

Thus,

$$(\mathfrak{N}_1 : \zeta) : \eta = \mu_0 \langle (\mu_0 \mathbb{D}(y, \boldsymbol{Y}_\eta) + \zeta) : (\mu_0 \mathbb{D}(y, \boldsymbol{Y}_\zeta) + \eta)\rangle_{Y_f},$$

which implies the positive definiteness of $\mathfrak{N}_1$. The proof is similar for $\mathfrak{N}_1^0$.

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