Z. Geomorph. N. F.	29	3	257–263	Berlin · Stuttgart	September 1985

Optimal control over geomorphological systems

A. M. Trofimov and V. M. Moskovkin, Kazan, USSR

with 3 figures

Zusammenfassung. Ehe man küstenmorphologische Prozeßreaktionssysteme künstlich beeinflussen kann, ist es notwendig zu wissen, wieviel Zeit für den Übergang eines solchen Systems von einem „Anfangsstadium" zu einem dynamischen Gleichgewichtszustand erforderlich ist. Diese Fage wird hier für ein aus Kliff und Strand bestehendes System untersucht; das dabei benutzte theoretische Modell beruht auf einer normalen Differentialgleichung, welche die Massenbilanz am Fuß des Kliffs beschreibt. Dabei dient die Zu- oder Wegführung von Schutt als Kontrollfaktor.

Die hier vorgestellte Methode läßt sich in der Planung von Küstenschutzmaßnahmen ohne Schutzmauern oder andere Kliffschutzkonstruktionen anwenden; sie ist besonders dort geeignet, wo sich das System schließlich durch das Einspielen auf ein dynamisches Gleichgewichtsregime selbst regulieren soll.

Abstract. In order to control coastal geomorphological process-response systems it is necessary to know how much time is required for such a system to change from an „initial state" to a state of dynamic equilibrium. This question is investigated here for a cliff-beach system by means of a theoretical model based on an ordinary differential equation that describes the mass balance at the foot of the cliff. The addition or removal of riprap serves as a control factor.

The method presented can be used for the planning of shore protection measures without seawalls or other cliff-protecting structures; it is particularly appropriate in cases where the system is eventually to control itself by adjustment to a dynamic equilibrium regime.

Résumé. Dans le but de contrôler des systèmes de réponses de processus de géomorphologie côtière, il est nécessaire de savoir combien de temps prend un système pour passer d'un ètat initial à un état d'équilibre dynamique. Cette question est étudiée ici sur un système plage-falaise au moyen d'un modèle théorique, basé sur une équation différentielle ordinaire qui décrit le bilan sédimentaire au pied de la falaise. L'accumulation ou l'enlèvement de "riprap" sert de facteur de contrôle. La méthode présentée peut être utilisée pour planifier les mesures de protection de la plage sans construction de digues ou d'autres structures de protection de falaises; elle est particulièrement appropriée dans des cas où le système est éventuellement auto contrôlé par ajustement à un régime d'équilibre dynamique.

0372-8854/85/0029-0257 $ 1.75

The mathematical theory of optimal control (PONTRYAGIN et al. 1961) very likely will be widely applied to the control of geomorphological systems in the future. The possibility of using this theory for problems of the control of exogenic processes has been discussed by the authors at the 24th International Geographical Congress (TROFIMOV & MOSKOVKIN 1980). This idea requires a definite level of formalization: it is necessary to construct models of the dynamics systems of these processes with due regard to a control factor. Under conditions of more or less constant inputs, the evolution of geosystems progresses toward a state of dynamic equilibrium because of negative feedbacks. For the purpose of control of such systems it is important to determine the minimum time they need to change from an initial state to a state of dynamic equilibrium (TROFIMOV & MOSKOVKIN 1980, 1983). This task is particularly relevant for rapidly evolving systems that are frequently subjected to control measures, such as coastal shore processes, stream processes and intensive slope processes.

This paper deals with the question of optimal control in the case of coastal cliff-beach systems; it has direct relevance to the intensity of the use of such coastal zones by man.

The model is based on the balance equation of the material present at the foot of the cliff (ESIN 1980)

$$\frac{dW}{dt} = af(W)H - kW, \tag{1}$$

where W is the volume of clastic beach material per unit of length (m^3/m); f(W) is the rate of cliff retreat as the function of the volume of material (m/year); H is the height of the cliff (m); a is the part of beach-forming material that is derived from bedrock; k is the abrading coefficient (hardness index for beach pebbles, 1/year); t is time (year).

Now we introduce the control factor ξ (t) in the right part of equation (1). This factor expresses (according to its sign) the intensity of addition of riprap material to the beach or the intensity of removal of such material from the beach. Differentiating equation (1) on time with the inclusion of the control factor ξ (t) we obtain the following system of equations (secondorder dynamic system)

$$\begin{aligned} \frac{dV}{dt} &= a\frac{df}{dW} VH - kW + u(t). \\ \frac{dW}{dt} &= V, \end{aligned} \tag{2}$$

where u(t) is the control factor connected with ξ (t) by $u(t) = d\,\xi(t)/dt$, $/u/ \leq \beta$ (some positive constant, $m^2/year^2$). The condition of restriction of the control factor is due to physical consideration.

The problem of optimal time control can now be expressed as a transfer of the dynamic system (2) from its initial state (W_0,V_0) to the state ($W_{St.}$, O) in a minimum of time. Here $W_{St.}$ is the stationary point of equation (1). It can be found from the solution of af(W)K–kW = O. It is possible to develop some non-linear equation as the function f(W) for easily-destroyed and for stable rock (ESIN 1980). We use instead the linear function (TROFIMOV & MOSKOVKIN 1983)

(3) $f(W) = \gamma(W_{max} - W)$,

where γ is some coefficient (1/m year); W_{max} is the maximum volume of material at which there is no abrasion (or some other value W which is needed for the linear approximation of the non-linear function f(W).

In order to tranform this problem of optimal control to the classical case, we must make a substitution of the variables in the system equations (2): $W' = (W-W_{St.})/\beta$, $V' = V/\beta$, $u' = u/\beta$. With the use of equation (3) $df/dW = -\gamma$, the classical problem is then expressed as

$$\frac{dV'}{dt} = -AV' + u',$$
(4)
$$\frac{dW'}{dt} = V', /u'/ \leq 1,$$

where $A = aH\gamma + k$.

The problem of optimal time control is now reformulated as the transform of the dynamic system (4) from the state $((W_0-W_{St.})/\beta, V_0/\beta)$ to the state (0.0). The last one is the coordinate set of the phass surface (W', V').

In accordance with PONTRYAGIN's maximum principle there exists as the only synthesis of optimal controls that which can be constructed on the basis of the solution of system equations (4) for $u' = 1$ and $u' = -1$ (u' has no more than two intervals of constancy). "Synthesis of optimal control" refers to the point trajectory on a phase surface which leads from a set state to the origin. It occurs in the minimum of time. We give here only final results for the region I (fig. 1), for initial variables when $V_0 = O$. Results for regions I and IV are analogues.

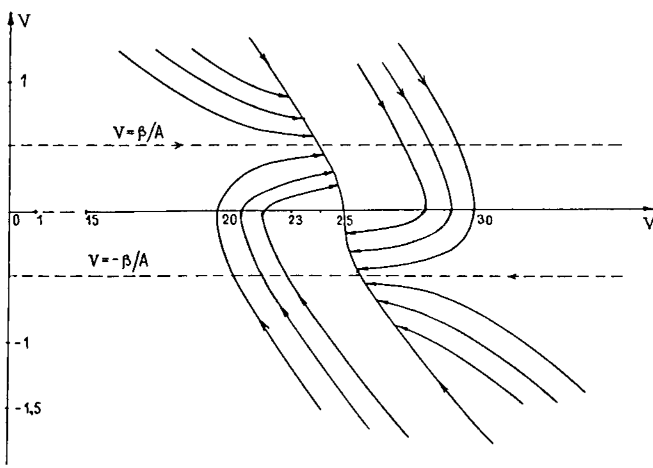

Fig. 1. A complete calculation synthesis of optimal control.

1. The equation of the point trajectory that leads from the initial state to the turning point B (fig. 1) is the solution of dynamic system (4) when u′ = –1

(5) $$W = W_0 + \frac{\beta}{A^2} \ln\left(1 + \frac{AV}{\beta}\right) - \frac{V}{A},$$

where $1 + \frac{AV}{\beta} \neq 0$.

2. The equation of the point trajectory that leads from the turning point to the final (stationary) state is the solution of dynamic system (4) when u′ = 1

(6) $$W = W_{St.} - \frac{\beta}{A^2} \ln\left(1 - \frac{AV}{\beta}\right) - \frac{V}{A},$$

where $1 - \frac{AV}{\beta} \neq 0$, $W^{St.} = \left(\frac{A-k}{A}\right) W^{max}$.

3. The coordinates of the turning point B (W_B, V_B) are

(7) $$W_B = -\frac{\beta}{A^2} \ln\left(1 - \frac{A}{\beta} V_B\right) - \frac{V_B}{A} + W_{St.},$$
$$V_B = -\frac{\beta}{A} (1 - \exp[-A^2(W_0 - W_{St.})/\beta])^{1/2}.$$

4. The optimal time of system transform from point (W_0, O) to the point B along the curve (5) is

(8) $$t_{W_{o},B} = -\frac{1}{A} \ln (1 - (1 - \exp[-A^2(W_o - W_{St.})/\beta])^{1/2}).$$

5. The optimal time of system transform from the point B to the stationary state along the curve (6) is

(9) $$t_{B,O} = -\frac{1}{A} \ln (1 + (1 - \exp[-A^2(W_o - W_{St.})/\beta])^{1/2}).$$

6. The general optimal time of system transform from the initial to the final point is

(10) $$T = t_{W_o,O} = t_{W_o,B} + t_{B,O}.$$

With the help of Loptile's law and when $W_o > W_{St.}$ we can find a limit

(11) $$\lim T = 2\,[(W_O - W_{St.})/\beta]^{1/2}.$$

It has been shown that the asymptotes $V = \pm\, \beta/A$ are the phase trajectories of the complete synthesis of optimal controls, i.e. the dynamic system that is studied here is controlled in the entire rectangle $O < W < W_{max}$, $/V/ <$ const. of phase surface (W,V). The direction of motion by these asymptotes is shown in fig. 1. For instance, the general optimal time of motion from the inital point (which lies on a straight line $V = -\beta/A$) to the stationary point can be written as follows:

(12) $$T = \frac{1}{A} (2 \ln 2 - 1) + A\,(W_O + W_{St.})/\beta.$$

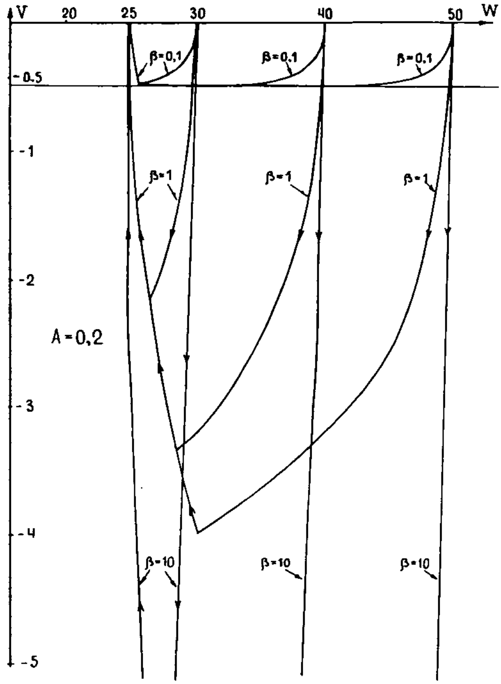

Fig. 2. The synthesis of optimal control for different values of β and W_o.

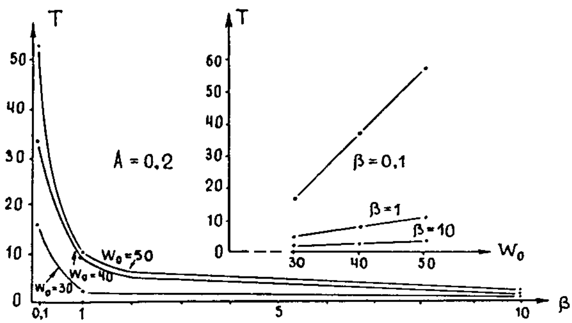

Fig. 3. The optimal time dependence of T from β and W_o.

For application to practical problems, which correspond to the geological conditions of the Novorosiysk region (ESIN 1980), experimental calculations according to expressions (5–10) have been made with the help of a computer. The results of one of these series are shown in figures 2 and 3. They were obtained with the following parameter values: $A = 0.2$ year^{-1} ($k = 0.1$ year^{-1}); $a = 0.3$; $H = 100$m; $\gamma = 1/300$ (m year)$^{-1}$; $W_{max} = 50$ m^2, $W_{St.} = 25$ m^2, $W_o = 30; 40; 50$ m^2; $\beta = 0.1; 10$ m^2/year^{-2}.

The calculations have shown that the most effective control region exists when $A = 0.2$ year^{-1}, $W_{St.} = 25$ m$^2 < W_o < 50$ m^2 occur with the range of the control parameter $u = \beta$ between 1 to 10 m^2/year^{-2}. In that case the optimal time T necessary for a transform of the shore system to a stationary state does not exceed 20 years.

For A, W_o = const., the ratio $t_{W_o,\beta}/t_{B,O}$ tends towards the value 1 with increasing β, and for A, β = const., this ratio increases with increasing W_o (the increase is large if β is small).

The optimal time T is a linear function of the initial volume of beach material W_o and increases with decreasing β; the functional relationship $T = f(\beta)$ is well approximated by hyperbolas. These relationships are shown in the diagrams of figure 3.

Figure 1 demonstrates the complete synthesis of optimal controls for $A = 0.2$ year^{-1}, $\beta = 0.1$ m^2/year^2 and $W_{max} = 50$ m^2. This synthesis appears to be sensitive to the change of A.

The physical sense of the initial control parameter ξ (t) is this: for a given case ($V_o = O$, $W_o > W_{St.}$) in a natural shore system it is necessary to remove the material artificially in the course of time $t_{W_o,B}$ (8) and then to slow down this process with the time $t_{B,O}$ (9), whereby the intensity changes in proportion to this time so that $\xi(t) = -\beta t$ and $\xi(t) = \beta(t-T)$.

It is important to note that the problem of control in a minimum of time is equivalent to the problem of optimal control with functional minimization $\int_o^T f(W)dt$, as the function $s(t) = \int_o^t f(W)dt$, $s(o) = O$ increases monotonously (the distance over which a cliff retreats, s(t), can only increase with time).

The approach considered here can be applied to the planning of shore protection measures that rely on the formation of beach materials rather than on the strengthening of cliffs by the construction of protective walls (in this latter case, the negative feedbacks are automatically removed). This approach is especially appropriate in cases where the control consists of the adjustment of undisturbed natural systems to a dynamic equilibrium regime.

References

ESIN, N. B. (1980): O roli oblomochnogo materiala v abrasionnom processe. – Oceanologia 20, N 1. [in Russian]

PONTRYAGIN, L. S., V. G. BOLTYANSKY, P. V. GAMKRELIDZE & E. PH. MITCHENKO (1961): Matematicheskaj teoria optimalnich protzessov. M. – Nauka. [in Russian]

TROFIMOV, A. M. & V. M. MOSKOVKIN (1980): Theoretical Preconditions of the Control over Exogenic Processes. – 24th IGC Main Session. Abstracts, vol. 1; Tokyo.

– (1983): Matematicheskae modelirovanie v geomorphologi sklonov. Kazan university, Kazan. [in Russian]

The addresses of the authors: Prof. Dr. A. M. TROFIMOV, Head of the Economical – Geography Department of Kazan State University, Lenin str 18, Kazan, USSR; Dr. V. M. MOSKOVKIN, Hydraulic Research Laboratory, Institute of Water Conservation, Bakulina, 6 Kharkov, USSR.