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Abstract. Given article presents a generalized equation for calculating the average coordination number from the density of a random sphere packing, supplemented by a dependence on the threshold value of the interparticle distance in two- and three-dimensional spaces. It is shown that the calculation of the average coordination numbers according to the proposed equation gives an unambiguous correspondence between the simulated, calculated and experimental data for threshold values of more than 1.02 particle diameters. An explanation of the weak dependence of the average coordinate number on the packing density for small threshold values of the interparticle distance is given in this work.

1. Introduction

Random sphere packing has been a topic of considerable attention for many decades, mainly because of their role in understanding the nature of dispersed materials [1]. A number of physical parameters, such as packing density (porosity) and coordination number characterize the structure of sphere packing. The coordination number is one of the most important parameters used to describe the spatial structure of random sphere packing. This concept can be defined as the number of nearest particles in a close-packed system of particles located around a selected particle allowing us to estimate the degree of proximity of particles to each other. It is customary to distinguish several varieties of the coordination number. In the case of direct contact between particles, the coordination number is usually considered as the so-called “contact number” or “Newtonian number”. An additional account of nearby, but not contacting, particles belonging to the second coordination sphere allows us to speak of a “local coordination number”. So, for example, the local coordination number of the body-centered packing can be considered as 8+(6), that is, 8 particles belonging to the first and 6 particles to the second coordination sphere, which gives a numerical value equal to: $Z = 10.125$ [2]. The averaged statistical calculation of the total number of neighboring particles near other packing particles is considered to be the “average coordination number”, and the allocation of a region with particles included in the selected particle is the “effective coordination number” of the spherical packing. In this paper, we restrict ourselves to considering only such a concept as the average coordination number.

The information on the average coordination number for various types of random sphere packings is rather well reflected in a number of earlier reviews [3, 4]. In addition, in the scientific literature there are many attempts to relate the coordination number to the packing density [2, 5-14]. Therefore, the authors of a number of presented works tried to find the relationship between the average coordination number and packing density, which was determined by selecting empirical dependencies taking into account the intervals of their changes (table 1). As a basis, they used the calculated data for regu-

lar packings of monodisperse spheres, as well as experimental data for determining the parameters of random sphere packings.

Table 1. The calculated values of the coordination numbers obtained on the basis of the formulas for the dependence of the coordination number *Z* on the density of the 3-D sphere packing η.

Author	Source	Formula	Packing type						
			Cubic simple	Hexagonal dense	Volume-centered cubic	Tetragonal	Cubic dense	Loose	Close
Smith W *et al*. (1929)	[5]	$26.49-10.73/\eta$	6.0	8.74	10.71	11.12	12	8.9	9.72
Kolmogorov A N (1937)	[6]	$-8\ln(1-\eta)$	5.93	7.42	9.12	9.58	10.79	7.53	8.17
Rumpf H (1958)	[7]	$3.1/(1-\eta)$	6.51	7.84	9.69	10.27	11.95	7.95	8.61
Field (1963)	[8]	$12/(2-\eta)$	8.13	8.6	9.09	9.22	9.53	8.63	8.82
Meissner H *et al.* (1964)	[9]	$2\exp(2.4\eta)$	7.03	8.54	10.23	10.68	11.83	8.65	9.29
Ridgway K and Tarbuck K (1967)	[10]	$\eta=\frac{Z}{8.38}-\frac{Z^2}{232}-0.07$	6.54	7.96	9.71	10.25	12.1	8.07	8.71
Nakagaki M and Sunada H (1968)	[11]	$1.61(1-\eta)^{-1.48}$	4.82	6.36	8.7	9.48	11.85	6.49	7.3
Haughey D and Beveridge G (1969)	[12]	$22.47-39.39(1-\eta)$	3.7	6.9	9.87	10.58	12.25	7.11	8.29
Gotoh K (1978)	[13]	$20.7\eta-4.35$	6.49	8.17	9.73	10.1	10.98	8.28	8.9
Timofeev V N *et al.* (2011)	[14]	13.4η	7.02	8.1	9.11	9.35	9.92	8.17	8.58
Bondarev V G *et al.* (2015)	[2]	$6(6\eta/\pi)^2$	6.0	8.0	10.13	10.67	12.0	8.14	8.96
Coordination number, *Z* (geometrical and experimental data)			6.0	8.0	8+(6)	10+(4)	12.0	7.5	8.9
Packing density, η			0.524	0.605	0.68	0.698	0.74	0.58	0.64

[a] The intervals for changing the parameters of sphere packings are not presented in the table.
[b] Data on experimental values of random spherical packing are taken from [15].

An analysis of the formulas for the dependence of the coordination number *Z* on the packing density η presented in Table 1 shows that most of them were obtained empirically. One can see a significant difference between the calculated values obtained from the proposed formulas and the geometric data for regular packings. There is also a large scatter of the obtained values for random packages. To be fair, it should be noted that their application for practical purposes has strict boundary conditions for the ranges of the considered package densities. The exception is the equations for the coordination number, theoretically determined by A N Kolmogorov in 1937, V G Bondarev *et al.* in 2015. As can be seen from the calculated data on the coordination numbers, the first equation can be relatively well applied for random packings, and the second equation, which allows one to uniquely determine the analytical dependence of the coordination number $Z_e$on the packing density η for regular spherical packings, located both in two- and three-dimensional spaces

$$Z_e(n)=n_m[(\frac{2n}{\pi}\eta(n))^2-1]+2n\,, \qquad (1)$$

where n is the dimension of the space under consideration; n_m is the largest dimension of the space ($n_m=6$). It would also be useful to clarify the possibility of applying equation (1) not only for regu-

lar, but also for random packings. For this reason, the purpose of the research presented in this article is to build a mathematical model of the coordination number when considering random sphere packings taking into account the packing density and dimension of the space under consideration.

2. Problem definition

The average coordination number of random sphere packing can be estimated by applying four different methods [15]:

– the threshold value method based on the choice of a certain radial distance *L* between the centers of neighboring particles, on which the "belonging" of a particle to the number of neighbors of the selected particle is determined. It is customary to consider strictly defined discrete limits of distance *L*, which are fixed on values in the range $1.02 \div 1.1\sigma$ (σ is the particle diameter);

– the method of deconvolution, which makes it possible to distinguish between particles that are in direct contact with each other and particles that are fairly close but do not have contacts with each other statistically;

– a geometric method that uses information obtained from tomography or computer simulation to determine the distances between adjacent particles in a package rigorously, based on which the coordination number is estimated. This method is also implemented by introducing a new concept – the coordination coefficient [16], which allows to determine the coordination numbers based on the consideration of the relative positions of particles relative to the central (base) particle;

– the analytical method used to calculate coordination numbers from known values of packing density (porosity) or interparticle distances by using some theoretical or empirical equation that determines the relationship of these parameters.

Now, after we have decided upon methods used we move to the goal setting. Let's consider a random sphere packing located in some limited region of a space of dimension n. When conducting the study, we take equation (1) as a function of the dependence of the coordination number Z_e on the density of the sphere packing η obtained for the case of regular packing. We choose the threshold value method as the basic approach for conducting the study. In the case of 2-D packing, let\s conduct the computational experiment to obtain the characteristics of a random sphere packing. In addition, we include the experimental data on coordination numbers for the selected threshold values of the limiting "contact" distance between particles, and the corresponding packing densities obtained earlier by T. Aste et al. [15] for 3-D packages. Based on the information presented, the statement of the problem can be formulated as follows. Let the source data be pairs of values (Z_i, η_i), $i = \overline{1, m_j}$ (m_j is the number of measurements performed for the *j*-th threshold value of the interparticle distance), which are the results of measurements obtained by computer simulation for 2-D and experimental for 3-D packages. It is necessary to determine the equation for random spherical packing on the basis of the modelling data, which allows to estimate the coordination number from the values of packing density, depending on the value of threshold values, and taking into account the dimensionality of space.

The main strategy for developing a mathematical model of the coordination number of random sphere packing will be to take into account the threshold values of interparticle distances when considering analytical equation (1) for packing in spaces of different dimensions. To obtain a generalized equation of the average coordination number, it is also necessary to find those laws that will allow transforming the constants obtained in the final equation into a form that takes into account the dimensions of the spaces considered.

3. Materials and methods

To study the coordination numbers of random sphere packing, it was decided to use, in addition to mathematical and computer modeling, statistical analysis methods as well. Therefore, we will simulate 2-D sphere packing by the method of layer-by-layer packing proposed in work [17] and based on a dynamic approach. Using this approach, a leading front of the installation of added particles moving in the chosen direction [18] is used to form the packing. We describe this approach in more detail. Let us consider some configuration of *N* installed particles with a certain frontal surface. Further, let's take

some set $g = (p_1, p_2, \ldots, p_m)$, which is a set of positions with coordinates (x_i,y_i), near this surface, in which the centers of new particles can be located. Finally, let us choose two adjacent positions, the distance between which is less than the particle diameter σ. It is required to find the position that will be occupied by the particle, in this case another position should subsequently be excluded from the further consideration. This problem is solved by applying the concept of the probability of installing a particle in a selected position. So, we assume that if the probability P of occupying separate position is 0.5, then this gives equally possible variants of placing the particle in one of both positions. We will also assume that with probability $P = 1.0$, the position with the smallest vertical y-coordinate will be taken.

Let's consider the implementation of the growth algorithm developed on the basis of the layer-by-layer packing method. Compared with the known advantages of the proposed method it can be listed as follows:

1. The generated particle systems are presented in the form of random packings of particles having geometric anisotropy caused by unidirectional weak force.
2. Particles in the installation area are placed without overlays.
3. The installation area is limited by "transparent" walls, which allows the boundary particles to be partially located outside this area.

The general algorithm of computer simulation in this case splits into three subtasks:

1. The construction of the base (initial) layer in the strip is the task of determining the coordinates of the centers of the particles of the layer.
2. Determination of the mechanism for installing particles is the task of choosing options for the location of particles, depending on the value of the probability of their participation in the construction of the package.
3. Analysis of the position of the boundary particles is the task of controlling the placement of particles near the boundaries of the installation area.

The process of forming random packing can be described as a set of the following steps. At the first stage, within the lower boundary of the installation area, there is a base layer obtained from a close-packed regular chain of particles having random displacements Δy in y-coordinate in the range: $0 \le \Delta y \le \sigma\sqrt{2}$ (σ is the diameter of the particle). The size of the displacement is determined so that the distance between the particles does not exceed a value that would lead to overlap of the adjacent neighboring particles. The array of particles of the base layer forms the lower boundary of the random packing (Figure 1).

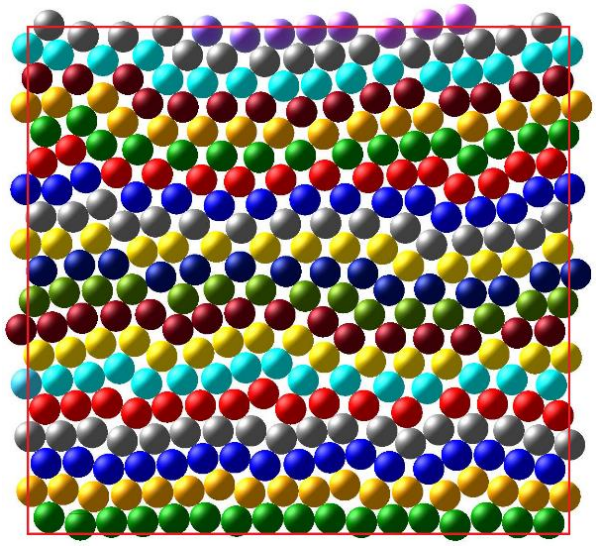

Figure 1. An example of the spatial structure of a random 2-D sphere package in the particle installation area (highlighted by a red line) size 16×16σ.

As soon as the base layer is formed, at the second stage the particles of the new layer are installed by performing a number of steps.

Step 1. A particle with the smallest *y*-coordinate is selected and a set of particles is formed, including the selected particle together with its neighbors.

Step 2. Near this particle, the coordinates of the possible positions are determined, in which a new particle can be installed. Positions for installing new particles are determined by geometric calculations and based on the considered condition of random packing. The choice of a specific position of a new particle is made on a competitive basis, taking into account the probability of its possible installation in this position.

Step 3. A check is made for the possibility of overlapping the new particle with other previously installed particles, and if there is such overlap, then the coordinates of the center of the new particle are recalculated, taking into account its contact with another particle, in order to remove the detected overlap.

Step 4. If the generated particle does not go beyond the boundaries of the installation area, as well as if there are no intersections with previously placed particles, then the coordinates of its center are fixed, and the generated particle is "included" in the package structure. Its close neighbors are determined and the particle itself is also included in the composition of the neighbors of the particles surrounding it. Exit particle position beyond the boundaries of the installation area gives the right to exclude this particle from the system. However, when installing subsequent particles, such a particle can also take part as a boundary particle, even without taking it into account in the package structure.

Step 5. Steps 1-4 are repeated many times until the next new particle is outside the upper boundary of the installation area. If the installed particle goes beyond the upper boundary of the installation area, the rules for stopping the process of forming random packaging are determined.

To conduct computational experiments to determine the characteristics of random packing, using the considered algorithm, we used the PackLD software package [17], which was developed on the basis of the layer-by-layer packing method chosen by us. The use of an improved modification of the complex allowed us to carry out a number of computational experiments to determine the characteristics of random packing. To this end, 60 different packings were generated in six different structural states. Packings were generated in the installation area of particles, size $50\times50\sigma$, which made it possible to study the volume of the aggregate of the order of 3000 or more particles, as well as to obtain stable values of structural characteristics. According to the results of computer calculations, the tables of the behavior of the integral packing density and the average coordination number were compiled depending on the probability values *P* of the choice of positions of the installed particles (see Appendix). The output also created a visualization of the spatial structure of random packing.

4. Results and discussion

4.1. 2-D packing

Let's consider a random ball packing if it is placed in a two-dimensional space. Such packing can be considered as a close-packed system of hard particles. Equation (1) of the coordination number of sphere packing for a 2-D space can be represented as

$$Z_e(2) = n_m(\frac{4}{\pi}\eta(2))^2 - 2\,. \tag{2}$$

However, this equation does not take into account the influence of threshold values of interparticle distances, therefore, to determine the dependence of the coordination number on the value of threshold values, a computer simulation of a random 2-D ball packing was performed.

The results of computer experiments obtained using the PackLD software package [16] made it possible to determine the range of change in coordination numbers with a change in packing densities (table 2). So, for free packing (RLP) we have $\eta_{RLP} = 0.815\pm0.001$, and for bound packing (RCP): $\eta_{RCP} = 0.832\pm0.002$.

Table 2. 2-D sphere packing densities and coordination numbers at various threshold values of the interparticle distance.

Packing density, η	Coordination number, Z			
	L=1.02σ	L=1.05σ	L=1.10σ	Theory
0.815±0.001	4.07±0.01	4.17±0.02	4.34±0.02	4.46±0.01
0.816±0.002	4.07±0.01	4.18±0.02	4.37±0.01	4.48±0.01
0.817±0.003	4.08±0.01	4.19±0.02	4.38±0.03	4.49±0.04
0.821±0.003	4.11±0.02	4.24±0.05	4.42±0.07	4.55±0.05
0.823±0.002	4.12±0.02	4.27±0.02	4.51±0.05	4.59±0.04
0.832±0.002	4.18±0.04	4.39±0.06	4.64±0.09	4.73±0.03

A comparison of the results of computer experiments performed with the calculated data obtained in accordance with equation (2) shows that the computational experiment gives somewhat lower values of the coordination number. The reason for this situation is the ambiguous correspondence between the average coordination number and packing density, which occurs in the case of computer simulation. To eliminate this discrepancy, we took into account the threshold value, which will allow us to consider experimental and calculated data based on a single equation. For this purpose, equation (2) for the coordination number was supplemented by a dependence on the threshold distance L, which can be estimated by constructing the equation: $Z(2) = Z_e(2) - \Delta Z$, where ΔZ is a function that takes into account the influence of the threshold value of the distance L between particles.

For further analysis, we establish the relationship between ΔZ and L. Based on the assumption of the linear nature of this function, we write

$$\Delta Z = A(2)\,(L_m(2) - L)/\sigma\,, \tag{3}$$

where $A(2)$ is a constant; $L_m(2)$ is the limit distance between particles at which an unambiguous correspondence is established between the coordination number and the 2-D packing density. The calculation shows that the constant in formula (3) has a value: $A(2) = 4$, and the limiting distance between particles: $L_m(2) = 1.131\sigma$. It follows that, the general formula for determining the coordination number of two-dimensional packing will have the following form

$$Z(2) = n_m\left(\frac{4}{\pi}\eta(2)\right)^2 - 2 - A(2)\,(L_m(2) - L)/\sigma\,. \tag{4}$$

With the help of formula (4), it is possible to calculate the coordination number for any two-dimensional sphere packing. Moreover, even despite the quadratic dependence on the packing density, the calculation results can be approximated by linear regression equations, due to the small range of variation in the packing density. Figure 2 presents the results of a theoretical study and computational experiments to assess the dependence of the coordination number on the packing density, taking into account the value of the threshold value L.

The analysis of the results presented in Figure 2 allows us to note the sufficient accuracy of the calculations, in which the correlation coefficient shows a strong direct relation of the studied parameters (r=0.998), which allows us to confirm the correctness of our assumption about the linear nature of the relation the coordination number and threshold value of L is bigger than 1.02 particle diameter. At the same time, at low threshold values, this dependence significantly decreases. This fact can be explained by the discrete nature of the inclusion of neighboring particles at such small distances in the "influence" region of the selected particle. Let us explain this fact by conducting a thought experiment. Suppose we have a fragment of a square package consisting of four particles, the coordination numbers of which are the same and equal to two. Now we will slowly bring together two opposite particles of this fragment.

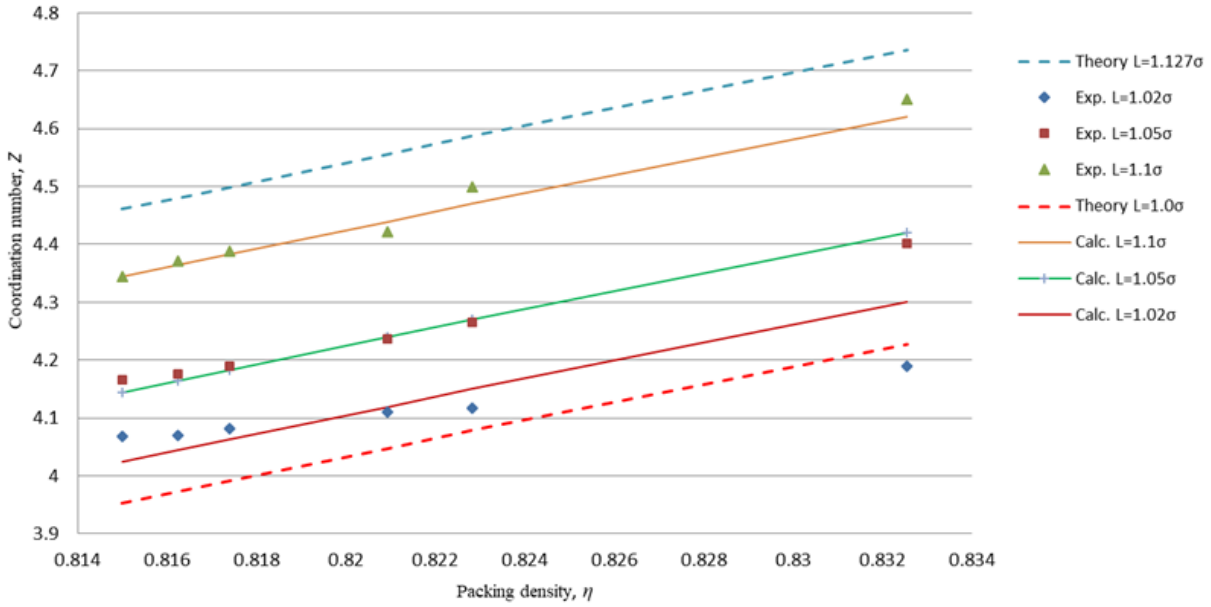

Figure 2. Dependence of the coordination number on the 2-D packing density at various threshold interparticle distances.

We note here that this process will in no way affect the coordination numbers of individual particles, although the packing density will increase during this process. Consequently, it can be argued that there is no relationship between the considered parameters in this example. However, if we choose large threshold values, then we can "notice" a change in the coordination number earlier and, therefore, we can already assert the emergence of a relationship between the coordination number and the packing density.

4.2. 3-D packing

For three-dimensional space, formula (1) for the coordination number of the ball packaging can be presented in the following form

$$Z_e(3) = n_m(\frac{6}{\pi}\eta(3))^2 . \quad (5)$$

Let us use the additional experimental data obtained by T. Aste et al. [15] for random sphere packing (table 3). Here, the known values of the average coordination number range from 5.5 to 8.9. Relative errors of data on the coordination number are not presented in the article under consideration. The data on the density of 3-D packings also have a range of changes ranging from 0.586 ± 0.005 for RLP packing to 0.640 ± 0.005 for RCP packing.

Having constructed the regression equations, similarly to formula (3), for the three-dimensional case we obtain the numerical values of the constant: $A(3)$ = 18 and the limiting distance between particles – $L_m(3)$ = 1.109σ. These results allow us to present the general equation for 3-D packaging in the following form

$$\mathrm{Z}(3) = n_m(\frac{6}{\pi}\eta(3))^2 - A(3)(L_m(3) - L)/\sigma. \quad (6)$$

Figure 3 shows the dependence of the coordination number Z on the density of a random spherical packing η at various threshold values of the interparticle distance L for three-dimensional space.

Table 3. Densities of 3-D sphere packing and coordination numbers at various threshold values of the interparticle distance.

Packing density, η	Coordination number, *Z*				
	L=1.02σ	Deconvolution	*L*=1.05σ	*L*=1.1σ	Theory
0.586±0.005	5.5	5.81	6.7	7.5	7.62
0.596±0.006	5.9	5.91	6.8	7.7	7.88
0.619±0.005	6.4	6.77	7.5	8.4	8.41
0.626±0.008	6.0	6.78	7.5	8.4	8.55
0.630±0.010	6.3	6.95	7.6	8.6	8.69
0.640±0.005	6.9	6.97	7.9	8.9	8.96

An analysis of the function *Z*(η) shows that for almost all threshold values of *L*, its graph is close to its linear position, that is proved by the correlation coefficient *r*=0.998). Except for the data related to the threshold value: *L* = 1.02, where the experimental data on the coordination number have a larger scatter (correlation coefficient *r* = 0.768) and are significantly less than for the other threshold values. For this reason, it was also decided to include in the consideration the data obtained by the method of deconvolution, allowing to clarify the value of the coordination number at such a threshold value.

It can be seen from the graph in Figure 3 that the deconvolution method also has a significant scatter of data, indicating a weak dependence of the coordination number on the packing density at low threshold values of the interparticle distance. Overall, the linear nature of the dependence of the coordination number on the threshold values is clearly visible when their value is more than 1.02 particle diameters, which are described with a certain accuracy for the practice by equation (6) for the coordination number of 3-D packing.

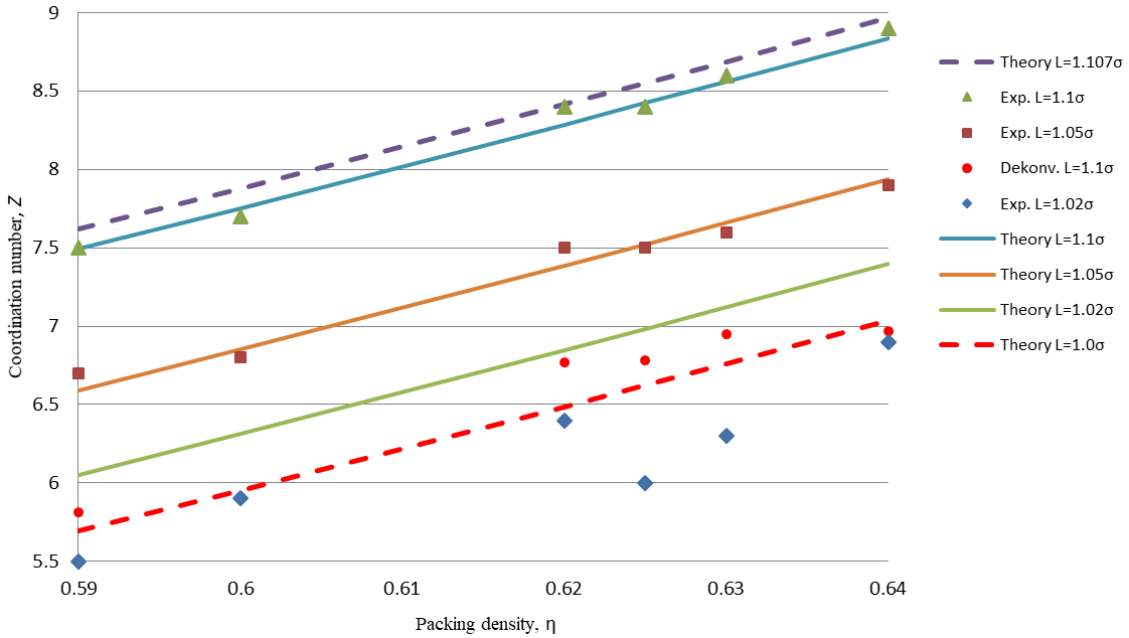

Figure 3. Dependence of the coordination number on the 3-D packing density at various threshold interparticle distance values.

Comparison of equation (4) for 2-D packing and expression (6) for 3-D packing shows that they differ from each other only by constants depending on the dimension of the selected space and the largest distance for particles of the first coordination sphere, numerically equal to $\sqrt{2}\sigma$. To summarize these equations, we can choose the appropriate expressions to explain the value of these constants. So,

the constant A, taking into account the dimension of space, can be represented as follows: $A(n) = 2n^{n-1}$, and the maximum threshold value for 2-D packaging can be written on the basis of the maximum dimension of space: $L_m(2)=\sqrt{2}\sigma\,(n_m-2)/(n_m-1)$. When considering the maximum threshold value for 3-D packaging, it is necessary to reduce its value by including an additional parameter: $L_m(3)=L_m(2)-\sigma/(n_m-1)^2$. In this case, the maximum threshold value of the *n*-dimensional packing can be represented as

$$L_m(n) = \sigma(\sqrt{2}\frac{n_m-2}{n_m-1} - \frac{n-2}{(n_m+1)^2}). \quad (7)$$

Now, having determined expression (7) for $L_m(n)$, the equation for the coordination number of the n-dimensional random sphere packing, taking into account formula (1), can finally be represented as follows

$$Z(n) = n_m[(\frac{2n}{\pi}\eta(n))^2 - 1] + 2n - 2n^{n-1}[(\sqrt{2}\frac{n_m-2}{n_m-1} - \frac{n-2}{(n_m+1)^2}) - L/\sigma]\,. \quad (8)$$

The resulting generalized analytical equation (8) can then be used to calculate the average coordination number of random sphere packings based on the knowledge of packing densities at various values of the interparticle distance in two- and three-dimensional spaces.

5. Conclusion

Our studies made it possible to expand the constructed mathematical model for the coordination number of sphere packing, additionally including the interparticle distance into consideration. Here we have shown that the coordination number is a function that uniquely depends on the packing density, the dimension of the space under consideration, and the magnitude of the interparticle distance in the case if $L > 1.02$, and described by equation (8). The validity of this equation was confirmed by the results of computer modeling and known experimental data.

We also explained the linear relationships between the coordination number and packing densities at various threshold interparticle distances. Additionally, we analyzed the experimental data on the coordination number. Also, it was confirmed by computer simulation that the coordination number in random packing is determined not only by changes in packing density, since the dimension of space and threshold values of interparticle distances also play an important role. We also verified the assertion that the ratios of the coordination number and packing density at small values of the interparticle distance are weakly dependent on each other and, using the example of a mental experiment, we demonstrated the reasons responsible for the lack of the dependence between the given parameters.

Appendix

Table 4. Results of a computational experiment to evaluate packing densities η and coordination numbers *Z* for 2-D packs for various probabilities *P* of choosing the positions of mounted particles.

P = 0.5					*P* = 0.6				
η	$L = 1.02\sigma$	$L = 1.05\sigma$	$L = 1.10\sigma$	Calculation	η	$L = 1.02\sigma$	$L = 1.05\sigma$	$L = 1.10\sigma$	Calculation
0.8137	4.0553	4.1521	4.3368	4.4402	0.8150	4.0542	4.1500	4.3333	4.4608
0.8141	4.0700	4.1726	4.3121	4.4465	0.8167	4.0748	4.1951	4.3826	4.4878
0.8149	4.0649	4.1475	4.3297	4.4592	0.8168	4.0814	4.1840	4.3796	4.4894
0.8153	4.0682	4.1603	4.3491	4.4656	0.8156	4.0650	4.1650	4.3760	4.4703
0.8161	4.0717	4.1614	4.3583	4.4783	0.8162	4.0746	4.1800	4.3620	4.4798
0.8143	4.0700	4.1501	4.3500	4.4497	0.8172	4.0730	4.1790	4.3930	4.4957
0.8147	4.0802	4.2004	4.3430	4.4561	0.8177	4.0701	4.1715	4.3955	4.5037

0.8149	4.0704	4.1703	4.3400	4.4592	0.8193	4.0923	4.2141	4.4079	4.5292
0.8150	4.0610	4.1602	4.3270	4.4608	0.8134	4.0806	4.1730	4.3209	4.4355
0.8170	4.0680	4.1840	4.3900	4.4926	0.8121	4.0591	4.1338	4.2848	4.4149
				Average					
0.8150	4.0679	4.1658	4.3436	4.4608	0.8163	4.0705	4.1755	4.3711	4.4806
				Standard deviation					
0.0010	0.0066	0.0164	0.0210	0.0153	0.0021	0.0096	0.0158	0.0211	0.0131
		$P = 0.7$					$P = 0.8$		
0.8169	4.0814	4.1916	4.3740	4.4910	0.8180	4.1003	4.2019	4.3775	4.5085
0.8190	4.0828	4.2060	4.4178	4.5244	0.8202	4.0867	4.1875	4.3842	4.5435
0.8171	4.0662	4.1668	4.3412	4.4941	0.8213	4.1392	4.3066	4.4996	4.5611
0.8161	4.0717	4.1614	4.3583	4.4783	0.8216	4.1250	4.2218	4.3473	4.5659
0.8176	4.0942	4.1947	4.3893	4.5021	0.8236	4.1014	4.2655	4.5031	4.5979
0.8185	4.0805	4.1982	4.3990	4.5164	0.8232	4.0928	4.2686	4.4791	4.5915
0.8141	4.0630	4.1638	4.3393	4.4465	0.8245	4.1182	4.2935	4.5300	4.6123
0.8170	4.0608	4.1565	4.3604	4.4926	0.8228	4.1113	4.2537	4.4657	4.5851
0.8120	4.0668	4.1424	4.2992	4.4130	0.8153	4.0657	4.1598	4.3385	4.4656
0.8220	4.1057	4.2625	4.4954	4.5718	0.8170	4.0863	4.2014	4.3987	4.4926
				Average					
0.8174	4.0825	4.1884	4.3849	4.4989	0.8209	4.1105	4.2367	4.4223	4.5554
				Standard deviation					
0.0012	0.0092	0.0190	0.0254	0.0196	0.0021	0.0211	0.0489	0.0735	0.0328
		$P = 0.9$					$P = 1.0$		
0.8236	4.1177	4.2699	4.5224	4.5979	0.8294	4.1120	4.2994	4.5865	4.6911
0.8194	4.0968	4.2523	4.4376	4.5308	0.8302	4.1308	4.3853	4.4678	4.7040
0.8239	4.1391	4.3014	4.5219	4.6027	0.8309	4.1736	4.3460	4.6355	4.7154
0.8248	4.1135	4.2751	4.5410	4.6171	0.8321	4.2126	4.4579	4.7173	4.7348
0.8215	4.1189	4.2578	4.4915	4.5643	0.8325	4.1912	4.4783	4.7105	4.7412
0.8222	4.1165	4.2511	4.4878	4.5755	0.8332	4.1837	4.3457	4.6935	4.7526
0.8199	4.0936	4.2037	4.4175	4.5387	0.8341	4.2461	4.3974	4.6774	4.7672
0.8253	4.1517	4.3240	4.5686	4.6251	0.8302	4.1300	4.3853	4.6775	4.7040
0.8261	4.1162	4.2774	4.5610	4.6380	0.8357	4.3277	4.1930	4.4772	4.7932
0.8216	4.1127	4.2359	4.4359	4.5659	0.8308	4.1326	4.3797	4.6695	4.7137
				Average					
0.8229	4.1168	4.2747	4.5057	4.5871	0.8319	4.1786	4.3871	4.6412	4.7295
				Standard deviation					
0.0024	0.0174	0.0203	0.0463	0.0384	0.0020	0.0459	0.0639	0.0917	0.0272

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