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On a digital approximation for pseudo-differential operators

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1 Introduction

We introduce a concept of a discrete pseudo-differential operator using general ideas of the theory and would like to show correlations between continuous and discrete cases.

2 Digital pseudo-differential operators

2.1 Digital Fourier transform

Given function u_d of a discrete variable $\tilde{x} \in h\mathbf{Z}^m, h > 0$, we define its discrete Fourier transform by the series

$$(F_d u_d)(\xi) \equiv \widetilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbf{Z}^m} e^{i\tilde{x}\cdot\xi} u(\tilde{x}) h^m, \ \xi \in \hbar\mathbf{T}^m,$$

where $\mathbf{T}^m = [-\pi, \pi]^m, \hbar = (2\pi h)^{-1}$, partial sums are taken over cubes

$$Q_N = \{\tilde{x} \in \mathbf{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \max_{1 \le k \le m} |\tilde{x}_k| \le N\}.$$

2.2 h-operators and $\hbar$-symbols

Let $D \subset \mathbf{R}^m$ be a domain, and $D_d = D \cap h\mathbf{Z}^m$.

We consider the following operators

$$(A_d u_d)(\tilde{x}) = \int\limits_{\hbar\mathbf{T}^m} \sum_{\tilde{y} \in D_d} e^{i(\tilde{y}-\tilde{x})\cdot\xi} \widetilde{A}_d(\xi) \tilde{u}_d(\xi) d\xi, \ \tilde{x} \in hD_d, \tag{1}$$

and the function $\tilde{A}_d(\xi), \xi \in \hbar\mathbf{T}^m$ is called a symbol of the operator A_d.

Also the function

$$A_d(\tilde{x}) = \int\limits_{\hbar\mathbf{T}^m} e^{i\tilde{x}\cdot\xi} \tilde{A}_d(\xi) d\xi.$$

is called a kernel of the operator A_d.

Definition 2.1 The symbol $\widetilde{A}_d(\xi)$ is called an elliptic symbol of the operator A_d if $\operatorname{ess}\inf_{\xi \in \hbar\mathbf{T}^m} |\tilde{A}_d(\xi)| > 0$.

Example 2.2 The digital Laplacian is the following

$$(\Delta_d u_d)(\tilde{x}) = h^{-2} \sum_{k=1}^{m} (u_d(x_1, \cdots, x_k + 2h, \cdots, x_m) - 2u_d(x_1, \cdots, x_k + h, \cdots, x_m) + u_d(x_1, \cdots, x_k, \cdots, x_m)),$$

and its symbol is the function

$$\tilde{\Delta}_d(\xi) = h^{-2} \sum_{k=1}^{m} (e^{ih\xi_k} - 1)^2.$$

Example 2.3 The digital Calderon–Zygmund operator is defined as follows [4]

$$(K_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in hD_d} K_d(\tilde{x} - \tilde{y}) u_d(\tilde{y}) h^m, \ \tilde{y} \in hD_d,$$

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3 A comparison between discrete and continual cases

3.1 An approximation rate

Let P_h be a projection $\mathbf{R}^m \to \mathbf{Z}^m$ so that a function u defined on $\mathbf{R}^m$ corresponds to a function u_d of a discrete variable defined on $h\mathbf{Z}^m$, $P_h u = u_d$. If we consider the equation

$$(Au)(x) = v(x),\ x \in D, \tag{2}$$

where A is a classical pseudo-differential operator with the symbol $\tilde{A}(\xi)$ [1–3] of the form

$$(Au)(x) = \int\limits_{D}\int\limits_{\mathbf{R}^m} e^{i(x-y)\cdot\xi}\tilde{A}(\xi)u(y)dyd\xi,$$

which acts in certain functional spaces $X \to Y$, for example Sobolev–Slobodetskii spaces [3]. We say that an element $u \in X$ is an admissible element if $P_h u$ is defined.

Definition 3.1 An approximation rate for operators A and A_d on an admissible element $u \in X$ is called the following norm

$$\mu_h(A, A_d, u) = ||(A_d P_h - P_h A)u||_{X_h},$$

where X_h is so-called digital realization of the space X so that the operator $A_d : X_h \to Y_h$ is a linear bounded operator.

One of main problems is the following. How we can choose the operator A_d to obtain a good approximation rate for the operator A? We need to fix a domain D and spaces X, Y.

Theorem 3.2 *Let D be a domain with a Lipschitz boundary and $X = Y = L_2(D), X_h = Y_h = L_2(D_d)$. If $\tilde{A}(\xi)$ is a smooth bounded function on $\mathbf{R}^m$ and*

$$A_d(\tilde{x}) = \int\limits_{\mathbf{R}^m} e^{i\tilde{x}\cdot\xi}\tilde{A}(\xi)d\xi$$

then $\mu_h(A, A_d, u) \le c_u h$ for arbitrary smooth function $u \in L_2(D), c_u$ is a constant.

3.2 Digital solution and comparison

Definition 3.3 A digital solution for the equation (2) is called a solution of the equation

$$(A_d u_d)(\tilde{x}) = (P_h v)(\tilde{x}),\ \tilde{x} \in D_d, \tag{3}$$

if it exists.

Remark 3.4 It is not evidently that a digital solution always exists. Thus, second of main problems is obtaining a solvability for the equation (3) in the space X_h at least for small h from the solvability of the equation (2) in the space X. For this purpose we need to study a solvability of discrete equations, some steps in this direction were done in [6,7] for special conical domains D and for the whole space $\mathbf{R}^m$ and the half-space $\mathbf{R}^m_+$ [4].

Theorem 3.5 *Let D be $\mathbf{R}^m$ or $\mathbf{R}^m_+$, the conditions of above theorem hold, A be an elliptic invertible operator, u be a solution of the equation* (2) *with a smooth right-hand side v, u_d be a solution of the equation* (3). *Then*

$$||P_h u - u_d||_{X_h} \le ch.$$

4 Conclusion

In authors' opinion these considerations will be useful for studying certain applied problems [5] because such operators and equation are very typical for these problems.

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References

[1] Taylor, M.E.: Pseudodifferential Operators. Princeton Univ. Press, Princeton (1981)
[2] Treves, F.: Introduction to Pseudodifferential Operators and Fourier Integral Operators. Springer, New York (1980)
[3] Eskin, G.: Boundary Value Problems for Elliptic Pseudodifferential Equations. AMS, Providence (1981)
[4] Vasilyev, A.V., Vasilyev, V.B.: Approximation rate and invertibility for some singular integral operators. Proc. Appl. Math. Mech. 13, 373–374 (2013)
[5] Oppenheim, A.V., Schafer, R.W.: Digital Signal Processing. Prentice Hall, Englewood Cliffs, NJ (1975)
[6] Vasilyev, V.B.: Discrete equations and periodic wave factorization. AIP Conf. Proc. 1759, 5 pp. (2016)
[7] Vasilyev, V.B.: Discrete operators in canonical domains. WSEAS Trans. Math. 16, 197–201 (2017)

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