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On homogenized equations of filtration in two domains with common boundary

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Abstract. We consider an initial-boundary value problem describing the process of filtration of a weakly viscous fluid in two distinct porous media with common boundary. We prove, at the microscopic level, the existence and uniqueness of a generalized solution of the problem on the joint motion of two incompressible elastic porous (poroelastic) bodies with distinct Lamé constants and different microstructures, and of a viscous incompressible porous fluid. Under various assumptions on the data of the problem, we derive homogenized models of filtration of an incompressible weakly viscous fluid in two distinct elastic or absolutely rigid porous media with common boundary.

Keywords: heterogeneous media, periodic structure, Lamé equations, Stokes equations, homogenization, two-scale convergence.

§ 1. Introduction

This paper is devoted to the homogenization of systems of differential equations describing the process of filtration in heterogeneous media, that is, in media consisting of two or more distinct components that occur in every sufficiently small volume of the medium. Our approach to the description of heterogeneous media is based on constructing a maximally precise mathematical model, which is then simplified using the methods of calculus. As a rule, the differential equations of precise mathematical models contain a small parameter. Hence, to simplify these models, we mainly use the methods of linearization and homogenization as the small parameter tends to zero.

Homogenization theory began with the papers by Burridge and Keller [1] and Sanchez–Palencia [2] and then developed extensively in [3]–[9]. The authors suggested different methods of solution, and each problem required a separate approach and many efforts.

The publication of Nguetseng's paper [8] in 1989 basically defined the theory of homogenization as a separate branch of calculus. Therefore mainstream research in homogenization theory moved from theoretical investigations to applications in mechanics, physics, biology and so on (see [10]–[16]).

In particular, Jäger and Mikelić [16] studied the problem of planar filtration (that is, on domains in $\mathbb{R}^2$) for a fluid in two distinct porous media filling domains

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Ω and Ω^0 with common boundary S^0 provided that the porous space in each domain has a special geometry. At the microscopic level, the rigid skeleton in each medium was assumed to be disconnected (isolated inclusions) and absolutely rigid (the velocity of the medium in an absolutely rigid body is identically equal to zero).

It is known [2], [1] that Darcy's system of filtration equations is a homogenization in dimensionless variables (a limit as $\varepsilon \to 0$) of the Stokes system of equations for a *weakly viscous incompressible fluid* with dimensionless viscosity $\mu = \varepsilon^2 \mu_1$ in a perforated domain (filled by the liquid) with ε-periodic structure.

Distinct domains Ω and Ω^0 obviously have distinct permeability matrices in Darcy's law. The resulting differential equations are equivalent to distinct Poisson equations in each domain, and each equation requires a single boundary condition on the common boundary S^0. The first condition, continuity of the normal components of the velocity vector, holds for any geometry of the porous spaces in Ω and Ω^0. This condition follows from the continuity equation.

The second boundary condition on S^0 depends on the geometry of the porous spaces in Ω and Ω^0. We shall prove that there are not many options to choose from. Either the normal component of the velocity vector vanishes (impenetrable boundary), or the pressure is continuous (penetrable boundary).

Jäger and Mikelić studied the case of a disconnected rigid skeleton in the domains

$$\Omega = \{\boldsymbol{x} \colon 0 < x_1 < L,\, 0 < x_2 < \infty\} \quad \text{and} \quad \Omega^0 = \{\boldsymbol{x} \colon 0 < x_1 < L,\, -\infty < x_2 < 0\}.$$

The rigid skeleton in Ω is the periodic replication of a domain εY_s completely contained in εY, $Y_s \subset Y = (0,1)^2 \subset \mathbb{R}^2$. The rigid skeleton in Ω^0 is the periodic replication of a domain εY_s^0 completely contained in εY, $Y_s^0 \subset Y = (0,1)^2 \subset \mathbb{R}^2$.

For this geometry of porous spaces, we always have $S^0 = \partial\Omega_f^\varepsilon \cap \partial\Omega_f^\varepsilon$ (penetrable boundary) and the only possible version of the second boundary condition on S^0 is continuity of the pressure.

We study the filtration of a weakly viscous fluid in two poroelastic domains Ω and Ω^0 in $\mathbb{R}^3$ with common boundary S^0 and with different properties of the rigid skeleton. For simplicity we assume that

$$\Omega = \{\boldsymbol{x} \colon 0 < x_1, x_2, x_3 < 1\}, \qquad \Omega^0 = \{\boldsymbol{x} \colon 0 < x_1, x_2 < 1,\, -1 < x_3 < 0\}.$$

The pores of each domain are assumed to be filled with the same viscous liquid of dimensionless viscosity α_μ, and the rigid skeleton in each domain is regarded as an incompressible elastic body described by the Lamé equations with Lamé constants λ and λ^0 in Ω and Ω^0 respectively. The conditions of continuity for the displacements and for the normal stress hold on the common boundary S^0. The problem is closed by boundary conditions on the exterior boundaries of Ω and Ω^0 and by initial conditions.

For every $\varepsilon > 0$ we prove the existence and uniqueness of a generalized solution and study its limit (homogenization) as $\varepsilon \to 0$. For fixed constants λ and λ^0, the limiting problem (the homogenized system) depends on the function $\alpha_\mu = \alpha_\mu(\varepsilon)$ or, more precisely, on the following limits μ_0 and μ_1 as $\varepsilon \to 0$:

$$\mu_0 = \lim_{\varepsilon \to 0} \alpha_\mu(\varepsilon), \qquad \mu_1 = \lim_{\varepsilon \to 0} \frac{\alpha_\mu(\varepsilon)}{\varepsilon^2}.$$

We restrict ourselves to the case when $\mu_0 = 0$ and the common rigid skeleton is connected (see § 2). When $0 < \mu_1 < \infty$ and $0 < \lambda$, $\lambda^0 < \infty$, the displacements of the elastic skeleton in Ω and Ω^0 satisfy the homogenized Lamé equations and the continuity conditions for the limiting displacements and for the normal stress on the common boundary S^0, while the motions of the liquid in Ω and Ω^0 are described by separate Darcy's laws. Moreover, the conditions of continuity for the normal components of the velocity of the liquid hold on the common boundary. As already mentioned, another scalar boundary condition is lacking and this condition will depend on the structure of the common porous space. Namely, *the pressure is continuous for a connected common porous space* (see § 2) and *the normal component of the velocity vector will vanish for a disconnected common porous space* (there is no flux of the liquid from Ω to Ω^0 and back).

If we put $\lambda = \lambda^0 = k$ in the homogenized equations, then the limit of the corresponding displacements of the rigid skeletons as $k \to \infty$ will be equal to zero (absolutely rigid body) and the limit *of the velocities and pressures of the liquid as $k \to \infty$ is a solution of the system of filtration equations in Ω and Ω^0 along with the condition of continuity of the normal component of the velocity on S^0. Moreover, *the pressure will be continuous for a connected common porous space, and the normal component of the velocity vector will vanish for a disconnected common porous space.*

§ 2. Notation

The dimensionless parameter $\overline{\alpha}_\mu$ characterizes the viscosity of the liquid:

$$\overline{\alpha}_\mu = \frac{2\mu}{\tau L g \rho^0}, \qquad \varepsilon_0 = \frac{l}{L}, \quad \varepsilon_0 \ll 1,$$

where L is the characteristic size of the physical domain under consideration, τ is the time of the physical process, ρ^0 is the density of water, g is the acceleration due to gravity, and μ is the dynamical coefficient of viscosity of the fluid.

Consider a function $\alpha_\mu(\varepsilon)$ such that $\alpha_\mu(\varepsilon_0) = \overline{\alpha}_\mu$ and there are finite or infinite limits

$$\lim_{\varepsilon \to 0} \alpha_\mu(\varepsilon) = \mu_0, \qquad \lim_{\varepsilon \to 0} \frac{\alpha_\mu(\varepsilon)}{\varepsilon^2} = \mu_1.$$

Let Ω and Ω^0 be bounded domains with Lipschitz boundaries $\partial\Omega$ and $\partial\Omega^0$ such that

$$\Omega \cap \Omega^0 = \varnothing, \qquad \overline{\Omega^0} \cap \overline{\Omega} \neq \varnothing, \qquad S^0 = \partial\Omega^0 \cap \partial\Omega, \qquad Q = \operatorname{Int}\big(\overline{\Omega^0} \cup \overline{\Omega}\big), \qquad S = \partial Q.$$

We now define domains Q_f^ε and Q_s^ε to be the common liquid and rigid components of the medium for Ω^0 and Ω. Namely, we put

$$Q_s^\varepsilon = \Omega_s^{0,\varepsilon} \cup S_s^{0,\varepsilon} \cup \Omega_s^\varepsilon, \qquad S_s^{0,\varepsilon} = \partial\Omega_s^{0,\varepsilon} \cap \partial\Omega_s^\varepsilon,$$

where Q_s^ε is the *common rigid skeleton*, which we assume to be a connected set, $\Omega_s^{0,\varepsilon}$ is the rigid skeleton of Ω^0, and Ω_s^ε is the rigid skeleton of Ω.

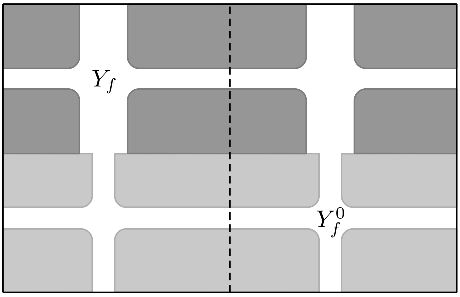

Figure 1. A connected common porous space

By Q_f^ε we mean the *common porous space*:

$$Q_f^\varepsilon = \Omega_f^{0,\varepsilon} \cup S_f^{0,\varepsilon} \cup \Omega_f^\varepsilon, \qquad S_f^{0,\varepsilon} = \partial\Omega_f^{0,\varepsilon} \cap \partial\Omega_f^\varepsilon, \qquad Q_f^\varepsilon = Q \setminus \overline{Q_s^\varepsilon},$$

where $\Omega_f^{0,\varepsilon}$ is the porous space in Ω^0 and Ω_f^ε is the porous space in Ω. The common porous space may be connected or disconnected.

We also introduce the following restrictions:

$$\Gamma^\varepsilon = \partial Q_s^\varepsilon \cap \partial Q_f^\varepsilon, \qquad S_s^\varepsilon = \partial Q_s^\varepsilon \cap S, \qquad S_f^\varepsilon = \partial Q_f^\varepsilon \cap S,$$

$$Y^{\varepsilon,\boldsymbol{k}} = \{\boldsymbol{x} \colon \boldsymbol{x} = \varepsilon\boldsymbol{k} + \varepsilon\boldsymbol{y},\ \boldsymbol{y} \in Y,\ \boldsymbol{k} = (k_1,k_2,k_3),\ k_1,k_2,k_3 \in \mathbb{N}\},$$

$$Y_f^{\varepsilon,\boldsymbol{k}} = \{\boldsymbol{x} \colon \boldsymbol{x} = \varepsilon\boldsymbol{k} + \varepsilon\boldsymbol{y},\ \boldsymbol{y} \in Y_f,\ \boldsymbol{k} = (k_1,k_2,k_3),\ k_1,k_2,k_3 \in \mathbb{N}\},$$

$$Y_s^{\varepsilon,\boldsymbol{k}} = \{\boldsymbol{x} \colon \boldsymbol{x} = \varepsilon\boldsymbol{k} + \varepsilon\boldsymbol{y},\ \boldsymbol{y} \in Y_s,\ \boldsymbol{k} = (k_1,k_2,k_3),\ k_1,k_2,k_3 \in \mathbb{N}\},$$

$$Q^{\varepsilon,\boldsymbol{k}} = Q \cap Y^{\varepsilon,\boldsymbol{k}}, \qquad Q_f^{\varepsilon,\boldsymbol{k}} = Q \cap Y_f^{\varepsilon,\boldsymbol{k}}, \qquad Q_s^{\varepsilon,\boldsymbol{k}} = Q \cap Y_s^{\varepsilon,\boldsymbol{k}}, \qquad \Gamma^{\varepsilon,\boldsymbol{k}} = \Gamma^\varepsilon \cap Y^{\varepsilon,\boldsymbol{k}}.$$

Moreover, let λ^0 and λ be the dimensionless Lamé constants of the rigid components in Ω^0 and Ω respectively, and let ϱ_s^0 and ϱ_s be the dimensionless densities of the rigid components in Ω^0 and Ω.

We write $Y = (0,1)^3$ for the periodicity cell, and $\langle\,\cdot\,\rangle_Y$ for the integral over Y.

Let $\chi(\boldsymbol{y})$ be a 1-periodic function of $\boldsymbol{y} \in \mathbb{R}^3$ (that is, $\chi(\boldsymbol{y}+\boldsymbol{k}) = \chi(\boldsymbol{y})$ for all $\boldsymbol{k} = (k_1,k_2,k_3)$, $k_1,k_2,k_3 \in \mathbb{N}$) such that $\chi(\boldsymbol{y}) = 1$ when $\boldsymbol{y} \in Y_f \subset Y$ and $\chi(\boldsymbol{y}) = 0$ when $\boldsymbol{y} \in Y_s \subset Y$, where $Y_f \cup \gamma \cup Y_s = Y$, $Y_f \cap Y_s = \varnothing$, $\partial Y_f \cap \partial Y_s = \gamma$, $\gamma \in \mathbb{C}^\infty$.

We call $\chi(\boldsymbol{y})$ the characteristic function of the domain Y_f.

We similarly define the characteristic function $\chi^0(\boldsymbol{y})$ of the domain $Y_f^0 \subset Y$, that is, $\chi^0(\boldsymbol{y}+\boldsymbol{k}) = \chi^0(\boldsymbol{y})$ for all $\boldsymbol{k} = (k_1,k_2,k_3)$, $k_1,k_2,k_3 \in \mathbb{N}$, $\chi^0(\boldsymbol{y}) = 1$ when $\boldsymbol{y} \in Y_f^0 \subset Y$ and $\chi^0(\boldsymbol{y}) = 0$ when $\boldsymbol{y} \in Y_s^0 \subset Y$, where $Y_f^0 \cup \gamma^0 \cup Y_s^0 = Y$, $Y_f^0 \cap Y_s^0 = \varnothing$, $\partial Y_f^0 \cap \partial Y_s^0 = \gamma^0$, $\gamma^0 \in \mathbb{C}^\infty$.

We now define the characteristic function $\zeta(\boldsymbol{x})$ of the domain Ω^0 in Q: $\zeta(\boldsymbol{x}) = 1$ when $\boldsymbol{x} \in \Omega^0$ and $\zeta(\boldsymbol{x}) = 0$ when $\boldsymbol{x} \in \Omega$. Then

$$\chi^\varepsilon(\boldsymbol{x}) = \chi\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\big(1 - \zeta(\boldsymbol{x})\big)$$

is the characteristic function of the porous space Ω_f^ε and

$$\chi^{0,\varepsilon}(\boldsymbol{x}) = \chi^0\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\zeta(\boldsymbol{x})$$

is the characteristic function of the porous space $\Omega_f^{0,\varepsilon}$, while

$$\widehat{\chi}\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\right) = \widehat{\chi}^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) + \chi^{0,\varepsilon}(\boldsymbol{x}) = \chi\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\big(1 - \zeta(\boldsymbol{x})\big) + \chi^0\left(\frac{\boldsymbol{x}}{\varepsilon}\right)\zeta(\boldsymbol{x})$$

is the characteristic function of the common porous space Q_f^ε.

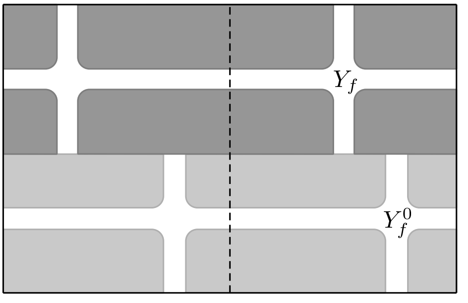

Figure 2. A disconnected common porous space

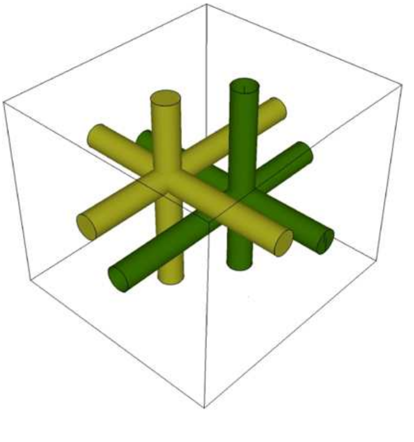

Figure 3. A disconnected common porous space

For a connected common porous space, the elementary domains Y_f^0 and Y_f have non-empty intersection in Y: $Y_f^0 \cap Y_f \neq \varnothing$. For a disconnected common porous space, they are disjoint in Y: $Y_f^0 \cap Y_f = \varnothing$.

Finally, we define the common elementary porous space $\widehat{Y}_f(\boldsymbol{x})$ as the set

$$\widehat{Y}_f(\boldsymbol{x}) = \{\boldsymbol{y} \in Y : \boldsymbol{y} \in Y_f \text{ if } \boldsymbol{x} \in \Omega;\ \boldsymbol{y} \in Y_f^0 \text{ if } \boldsymbol{x} \in \Omega^0\}.$$

The function $\widehat{\chi}(\boldsymbol{x},\boldsymbol{y})$ is the characteristic function of the set $\widehat{Y}_f(\boldsymbol{x})$ in $\widehat{Y}(\boldsymbol{x}) = \{\boldsymbol{y}\in Y:\ \boldsymbol{y}\in Y \text{ if } \boldsymbol{x}\in\Omega;\ \boldsymbol{y}\in Y \text{ if } \boldsymbol{x}\in\Omega^0\}$.

We put $\widehat{Y}_s = \widehat{Y}\setminus\overline{\widehat{Y}_f}$, $\gamma=\partial Y_f$, $\gamma^0=\partial Y_f^0$ and $\widehat{\gamma}=\partial\widehat{Y}_f$,

$$\varrho^\varepsilon(\boldsymbol{x}) = \bigl(1-\zeta(\boldsymbol{x})\bigr)\bigl(\varrho_f\chi^\varepsilon(\boldsymbol{x})+\varrho_s\bigl(1-\chi^\varepsilon(\boldsymbol{x})\bigr),$$
$$\varrho^{0,\varepsilon}(\boldsymbol{x}) = \zeta(\boldsymbol{x})\bigl(\varrho_f\chi^{0,\varepsilon}(\boldsymbol{x})+\varrho_s^0\bigl(1-\chi^{0,\varepsilon}(\boldsymbol{x})\bigr)\bigr),$$
$$\widehat{\varrho}^\varepsilon(\boldsymbol{x}) = \varrho^\varepsilon(\boldsymbol{x})+\varrho^{0,\varepsilon}(\boldsymbol{x}),$$
$$m=\int_Y\chi(\boldsymbol{y})\,dy\equiv\langle\chi\rangle_Y,\qquad m^0=\int_Y\chi^0(\boldsymbol{y})\,dy=\langle\chi^0\rangle_Y,$$
$$\widehat{m}(\boldsymbol{x})=\zeta(\boldsymbol{x})m^0+\bigl(1-\zeta(\boldsymbol{x})\bigr)m,$$
$$\widehat{\lambda}(\boldsymbol{x})=\lambda^0\,\zeta(\boldsymbol{x})+\lambda\bigl(1-\zeta(\boldsymbol{x})\bigr).$$

We shall generally write $\widehat{A}$ for a quantity which is different in Ω and in Ω^0: $\widehat{A}=A$ in Ω and $\widehat{A}=A^0$ in Ω^0, or $\widehat{A}=A\bigl(1-\zeta(\boldsymbol{x})\bigr)+A^0\zeta(\boldsymbol{x})$.

Here is some more notation:

$\mathbb{I}$ is the identity matrix;

the operator ∇ without subscripts stands for differentiation with respect to $\boldsymbol{x}$, and ∇_y is differentiation with respect to $\boldsymbol{y}$;

$\mathbb{D}(x,\boldsymbol{u})=(1/2)(\nabla\boldsymbol{u}+\nabla\boldsymbol{u}^T)$;

$\mathbb{B}:\mathbb{C}=\operatorname{tr}(\mathbb{B}\cdot\mathbb{C}^T)$, where $\mathbb{B}$, $\mathbb{C}$ are tensors of rank two;

$\boldsymbol{a}\otimes\boldsymbol{b}$ is a dyad: we have $(\boldsymbol{a}\otimes\boldsymbol{b})\cdot\boldsymbol{c}=\boldsymbol{a}(\boldsymbol{b}\cdot\boldsymbol{c})$ for all vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$:

$\mathbb{J}^{ij}=(1/2)(\boldsymbol{e}^i\otimes\boldsymbol{e}^j+\boldsymbol{e}^j\otimes\boldsymbol{e}^i)$, where $\{\boldsymbol{e}^1,\ \boldsymbol{e}^2,\ \boldsymbol{e}^3\}$ is the standard Cartesian basis of $\mathbb{R}^3$;

$\mathbb{A}\otimes\mathbb{B}$ is a tensor of rank four;

$(\mathbb{A}\otimes\mathbb{B}):\mathbb{C}=\mathbb{A}(\mathbb{B}:\mathbb{C})$ for all tensors $\mathbb{A}$, $\mathbb{B}$, $\mathbb{C}$ of rank two.

§ 3. Statement of the problem

Consider the joint motion in Q of two distinct poroelastic media filling the domains Ω^0 and Ω and having common boundary S^0.

The motion of the medium in Ω^0 for $t>0$ is described by the equations

$$\nabla\cdot\boldsymbol{w}^\varepsilon=0, \tag{3.1}$$
$$\nabla\cdot\mathbb{P}^0+\varrho^{0,\varepsilon}\boldsymbol{F}^\varepsilon=0, \tag{3.2}$$

where

$$\mathbb{P}^0=\chi^{0,\varepsilon}\alpha_\mu\mathbb{D}\biggl(x,\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}\biggr)+(1-\chi^{0,\varepsilon})\lambda^0\mathbb{D}(x,\boldsymbol{w}^\varepsilon)-p^\varepsilon\mathbb{I}=\mathbb{P}^{0,f}+\mathbb{P}^{0,s},$$
$$\mathbb{P}^{0,f}=\chi^{0,\varepsilon}\biggl(\alpha_\mu\,\mathbb{D}\biggl(x,\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}\biggr)-p^\varepsilon\mathbb{I}\biggr),\qquad \mathbb{P}^{0,s}=(1-\chi^{0,\varepsilon})\bigl(\lambda^0\mathbb{D}(x,\boldsymbol{w}^\varepsilon)-p^\varepsilon\mathbb{I}\bigr).$$

The motion of the medium in Ω for $t>0$ is described by a system consisting of the continuity equation (3.1) and the momentum balance equation

$$\nabla\cdot\mathbb{P}+\varrho^\varepsilon\boldsymbol{F}^\varepsilon=0, \tag{3.3}$$

where

$$\mathbb{P} = \chi^\varepsilon \alpha_\mu \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) + (1-\chi^\varepsilon)\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - p^\varepsilon \mathbb{I} = \mathbb{P}^f + \mathbb{P}^s,$$

$$\mathbb{P}^f = \chi^\varepsilon \Big(\alpha_\mu \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) - p^\varepsilon \mathbb{I}\Big), \qquad \mathbb{P}^s = (1-\chi^\varepsilon)\big(\lambda \mathbb{D}(x, \boldsymbol{w}^\varepsilon) - p^\varepsilon \mathbb{I}\big).$$

Continuity conditions hold for the displacements and for the normal stress on the common boundary S^0 for $t > 0$:

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}} \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}} \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t), \tag{3.4}$$

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}} \mathbb{P}^0(\boldsymbol{x}, t)\cdot \boldsymbol{n}(\boldsymbol{x}^0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}} \mathbb{P}(\boldsymbol{x}, t)\cdot \boldsymbol{n}(\boldsymbol{x}^0). \tag{3.5}$$

Here $\boldsymbol{n}(\boldsymbol{x}^0)$ is the exterior unit normal vector (with respect to Ω) to the boundary S^0 at a point $\boldsymbol{x}^0 \in S^0$.

We put

$$\widehat{\mathbb{P}} = \zeta \mathbb{P}^0 + (1-\zeta)\mathbb{P}, \qquad \widehat{\mathbb{P}}^f = \zeta \mathbb{P}^{0,f} + (1-\zeta)\mathbb{P}^f, \qquad \widehat{\mathbb{P}}^s = \zeta \mathbb{P}^{0,s} + (1-\zeta)\mathbb{P}^s.$$

Then the continuity conditions for the normal stress follow from (3.1)–(3.3):

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in Q_f^\varepsilon}} \widehat{\mathbb{P}}(\boldsymbol{x}, t)\cdot \boldsymbol{n}(\boldsymbol{x}^0) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in Q_s^\varepsilon}} \widehat{\mathbb{P}}(\boldsymbol{x}, t)\cdot \boldsymbol{n}(\boldsymbol{x}^0), \tag{3.6}$$

where $\boldsymbol{n}(\boldsymbol{x}^0)$ is the exterior (to Q_f^ε) unit normal vector to the boundary Γ^ε at a point $\boldsymbol{x}^0 \in \Gamma^\varepsilon$.

The problem is closed by the Dirichlet boundary condition

$$\boldsymbol{w}^\varepsilon(\boldsymbol{x}, t) = 0, \qquad \boldsymbol{x} \in S = \partial Q, \quad t > 0, \tag{3.7}$$

on the exterior boundary S for $t > 0$, by the initial condition

$$\widehat{\chi}^\varepsilon(\boldsymbol{x})\boldsymbol{w}^\varepsilon(\boldsymbol{x}, 0) = 0, \qquad \boldsymbol{x} \in Q, \tag{3.8}$$

and by the normalization condition

$$\int_0^T \int_Q p^\varepsilon(\boldsymbol{x}, t)\, dx\, dt = 0. \tag{3.9}$$

The function $\boldsymbol{F}^\varepsilon$ in (3.2) and (3.3) is a given density of distribution of the massive forces.

Assumption 3.1. The following assertions hold.

1) The boundaries $\gamma = \partial Y_f \cap \partial Y_s$ and $\gamma^0 = \partial Y_f^0 \cap \partial Y_s^0$ are connected infinitely differentiable surfaces.

2) Y_s, Y_f, Y_s^0 and Y_f^0 are connected sets.

Assumption 3.2. $\varepsilon = 1/n$, $n \in \mathbb{N}$ is a positive integer.

Assumption 3.3. The sets Ω_s^ε, $\Omega_s^{0,\varepsilon}$, Ω_f^ε, $\Omega_f^{0,\varepsilon}$ and Q_s^ε are connected.

Assumption 3.4. We have $\boldsymbol{F}^\varepsilon(\boldsymbol{x}, 0) = 0$.

By Assumption 3.1, the boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ which separates the domains Ω_f^ε and Ω_s^ε is an infinitely differentiable surface, and so is the boundary $\Gamma^{0,\varepsilon} = \partial\Omega_s^{0,\varepsilon} \cap \partial\Omega_f^{0,\varepsilon}$ which separates the domains Ω_f^ε and $\Omega_s^{0,\varepsilon}$.

For function spaces and their norms, we use the notation adopted in [17].

§ 4. Main results

Definition 4.1. A pair of functions $\{\boldsymbol{w}^\varepsilon, p^\varepsilon\}$ with

$$\boldsymbol{w}^\varepsilon \in \mathring{\mathbb{W}}_2^{1,0}\big(Q \times (0,T)\big), \qquad \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} \in \mathbb{W}_2^{1,0}\big(Q_f^\varepsilon \times (0,T)\big), \qquad p^\varepsilon \in \mathbb{L}_2\big(Q \times (0,T)\big),$$

is called a *generalized solution of the problem* (3.1)–(3.8) if they satisfy the condition (3.8) and the following integral identities hold:

$$\int_0^{t_0} \int_Q \boldsymbol{w}^\varepsilon \cdot \nabla\psi \, dx\, dt = 0 \tag{4.1}$$

and

$$\int_0^{t_0} \int_Q \bigg(\alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big) : \mathbb{D}(x, \boldsymbol{\varphi}) + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{w}^\varepsilon) : \mathbb{D}(x, \boldsymbol{\varphi})\bigg)\, dx\, dt$$
$$= \int_0^{t_0} \int_Q \big(p^\varepsilon \nabla \cdot \boldsymbol{\varphi} + \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon \cdot \boldsymbol{\varphi}\big)\, dx\, dt \tag{4.2}$$

for all t_0, $0 < t_0 < T$, all functions ψ vanishing on the boundary of Q, and all functions $\boldsymbol{\varphi} \in \mathring{\mathbb{W}}_2^{1,0}\big(Q \times (0,T)\big)$.

Lemma 4.1. *Suppose that* $\boldsymbol{w}^\varepsilon \in \mathring{\mathbb{W}}_2^{1,0}\big(Q \times (0,T)\big)$ *and Assumptions* 3.1–3.3 *hold. Then there is an extension operator*

$$E_{Q_s^\varepsilon} : \mathring{\mathbb{W}}_2^{1,0}\big(Q \times (0,T)\big) \to \mathring{\mathbb{W}}_2^{1,0}\big(Q \times (0,T)\big), \qquad \boldsymbol{w}_s^\varepsilon = E_{Q_s^\varepsilon}(\boldsymbol{w}^\varepsilon),$$

such that

$$\big(1 - \widehat{\chi}^\varepsilon(\boldsymbol{x})\big)\big(\boldsymbol{w}_s^\varepsilon(\boldsymbol{x}, t) - \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\big) = 0$$

and

$$\int_Q |\boldsymbol{w}_s^\varepsilon|^2 \, dx \leqslant C_0 \int_{Q_s^\varepsilon} |\boldsymbol{w}^\varepsilon|^2 \, dx, \qquad \int_Q |\mathbb{D}(x, \boldsymbol{w}_s^\varepsilon)|^2 \, dx \leqslant C_0 \int_{Q_s^\varepsilon} |\mathbb{D}(x, \boldsymbol{w}^\varepsilon)|^2 \, dx, \tag{4.3}$$

where C is independent of ε and $t \in (0,T)$.

Proof. See [18], [19]. □

Lemma 4.2. *Under the hypotheses of Lemma 4.1, put* $\boldsymbol{w}^{\varepsilon}_s = E_{Q^{\varepsilon}_s}(\boldsymbol{w}^{\varepsilon})$. *Then*

$$\int_Q |\boldsymbol{w}^{\varepsilon}_s|^2\,dx \leqslant C\int_Q |D(x,\boldsymbol{w}^{\varepsilon}_s)|^2\,dx \tag{4.4}$$

with a constant C *independent of* ε.

The proof of this lemma follows directly from the Friedrichs–Poincaré inequality

$$\int_Q |\boldsymbol{u}|^2\,dx \leqslant C\int_Q |\mathbb{D}(x,\boldsymbol{u})|^2\,dx \tag{4.5}$$

for functions $\boldsymbol{u}\in \overset{\circ}{\mathbb{W}}{}^1_2(\Omega)$.

Theorem 4.1. *Suppose that Assumptions 3.1–3.4 hold,* $\mu_0=0$, $0<\mu_1,\lambda,\lambda^0<\infty$,

$$\max_{0<t<T}\int_Q |\boldsymbol{F}^{\varepsilon}(\boldsymbol{x},t)|^2\,dx \leqslant F^2<\infty,$$

$$\int_0^t\int_Q\left|\frac{\partial \boldsymbol{F}^{\varepsilon}}{\partial t}(\boldsymbol{x},t)\right|^2 dx\,dt \leqslant F^2_{\varepsilon}<\infty, \qquad \boldsymbol{F}^{\varepsilon}\to\boldsymbol{F} \quad in \quad \mathbb{L}_2\big(Q\times(0,T)\big),$$

and

$$\boldsymbol{u}^{\varepsilon}(\boldsymbol{x},t)=\int_0^t \boldsymbol{w}^{\varepsilon}(\boldsymbol{x},\tau)\,d\tau, \qquad \pi^{\varepsilon}(\boldsymbol{x},t)=\int_0^t p^{\varepsilon}(\boldsymbol{x},\tau)\,d\tau.$$

Then for every $\varepsilon>0$ *and an arbitrary time interval* $[0,T]$ *there is a unique generalized solution of the problem* (3.1)–(3.8) *and*

$$\int_0^T\int_Q \alpha_{\mu}\widehat{\chi}^{\,\varepsilon}\left|\mathbb{D}\Big(x,\frac{\partial \boldsymbol{w}^{\varepsilon}}{\partial t}\Big)\right|^2 dx\,dt \leqslant C_{\varepsilon}\,F^2_{\varepsilon},$$

$$\max_{0<t<T}\varepsilon^2\bigg(\int_Q \widehat{\chi}^{\,\varepsilon}\left|\mathbb{D}\Big(x,\frac{\partial \boldsymbol{u}^{\varepsilon}}{\partial t}(\boldsymbol{x},t)\Big)\right|^2 dx+\int_Q\widehat{\chi}^{\,\varepsilon}\big|\mathbb{D}\big(x,\boldsymbol{w}^{\varepsilon}(\boldsymbol{x},t)\big)\big|^2\,dx\bigg)\leqslant CF^2,$$

$$\int_0^T\int_Q \widehat{\lambda}(\boldsymbol{x})\big(|\boldsymbol{w}^{\varepsilon}_s|^2+|\mathbb{D}(x,\boldsymbol{w}^{\varepsilon}_s)|^2\big)\,dx\,dt\leqslant CF^2,$$

$$\int_0^T\int_Q\bigg(|\boldsymbol{u}^{\varepsilon}|^2+|\boldsymbol{w}^{\varepsilon}|^2+\left|\frac{\partial\boldsymbol{u}^{\varepsilon}}{\partial t}\right|^2\bigg)\,dx\,dt\leqslant CF^2,$$

$$\int_0^t\int_Q|\pi^{\,\varepsilon}|^2\,dx\,dt\leqslant CF^2, \tag{4.6}$$

where $\boldsymbol{w}^{\varepsilon}_s=E_{Q^{\varepsilon}_s}(\boldsymbol{w}^{\varepsilon})$, *and* C *is independent of* ε.

Theorem 4.2. *Under the hypotheses of Theorem 4.1, let* $\{\boldsymbol{w}^{\varepsilon},p^{\,\varepsilon}\}$ *be the generalized solution of the problem* (3.1)–(3.8), *and let* $\boldsymbol{w}^{\varepsilon}_s=\mathbb{E}_{Q^{\varepsilon}_s}(\boldsymbol{w}^{\varepsilon})$ *be the extension of* $\boldsymbol{w}^{\varepsilon}$ *from* Q^{ε}_s *to* Q. *Define functions* $\pi^{\,\varepsilon}$ *and* $\boldsymbol{u}^{\varepsilon}$ *as in Theorem 4.1.*

Then, up to choosing a subsequence, the sequences $\{\widehat{\chi}^{\,\varepsilon}\pi^{\varepsilon}\}$, $\{\widehat{\chi}^{\,\varepsilon}\boldsymbol{u}^{\varepsilon}\}$ *and* $\{\widehat{\chi}^{\,\varepsilon}\boldsymbol{w}^{\varepsilon}\}$ *converge weakly in* $\mathbb{L}_2(Q\times(0,T))$ *as* $\varepsilon\to 0$ *to certain functions* π_f, $\boldsymbol{u}_f$ *and* $\boldsymbol{w}_f$

respectively, and the sequence $\{\boldsymbol{w}_s^\varepsilon\}$ *converges weakly in* $\mathring{\mathbb{W}}_2^{1,0}(Q\times(0,T))$ *as* $\varepsilon\to 0$ *to a function* $\boldsymbol{w}_s$*.*

These limiting functions are a solution of the compound system consisting of the integral identities

$$\int_0^T\int_Q(\boldsymbol{w}_f+\widehat{m}\boldsymbol{w}_s)\cdot\nabla\psi\,dx\,dt=0,$$

$$\int_0^T\int_Q\big(\widehat{\lambda}(\boldsymbol{x})\big(\widehat{\mathfrak{N}}^s:\mathbb{D}(x,\boldsymbol{w}_s)\big):\mathbb{D}(x,\boldsymbol{\varphi})\big)\,dx\,dt=\int_0^T\int_Q\big(p(\nabla\cdot\boldsymbol{\varphi})+\widehat{\varrho}\boldsymbol{F}\cdot\boldsymbol{\varphi}\big)\,dx\,dt,$$

which hold for all functions ψ *vanishing on the boundary of* Q *and all functions* $\boldsymbol{\varphi}\in\mathring{\mathbb{W}}_2^{1,0}(Q\times(0,T))$*, and of the Darcy law*

$$\boldsymbol{u}_f=\widehat{m}\int_0^t\boldsymbol{w}_s(\boldsymbol{x},\tau)\,d\tau+\frac{1}{\mu_1}\widehat{\mathbb{B}}(\boldsymbol{x})\cdot\big(-\nabla\pi_f+\varrho_f\boldsymbol{\Phi}\big)\tag{4.7}$$

in the domain $Q\times(0,T)$*.*

In terms of differential equations in the corresponding domains and boundary conditions on the common boundary S^0 *for* $t>0$*, the limiting functions* $\boldsymbol{w}_f$*,* $\boldsymbol{w}_s$ *and* π_f *in the domain* Q *for* $t>0$*, where* $\nabla\pi_f,\partial\pi_f/\partial t\in L_2(Q\times(0,T))$*, are a generalized solution* (*in the sense of distributions*) *of the homogenized system of differential equations which consists of the continuity equation*

$$\nabla\cdot\big(\boldsymbol{w}_f+(1-\widehat{m})\boldsymbol{w}_s\big)=0,\tag{4.8}$$

the momentum balance equation

$$\nabla\cdot\widehat{\mathbb{P}}^s+\widehat{\varrho}\boldsymbol{F}=0,\qquad\widehat{\mathbb{P}}^s=\widehat{\lambda}\,\widehat{\mathfrak{N}}^s:\mathbb{D}(x,\boldsymbol{w}_s)-p_f\,\mathbb{I}\tag{4.9}$$

for the rigid component, and the Darcy law (4.7) *for the liquid component.*

The problem is closed by the normalization condition (3.9)*, the boundary condition* (3.7) *for the displacements of the rigid skeleton* $\boldsymbol{w}_s$ *on the exterior boundary* S *for* $t>0$*, the boundary condition*

$$\boldsymbol{w}_f(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})=0\tag{4.10}$$

for the displacements of the liquid on the exterior boundary S *for* $t>0$*, and the continuity conditions*

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}}\boldsymbol{w}_s(\boldsymbol{x},t)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}}\boldsymbol{w}_s(\boldsymbol{x},t),\qquad\boldsymbol{x}^0\in S^0,\tag{4.11}$$

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}}\mathbb{P}^s(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}^0)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}}\mathbb{P}^{s,0}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}^0),\qquad\boldsymbol{x}^0\in S^0,\tag{4.12}$$

$$\begin{aligned}&\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}}\boldsymbol{n}(\boldsymbol{x}^0)\cdot\big(\boldsymbol{w}_f+(1-m)\boldsymbol{w}_s\big)(\boldsymbol{x},t)\\&\qquad=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}}\boldsymbol{n}(\boldsymbol{x}^0)\cdot\big(\boldsymbol{w}_f+(1-m^0)\,\boldsymbol{w}_s\big)(\boldsymbol{x},t),\qquad\boldsymbol{x}^0\in S^0,\end{aligned}\tag{4.13}$$

on the common boundary S^0 *for* $t>0$*.*

Finally, the last missing continuity condition on S^0 depends on the structure of the common porous space. If it is connected, then

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}} \pi(\boldsymbol{x},t) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}} \pi(\boldsymbol{x},t), \qquad \boldsymbol{x}^0\in S^0. \tag{4.14}$$

But if not, then

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}} \boldsymbol{n}(\boldsymbol{x}^0)\cdot\big(\boldsymbol{w}_f - m\,\boldsymbol{w}_s\big)(\boldsymbol{x},t) = 0, \qquad x^0\in S^0. \tag{4.15}$$

In (4.7)–(4.15) we have put $\boldsymbol{\Phi} = \int_0^t \boldsymbol{F}(\boldsymbol{x},\tau)\,d\tau$, the symmetric strictly positive definite matrix $\widehat{\mathbb{B}}(\boldsymbol{x})$ is given by (7.18), and the symmetric strictly positive tensor $\widehat{\mathfrak{N}}^s(\boldsymbol{x})$ of rank four is given by (7.28).

Theorem 4.3. *Under the hypotheses of Theorem 4.1, let $\{\boldsymbol{w}_s^k, \boldsymbol{w}_f^k, \boldsymbol{u}_f^k, \pi_f^k\}$ be the generalized solution of the problem (4.7)–(4.15) with $\lambda^0 = \lambda = k$.*

Then, up to choosing a subsequence, the sequences $\{\pi_f^k\}$, $\{\boldsymbol{u}_f^k\}$, $\{\boldsymbol{w}_f^k\}$ converge weakly in $\mathbb{L}_2(Q\times(0,T))$ as $k\to\infty$ to certain functions π_f, $\boldsymbol{u}_f$, $\boldsymbol{w}_f$ respectively, and the sequence $\{\boldsymbol{w}_s^k\}$ converges to zero strongly in $\mathbb{L}_2(Q\times(0,T))$.

The limiting functions in the domain Q for $t>0$ are a solution of the homogenized system consisting of the continuity equation

$$\nabla\cdot\boldsymbol{u}_f = 0 \tag{4.16}$$

and the Darcy law

$$\boldsymbol{u}_f = \frac{1}{\mu_1}\widehat{\mathbb{B}}\cdot(-\nabla\pi_f + \varrho_f\boldsymbol{\Phi}). \tag{4.17}$$

The problem is closed by the boundary condition (4.10) for the displacements of the liquid on the exterior boundary S for $t>0$, the normalization condition (3.9) for the pressure, and the continuity condition

$$\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega}} \boldsymbol{n}(\boldsymbol{x}^0)\cdot\boldsymbol{w}_f(\boldsymbol{x},t) = \lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}^0\\ \boldsymbol{x}\in\Omega^0}} \boldsymbol{n}(\boldsymbol{x}^0)\cdot\boldsymbol{w}_f(\boldsymbol{x},t), \qquad \boldsymbol{x}^0\in S^0, \tag{4.18}$$

on the common boundary S^0 for $t>0$.

If the common porous space is connected, then the boundary condition (4.14) holds on the common boundary S^0 for $t>0$. But if not, then the boundary condition (4.15) holds on S^0 for $t>0$.

The symmetric strictly positive definite matrix $\widehat{\mathbb{B}}(\boldsymbol{x})$ is defined in Theorem 4.2.

§ 5. Auxiliary assertions

5.1. Two-scale convergence.

Definition 5.1. A sequence $\{w^\varepsilon\}\subset\mathbb{L}_2(\Omega_T)$, $\Omega_T = \Omega\times(0,T)$, is said to be *two-scale convergent* to a 1-periodic function $W(\boldsymbol{x},t,\boldsymbol{y})\in\mathbb{L}_2(\Omega_T\times Y)$ of $\boldsymbol{y}\in Y$ ($w^\varepsilon \xrightarrow{\text{t.-s.}} W(\boldsymbol{x},t,\boldsymbol{y})$) if for every 1-periodic function $\sigma=\sigma(\boldsymbol{x},t,\boldsymbol{y})$ of $\boldsymbol{y}$ we have

$$\lim_{\varepsilon\to0}\int_{\Omega_T} w^\varepsilon(\boldsymbol{x},t)\sigma\Big(\boldsymbol{x},t,\frac{\boldsymbol{x}}{\varepsilon}\Big)\,dx\,dt = \int_{\Omega_T}\Big(\int_Y W(\boldsymbol{x},t,\boldsymbol{y})\sigma(\boldsymbol{x},t,\boldsymbol{y})\,dy\Big)\,dx\,dt,$$

where $Y=(0,1)^3$ is the periodicity cell.

The following theorem establishes the existence and basic properties of two-scale convergent sequences.

Theorem 5.1 (Nguetseng's theorem). 1) *Every bounded sequence* $\{\boldsymbol{w}^\varepsilon\}$ *in* $\mathbb{L}_2(\Omega_T)$ *contains a subsequence two-scale convergent to a function* $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, *where* $\boldsymbol{W} \in \mathbb{L}_2(\Omega_T \times Y)$ *is* 1*-periodic with respect to* $\boldsymbol{y}$.

2) *Let* $\{\boldsymbol{w}^\varepsilon\}$ *and* $\{\varepsilon\mathbb{D}(x,\boldsymbol{w}^\varepsilon)\}$ *be uniformly bounded sequences in* $\mathbb{L}_2(\Omega_T)$.

Then there are a 1*-periodic function* $\boldsymbol{W} = \boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$ *of* $\boldsymbol{y}$ *and a subsequence of* $\{\boldsymbol{w}^\varepsilon\}$ *such that* $\boldsymbol{W}, \nabla_y \boldsymbol{W} \in \mathbb{L}_2(\Omega_T \times Y)$, *and the subsequences of* $\{\boldsymbol{w}^\varepsilon\}$ *and* $\{\varepsilon\mathbb{D}(x,\boldsymbol{w}^\varepsilon)\}$ *are two-scale convergent in* $\mathbb{L}_2(\Omega_T)$ *to* $\boldsymbol{W}$ *and* $\mathbb{D}(y,\boldsymbol{W})$ *respectively.*

3) *Let* $\{\boldsymbol{w}^\varepsilon\}$ *and* $\{D(x,\boldsymbol{w}^\varepsilon)\}$ *be bounded sequences in* $\mathbb{L}_2(\Omega_T)$. *Then there are functions* $\boldsymbol{w}(\boldsymbol{x},t)$, $\boldsymbol{w} \in \mathbb{W}_2^{1,0}(\Omega_T)$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W} \in \mathbb{L}_2(\Omega_T \times Y) \cap \mathbb{W}_2^{1,0}(Y)$ *and a subsequence of* $\{\mathbb{D}(x,\boldsymbol{w}^\varepsilon)\}$ *such that* $\boldsymbol{W}$ *is* 1*-periodic with respect to* $\boldsymbol{y}$, $\mathbb{D}(x,\boldsymbol{w}) \in \mathbb{L}_2(\Omega_T)$, $D(y,\boldsymbol{W}) \in \mathbb{L}_2(\Omega_T \times Y)$, *and the subsequence of* $\{\mathbb{D}(x,\boldsymbol{w}^\varepsilon)\}$ *is two-scale convergent to* $\mathbb{D}(x,\boldsymbol{w}) + D(y,\boldsymbol{W})$.

Note that weak convergence and two-scale convergence are related as follows:

$$\text{if } \boldsymbol{w}^\varepsilon \xrightarrow{\text{t.-s.}} \boldsymbol{W}(\boldsymbol{x},\boldsymbol{y}) \quad \text{(two-scale)},$$

$$\text{then } \boldsymbol{w}^\varepsilon(\boldsymbol{x}) \rightharpoonup \int_Y \boldsymbol{W}(\boldsymbol{x},\boldsymbol{y})d\boldsymbol{y} \quad \text{(weakly)}.$$

Lemma 5.1. *Suppose that the geometry of periodic structures satisfies Assumptions 3.1–3.3 and the sequence* $\{\boldsymbol{w}_s^\varepsilon\}$ *converges weakly in* $\mathring{\mathbb{W}}_2^{1,0}(Q \times (0,T))$ *to* $\boldsymbol{w}_s(\boldsymbol{x},t)$. *Then* $\boldsymbol{w}_s \in \mathring{\mathbb{W}}_2^{1,0}(Q \times (0,T))$.

Proof. The proof is obvious. By construction, the solenoidal function $\boldsymbol{w}_s^\varepsilon$ vanishes on S. Therefore,

$$\int_0^T \int_Q \boldsymbol{w}_s^\varepsilon \cdot \nabla\varphi \, dx \, dt = 0$$

for an arbitrary function $\varphi \in \mathbb{W}_2^{1,0}\big(Q \times (0,T)\big)$. Letting $\varepsilon \to 0$ in this identity, we obtain that

$$\int_0^T \int_Q \boldsymbol{w}_s \cdot \nabla\varphi \, dx \, dt = 0.$$

Since $\varphi \in \mathbb{W}_2^{1,0}\big(Q \times (0,T)\big)$ is arbitrary, this is equivalent to the conclusion of the lemma. □

Lemma 5.2. *For every unit vector* $\boldsymbol{e}$ *there is a solenoidal vector-valued function* $\boldsymbol{u}(\boldsymbol{y})$ *with* $\operatorname{supp} \boldsymbol{u} \subset Y_f$ *such that*

$$\langle \boldsymbol{u} \rangle_Y = \int_Y \boldsymbol{u}(\boldsymbol{y}) \, dy = \boldsymbol{e}. \tag{5.1}$$

Proof. Take a ball $B \subset Y_f$ and let $\boldsymbol{v}(\boldsymbol{y})$ be a solution of the problem

$$\triangle \boldsymbol{v} - \nabla p = \lambda \boldsymbol{f}, \qquad \boldsymbol{y} \in B, \tag{5.2}$$

$$\nabla \cdot \boldsymbol{v} = 0, \qquad \boldsymbol{y} \in B, \tag{5.3}$$

$$\int_B p \, dy = 0, \qquad \boldsymbol{v}(\boldsymbol{y}) = 0, \quad \boldsymbol{y} \in \partial B, \tag{5.4}$$

with a fixed function $\boldsymbol{f}(\boldsymbol{y})$ and an arbitrary constant λ.

We choose the constant λ in such a way that

$$|\boldsymbol{e}_0| = 1, \qquad \text{where} \quad \boldsymbol{e}_0 = \int_B \boldsymbol{v}\, dy.$$

In what follows, let $\mathbb{T}$ be an orthogonal matrix such that $\mathbb{T} \cdot \boldsymbol{e}_0 = \boldsymbol{e}$.

In the new variables $\boldsymbol{z} = \mathbb{T} \cdot \boldsymbol{y}$, the function $\boldsymbol{u}(\boldsymbol{z}) = \mathbb{T} \cdot \boldsymbol{v}(\boldsymbol{y})$ is a solution of the problem

$$\triangle_z \boldsymbol{u} - \nabla_z q = \boldsymbol{F}(\boldsymbol{z}), \quad \nabla_z \cdot \boldsymbol{u} = 0, \qquad \boldsymbol{z} \in B,$$

$$\int_B q\, dy = 0, \qquad \boldsymbol{u}(\boldsymbol{z}) = 0, \quad \boldsymbol{z} \in \partial B,$$

where $\boldsymbol{F} = \mathbb{T} \cdot \boldsymbol{f}$ and $q(\boldsymbol{z}) = p(\boldsymbol{y})$. By construction,

$$\boldsymbol{e} = \mathbb{T} \cdot \boldsymbol{e}_0 = \int_B \mathbb{T} \cdot \boldsymbol{v}\, dy = \int_B \boldsymbol{u}\, dz. \qquad \square$$

§ 6. Proof of Theorem 4.1

For convenience, we split the proof of the theorem into several lemmas.

Lemma 6.1. *For every* $\varepsilon > 0$ *the problem* (3.1)–(3.8) *has a unique generalized solution such that*

$$\boldsymbol{w}^\varepsilon \in \mathring{\mathbb{W}}^{1,0}_2\big(Q \times (0,T)\big), \qquad \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t} \in \mathbb{W}^{1,0}_2\big(Q^\varepsilon_f \times (0,T)\big), \qquad p^\varepsilon \in \mathbb{L}_2(Q \times (0,T))$$

and the first bound in (4.6) *holds.*

Proof. The proof is standard. It uses a derivation of the *a priori* bounds (4.6) and Galerkin's method. Putting $t_0 = T$ and $\boldsymbol{\varphi}(\boldsymbol{x}, \tau) = \zeta_t(\tau) \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau}(\boldsymbol{x}, \tau)$ in (4.2), where $\zeta_t(\tau) = 1$ for $0 < \tau < t$ and $\zeta_t(\tau) = 0$ for $\tau > t$, we obtain that

$$\int_0^t \int_Q \alpha_\mu \widehat{\chi}^\varepsilon \Big| \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial \tau}\Big)\Big|^2 dx\, d\tau + \frac{1}{2} \int_Q (1 - \widehat{\chi}^\varepsilon) \widehat{\lambda}(\boldsymbol{x}) |\mathbb{D}(x, \boldsymbol{w}^\varepsilon)|^2 (\boldsymbol{x}, t)\, dx$$
$$= - \int_0^t \int_Q \widehat{\varrho}^\varepsilon \frac{\partial \boldsymbol{F}^\varepsilon}{\partial \tau}(\boldsymbol{x}, \tau) \cdot \boldsymbol{w}^\varepsilon\, dx\, d\tau \leqslant C F^2_\varepsilon \int_0^T \int_Q |\mathbb{D}(x, \boldsymbol{w}^\varepsilon)|^2\, dx\, dt.$$

This yields the desired bound

$$\int_0^T \int_Q \alpha_\mu \widehat{\chi}^\varepsilon \Big| \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\Big)\Big|^2 dx\, dt$$
$$+ \max_{0<t<T} \int_Q (1 - \widehat{\chi}^\varepsilon) \widehat{\lambda}(\boldsymbol{x}) |\mathbb{D}(x, \boldsymbol{w}^\varepsilon)|^2 (\boldsymbol{x}, t)\, dx \leqslant C F^2_\varepsilon. \tag{6.1}$$

The main integral identity in the construction of approximate solutions is obtained from (4.2) with $t_0 = t$ by integrating the resulting expression with respect to time and choosing test functions in the special class $\mathring{\mathbb{S}}^1_2(Q)$ of solenoidal functions in $\mathring{\mathbb{W}}^1_2(Q)$: $\boldsymbol{\varphi} \in \mathring{\mathbb{S}}^1_2(Q) \subset \mathring{\mathbb{W}}^1_2(Q)$:

$$\begin{aligned}\int_Q \alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(\boldsymbol{x})\Big) : \mathbb{D}\big(x, \boldsymbol{\varphi}(\boldsymbol{x}, t)\big)\, dx \\ + \int_Q (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\big) : \mathbb{D}\big(x, \boldsymbol{\varphi}(\boldsymbol{x}, t)\big)\, dx \\ = \int_Q \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon(\boldsymbol{x}, t) \cdot \boldsymbol{\varphi}(\boldsymbol{x}, t)\, dx. \end{aligned} \tag{6.2}$$

Here we have used Assumption 3.4.

Let $\{\boldsymbol{\varphi}_k\}$, $k = 1, 2, \dots$, be an orthonormal basis in $\mathring{\mathbb{S}}^1_2(Q)$. We put

$$\boldsymbol{w}^\varepsilon_N(\boldsymbol{x}, t) = \sum_{k=1}^N \frac{dc^k_N}{dt}(t)\boldsymbol{\varphi}_k(\boldsymbol{x}), \tag{6.3}$$

where the functions c^k_N are determined using the following system of linear differential equations with constant coefficients:

$$\begin{aligned} \sum_{k=1}^N a^{k,l}\frac{dc^k_N}{dt} + \sum_{k=1}^N b^{k,l}c^k_N = F^l, \qquad l = 1, 2, \dots, N; \\ a^{k,l} = \int_Q \alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}(x, \boldsymbol{\varphi}_k) : \mathbb{D}(x, \boldsymbol{\varphi}_l)\, dx, \\ b^{k,l} = \int_Q (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{\varphi}_k) : \mathbb{D}(x, \boldsymbol{\varphi}_l)\, dx, \qquad k, l = 1, 2, \dots, N; \\ F^l = \int_Q \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon \cdot \boldsymbol{\varphi}_l\, dx, \qquad l = 1, 2, \dots, N. \end{aligned} \tag{6.4}$$

Since

$$\sum_{k,l=1}^N a^{k,l}\xi^k\xi^l = \int_Q \alpha_\mu \widehat{\chi}^\varepsilon |\mathbb{D}(x, \boldsymbol{e})|^2\, dx > 0$$

for all $\boldsymbol{e} = \sum_{k=1}^N \xi^k \boldsymbol{\varphi}_k$ with $\sum_{k=1}^N |\xi^k|^2 = 1$, the system (6.4) has a unique solution. Thus we have constructed approximate solutions for all $N = 1, 2, \dots$.

Consider also the function

$$\boldsymbol{u}^\varepsilon_N(\boldsymbol{x}, t) = \sum_{k=1}^N c^k_N(t)\boldsymbol{\varphi}_k(\boldsymbol{x}), \quad l = 1, 2, \dots, N, \qquad \boldsymbol{w}^\varepsilon_N = \frac{\partial \boldsymbol{u}^\varepsilon_N}{\partial t}. \tag{6.5}$$

We easily see that $\boldsymbol{w}^\varepsilon_N$ and $\boldsymbol{u}^\varepsilon_N$ satisfy the integral identities

$$\begin{aligned}\int_0^{t_0}\int_Q \bigg(\alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\Big(x, \frac{\partial \boldsymbol{w}^\varepsilon_N}{\partial t}\Big) + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{w}^\varepsilon_N)\bigg) : \mathbb{D}(x, \boldsymbol{\varphi}^N)\, dx\, dt \\ = \int_0^{t_0}\int_Q \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon \cdot \boldsymbol{\varphi}^N\, dx\, dt, \end{aligned} \tag{6.6}$$

$$\int_0^{t_0}\int_Q\Big(\alpha_\mu\widehat{\chi}^\varepsilon\mathbb{D}\Big(x,\frac{\partial\boldsymbol{u}_N^\varepsilon}{\partial t}\Big)+(1-\widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x,\boldsymbol{u}_N^\varepsilon)\Big):\mathbb{D}(x,\boldsymbol{\varphi}^N)\,dx\,dt$$
$$=\int_0^{t_0}\int_Q\widehat{\varrho}^\varepsilon\boldsymbol{\Phi}^\varepsilon\cdot\boldsymbol{\varphi}^N\,dx\,dt, \tag{6.7}$$

where

$$\boldsymbol{\Phi}^\varepsilon(\boldsymbol{x},t)=\int_0^t\boldsymbol{F}^\varepsilon(\boldsymbol{x},\tau)\,d\tau,\qquad \boldsymbol{\varphi}^N=\sum_{k=1}^N\xi^k(t)\boldsymbol{\varphi}_k(\boldsymbol{x}),$$

with arbitrary functions $\xi^k(t)$, and the following estimates hold:

$$\int_0^T\int_Q\alpha_\mu\widehat{\chi}^\varepsilon\Big|\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}_N^\varepsilon}{\partial t}\Big)\Big|^2\,dx\,dt$$
$$+\max_{0<t<T}\int_Q(1-\widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\big|\mathbb{D}\big(x,\boldsymbol{w}_N^\varepsilon(\boldsymbol{x},t)\big)\big|^2\,dx\leqslant C_\varepsilon F_\varepsilon^2, \tag{6.8}$$

$$\int_0^T\int_Q\alpha_\mu\widehat{\chi}^\varepsilon\Big|\mathbb{D}\Big(x,\frac{\partial\boldsymbol{u}_N^\varepsilon}{\partial t}\Big)\Big|^2\,dx\,dt$$
$$+\max_{0<t<T}\int_Q(1-\widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\big|\mathbb{D}\big(x,\boldsymbol{u}_N^\varepsilon(\boldsymbol{x},t)\big)\big|^2\,dx\leqslant C_\varepsilon F^2, \tag{6.9}$$

where the constant C_ε, generally speaking, depends on ε.

Passing to the limit as $N\to\infty$, we obtain the desired generalized solution of (3.1)–(3.8). It satisfies the integral identities

$$\int_0^{t_0}\int_Q\Big(\alpha_\mu\widehat{\chi}^\varepsilon\mathbb{D}\Big(x,\frac{\partial\boldsymbol{w}^\varepsilon}{\partial t}\Big)+(1-\widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x,\boldsymbol{w}^\varepsilon)\Big):\mathbb{D}(x,\boldsymbol{\varphi})\,dx\,dt$$
$$=\int_0^{t_0}\int_Q\widehat{\varrho}^\varepsilon\boldsymbol{F}^\varepsilon\cdot\boldsymbol{\varphi}\,dx\,dt, \tag{6.10}$$

$$\int_0^{t_0}\int_Q\Big(\alpha_\mu\widehat{\chi}^\varepsilon\mathbb{D}\Big(x,\frac{\partial\boldsymbol{u}^\varepsilon}{\partial t}\Big)+(1-\widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x,\boldsymbol{u}^\varepsilon)\Big):\mathbb{D}(x,\boldsymbol{\varphi})\,dx\,dt$$
$$=\int_0^{t_0}\int_Q\widehat{\varrho}^\varepsilon\boldsymbol{\Phi}^\varepsilon\cdot\boldsymbol{\varphi}\,dx\,dt \tag{6.11}$$

and the equality

$$\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)=\frac{\partial\boldsymbol{u}^\varepsilon}{\partial t}(\boldsymbol{x},t). \tag{6.12}$$

Note that we passed to the limit in the integral identity (4.2) with test functions of class $\mathring{\mathbb{S}}_2^1(Q)$, where the pressure is absent because the functions of this class are orthogonal in $\mathbb{L}_2\big(Q\times(0,T)\big)$ to the gradients of scalar-valued functions.

Therefore the pressure p^ε does not occur in the integral identity (6.10). This identity will take the form (4.2) with pressure $p^\varepsilon\in\mathbb{L}_2\big(Q\times(0,T)\big)$ if we are considering non-solenoidal test functions.

In a similar way, the identity (6.11) in the case of non-solenoidal test functions $\boldsymbol{\psi} \in \mathring{\mathbb{W}}^1_2(Q)$ will take the form

$$\int_0^{t_0} \int_Q \left(\alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\left(x, \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t}\right) + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{u}^\varepsilon) \right) : \mathbb{D}(x, \boldsymbol{\psi})\, dx\, dt$$
$$= \int_0^{t_0} \int_Q (\pi^\varepsilon\, \nabla \cdot \boldsymbol{\psi} + \widehat{\varrho}^\varepsilon \boldsymbol{\Phi}^\varepsilon \cdot \boldsymbol{\psi})\, dx\, dt, \tag{6.13}$$

where

$$\pi^\varepsilon(\boldsymbol{x}, t) = \int_0^t p^\varepsilon(\boldsymbol{x}, \tau)\, d\tau \in \mathbb{L}_2\big(Q \times (0, T)\big). \tag{6.14}$$

The proof of (6.14) is standard (see [17]): choose $\boldsymbol{\psi} = \partial \boldsymbol{\varphi}/\partial t$ in (6.13), integrate by parts and compare with (4.2).

Finally, the function π^ε belongs to $\mathbb{L}_2\big(Q \times (0,T)\big)$ because this space is the direct sum of the closure of the space of solenoidal functions and the closure of the space of gradients of scalar functions [20]. □

Lemma 6.2. *The bounds* (4.6) *hold under the hypotheses of Theorem* 4.1.

Proof. Put

$$\beta^\varepsilon(t) = \int_Q \left(\alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\left(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\right) : \mathbb{D}(x, \boldsymbol{\varphi}) \right.$$
$$\left. + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{w}^\varepsilon) : \mathbb{D}(x, \boldsymbol{\varphi}) - \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon \cdot \boldsymbol{\varphi} \right) dx.$$

Then (6.5) takes the form

$$\int_0^{t_0} \beta^\varepsilon(t)\, dt = 0 \quad \text{for all } t_0 < T,$$

or

$$\beta^\varepsilon(t_0) = 0 \quad \text{for almost all } t_0 < T$$

or

$$\int_Q \left(\alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\left(x, \frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}(\boldsymbol{x}, t_0)\right) : \mathbb{D}\big(x, \boldsymbol{\varphi}(\boldsymbol{x}, t_0)\big) \right.$$
$$\left. + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t_0)\big) : \mathbb{D}\big(x, \boldsymbol{\varphi}(\boldsymbol{x}, t_0)\big) - \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon \cdot \boldsymbol{\varphi}(\boldsymbol{x}, t_0) \right) dx = 0$$

for almost all $t_0 < T$. The last identity also holds for all functions $\boldsymbol{\varphi}(\boldsymbol{x}, t)$ that are solenoidal with respect to the spatial variable and vanish for $\boldsymbol{x} \in S$.

Putting $\boldsymbol{\varphi} = \boldsymbol{w}^\varepsilon$ in this identity, we obtain that

$$\frac{1}{2} \int_Q \alpha_\mu \widehat{\chi}^\varepsilon \big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t_0)\big)\big|^2 dx + \int_0^{t_0} \int_Q (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 dx\, dt$$
$$= \int_0^{t_0} \int_Q \widehat{\varrho}^\varepsilon \boldsymbol{F}^\varepsilon \cdot \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\, dx\, dt,$$

or

$$\int_Q \alpha_\mu \widehat{\chi}^\varepsilon \big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t_0)\big)\big|^2 \, dx + \int_0^{t_0} \int_Q (1 - \widehat{\chi}^\varepsilon) \widehat{\lambda}(\boldsymbol{x}) \big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 \, dx \, dt$$
$$\leqslant \frac{F^2}{\delta} + \delta \int_0^{t_0} \int_Q |\boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)|^2 \, dx \, dt.$$

Put $\boldsymbol{w}_s^\varepsilon = E_{Q_s^\varepsilon}(\boldsymbol{w}^\varepsilon)$. Then

$$\int_Q \alpha_\mu \widehat{\chi}^\varepsilon \big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t_0)\big)\big|^2 \, dx + \int_0^{t_0} \int_Q \widehat{\lambda}(\boldsymbol{x}) \big|\mathbb{D}\big(x, \boldsymbol{w}_s^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 \, dx \, dt$$
$$\leqslant \frac{F^2}{\delta} + \delta \int_0^{t_0} \int_Q |\boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)|^2 \, dx \, dt. \tag{6.15}$$

Consider the function $\boldsymbol{u}^{\varepsilon,\boldsymbol{k}} = \boldsymbol{w}^\varepsilon - \boldsymbol{w}_s^\varepsilon$ on $Q_f^{\varepsilon,\boldsymbol{k}}$, where $\boldsymbol{w}_s^\varepsilon = E_{Q_s^\varepsilon}(\boldsymbol{w}^\varepsilon)$ (see Lemma 4.1). By construction, $\boldsymbol{u}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{x}, t) = 0$ for $\boldsymbol{x} \in Q_s^{\varepsilon,\boldsymbol{k}}$. Therefore,

$$\int_{Q_f^{\varepsilon,\boldsymbol{k}}} |\boldsymbol{u}^{\varepsilon,\boldsymbol{k}}|^2 \, dx \leqslant C\varepsilon^2 \int_{Q_f^{\varepsilon,\boldsymbol{k}}} |D(x, \boldsymbol{u}^{\varepsilon,\boldsymbol{k}})|^2 \, dx, \tag{6.16}$$

where C is independent of ε.

Indeed, put $\widehat{\boldsymbol{u}}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{y}, t) = \boldsymbol{u}^{\varepsilon,\boldsymbol{k}}(\varepsilon\boldsymbol{k} + \varepsilon\boldsymbol{y}, t)$. Then $D(y, \widehat{\boldsymbol{u}}^{\varepsilon,\boldsymbol{k}}) = \varepsilon D(x, \boldsymbol{u}^{\varepsilon,\boldsymbol{k}})$. Using the Poincaré–Friedrichs inequality (4.5), we see that

$$\int_{Q_f^{\varepsilon,\boldsymbol{k}}} |\boldsymbol{u}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{x}, t)|^2 \, dx = \int_{Y^{\varepsilon,\boldsymbol{k}}} |\widehat{\boldsymbol{u}}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{y}, t)|^2 \, dy$$
$$\leqslant C \int_{Y^{\varepsilon,\boldsymbol{k}}} \big|\mathbb{D}\big(y, \widehat{\boldsymbol{u}}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{y}, t)\big)\big|^2 \, dy = C\varepsilon^2 \int_{Q_f^{\varepsilon,\boldsymbol{k}}} \big|\mathbb{D}\big(x, \boldsymbol{u}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{x}, t)\big|^2 \, dx,$$

$$\int_Q |\boldsymbol{w}^\varepsilon(\boldsymbol{x}, t) - \boldsymbol{w}_s^\varepsilon(\boldsymbol{x}, t)|^2 \, dx = \sum_{\boldsymbol{k}} \int_{Q_f^{\varepsilon,\boldsymbol{k}}} |\boldsymbol{u}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{x}, t)|^2 \, dx$$
$$= \sum_{\boldsymbol{k}} \int_{Y^{\varepsilon,\boldsymbol{k}}} |\widehat{\boldsymbol{u}}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{y}, t)|^2 \, dy \leqslant C \sum_{\boldsymbol{k}} \int_{Y^{\varepsilon,\boldsymbol{k}}} \big|\mathbb{D}\big(y, \widehat{\boldsymbol{u}}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{y}, t)\big)\big|^2 \, dy$$
$$= C\varepsilon^2 \sum_{\boldsymbol{k}} \int_{Q_f^{\varepsilon,\boldsymbol{k}}} \big|\mathbb{D}\big(x, \boldsymbol{u}^{\varepsilon,\boldsymbol{k}}(\boldsymbol{x}, t)\big)\big|^2 \, dx = C\varepsilon^2 \int_Q \big|\mathbb{D}\big(x, (\boldsymbol{w}^\varepsilon - \boldsymbol{w}_s^\varepsilon)(\boldsymbol{x}, t)\big)\big|^2 \, dx$$
$$\leqslant C\varepsilon^2 \int_Q \big|\mathbb{D}\big(x, \boldsymbol{w}_s^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 \, dx + C \frac{\varepsilon^2}{\alpha_\mu} \int_Q \widehat{\chi}^\varepsilon \alpha_\mu \big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 \, dx.$$

Thus,

$$\int_0^{t_0} \int_Q |\boldsymbol{w}^\varepsilon(\boldsymbol{x}, t) - \boldsymbol{w}_s^\varepsilon(\boldsymbol{x}, t)|^2 \, dx \, dt$$
$$\leqslant C\varepsilon^2 \int_Q \big|\mathbb{D}\big(x, \boldsymbol{w}_s^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 \, dx + C \frac{\varepsilon^2}{\alpha_\mu} \int_Q \widehat{\chi}^\varepsilon \alpha_\mu \big|\mathbb{D}\big(x, \boldsymbol{w}^\varepsilon(\boldsymbol{x}, t)\big)\big|^2 \, dx, \tag{6.17}$$

$$\int_Q |\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)|^2\,dx \leqslant \int_Q |\boldsymbol{w}_s^\varepsilon(\boldsymbol{x},t)|^2\,dx + \int_Q |(\boldsymbol{w}^\varepsilon - \boldsymbol{w}_s^\varepsilon)(\boldsymbol{x},t)|^2\,dx$$
$$\leqslant C\int_Q \bigl|\mathbb{D}\bigl(x,\boldsymbol{w}_s^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx + C\frac{\varepsilon^2}{\alpha_\mu}\int_Q \widehat{\chi}^\varepsilon\alpha_\mu\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx, \tag{6.18}$$

where the constant C is independent of ε.

Again taking (6.15) and choosing a sufficiently small δ, we obtain that

$$\int_Q \alpha_\mu\widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t_0)\bigr)\bigr|^2\,dx + \int_0^{t_0}\int_Q \widehat{\lambda}(\boldsymbol{x})\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}_s^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx\,dt$$
$$\leqslant CF^2 + C\frac{\varepsilon^2}{\alpha_\mu}\int_0^{t_0}\int_Q \alpha_\mu\widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx\,dt,$$

or

$$\begin{aligned}
&\int_Q \alpha_\mu\widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t_0)\bigr)\bigr|^2\,dx \\
&\qquad\leqslant CF^2 + C\frac{\varepsilon^2}{\alpha_\mu}\int_0^{t_0}\int_Q \alpha_\mu\widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx\,dt, \\
&\int_0^{t_0}\int_Q \widehat{\lambda}(\boldsymbol{x})\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}_s^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx\,dt \\
&\qquad\leqslant CF^2 + C\int_0^{t_0}\int_Q \varepsilon^2\widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx\,dt.
\end{aligned} \tag{6.19}$$

Putting

$$y(t) = \int_Q \alpha_\mu\widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx,$$

the first relation in (6.19) takes the form

$$\frac{dy}{dt}(t) \leqslant CF^2 + C\frac{\varepsilon^2}{\alpha_\mu}y(t), \qquad y(0)=0.$$

We can now use Gronwall's inequality [17]

$$y(t) \leqslant \frac{\alpha_\mu}{\varepsilon^2}CF^2$$

or

$$\max_{0<t<T}\varepsilon^2\int_Q \widehat{\chi}^\varepsilon\bigl|\mathbb{D}\bigl(x,\boldsymbol{w}^\varepsilon(\boldsymbol{x},t)\bigr)\bigr|^2\,dx \leqslant CF^2. \tag{6.20}$$

Along with the representation $\partial\boldsymbol{u}^\varepsilon/\partial t = \boldsymbol{w}^\varepsilon$, this proves the second bound in (4.6).

The third bound follows from (6.19), the second bound and the Friedrichs–Poincaré inequality (4.5). The fourth bound follows from (6.18) and the first three.

We again stress that the constants C in our calculations are distinct (generally speaking) but independent of the small parameter ε. Since there are finitely many bounds in our paper, we do not distinguish these constants and denote them all by C.

To prove the bound (4.6) for π^ε, we write the identity (6.13) in the form

$$\Pi(\boldsymbol{\psi}) \equiv \int_0^T \int_Q \pi^\varepsilon \nabla \cdot \boldsymbol{\psi}\, dx\, dt = \int_0^T \int_Q \big(\mathbb{A} : \mathbb{D}(x, \boldsymbol{\psi}) + \boldsymbol{a} \cdot \boldsymbol{\psi}\big)\, dx\, dt, \tag{6.21}$$

where

$$\mathbb{A} = \alpha_\mu \widehat{\chi}^\varepsilon \mathbb{D}\bigg(x, \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t}\bigg) + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{u}^\varepsilon), \qquad \boldsymbol{a} = \widehat{\varrho}^\varepsilon \boldsymbol{\Phi}^\varepsilon,$$

$$\int_0^T \int_Q (\mathbb{A} : \mathbb{A} + |\boldsymbol{a}|^2)\, dx\, dt \leqslant \frac{\varepsilon^2}{\alpha_\mu} C F^2 \leqslant \mu_1 C F^2$$

by the bounds (4.6) and the hypotheses.

Since

$$\bigg|\int_0^T \int_Q \pi^\varepsilon \nabla \cdot \boldsymbol{\psi}\, dx\, dt\bigg| \leqslant \int_0^T \int_Q (\mathbb{A} : \mathbb{A} + |\boldsymbol{a}|^2)\, dx\, dt\, \|\boldsymbol{\psi}\|^{1,0}_{2, Q\times(0,T)} \leqslant \mu_1 C F^2,$$

the linear functional $\Pi(\boldsymbol{\psi})$, $\Pi\colon \mathring{\mathbb{W}}^{1,0}_2\big(Q \times (0,T)\big)\big) \to \mathbb{R}$ is bounded. By the theorem on the representation of continuous linear functionals on a Hilbert space [21], we conclude that

$$\int_0^T \int_Q |\pi^\varepsilon(\boldsymbol{x}, t)|^2\, dx\, dt \leqslant \int_0^T \int_Q (\mathbb{A} : \mathbb{A} + |\boldsymbol{a}|^2)\, dx\, dt \leqslant \mu_1 C F^2. \tag{6.22}$$

This completes the proof of Theorem 4.1. □

§ 7. Proof of Theorem 4.2

Suppose that $\{\boldsymbol{w}^\varepsilon, \pi^\varepsilon\}$ is a solution of the problem (3.1)–(3.8),

$$\boldsymbol{w}^\varepsilon_s = \mathbb{E}_{Q^\varepsilon_s}\big(\boldsymbol{w}^\varepsilon\big), \quad \boldsymbol{w}^\varepsilon_s = \frac{\partial \boldsymbol{u}^\varepsilon_s}{\partial t} \quad \text{and} \quad \boldsymbol{w}^\varepsilon = \frac{\partial \boldsymbol{u}^\varepsilon}{\partial t}.$$

The bounds (4.6) enable us to choose subsequences (we preserve the same subscripts here and in what follows)

$$\{\boldsymbol{w}^\varepsilon\}, \quad \{\widehat{\chi}^\varepsilon \boldsymbol{w}^\varepsilon\}, \quad \{\boldsymbol{u}^\varepsilon\}, \quad \{\widehat{\chi}^\varepsilon \boldsymbol{u}^\varepsilon\}, \quad \pi^\varepsilon, \quad \{\widehat{\chi}^\varepsilon \pi^\varepsilon\}, \quad \{(1 - \widehat{\chi}^\varepsilon)\pi^\varepsilon\},$$
$$\{\boldsymbol{u}^\varepsilon_s\}, \quad \{\boldsymbol{w}^\varepsilon_s\}, \quad \{\mathbb{D}(x, \boldsymbol{u}^\varepsilon_s)\} \quad \text{and} \quad \{\mathbb{D}(x, \boldsymbol{w}^\varepsilon_s)\},$$

converging weakly in $\mathbb{L}_2\big(Q \times (0,T)\big)$ to functions

$$\boldsymbol{w}, \quad \boldsymbol{w}_f, \quad \boldsymbol{u}, \quad \boldsymbol{u}_f, \quad \pi, \quad \pi_f, \quad \pi_s, \quad \boldsymbol{u}_s, \quad \boldsymbol{w}_s, \quad \mathbb{D}(x, \boldsymbol{u}_s) \quad \text{and} \quad \mathbb{D}(x, \boldsymbol{w}_s)$$

respectively.

By Theorem 5.1, there are functions 1-periodic (with respect to $\boldsymbol{y}$)

$$\widehat{\boldsymbol{W}} = (1 - \zeta)\boldsymbol{W} + \zeta \boldsymbol{W}^0, \qquad \widehat{\boldsymbol{W}}_f = \widehat{\boldsymbol{W}} \chi, \qquad \widehat{\boldsymbol{U}} = (1 - \zeta)\boldsymbol{U} + \zeta \boldsymbol{U}^0,$$
$$\widehat{\boldsymbol{U}}_f = \widehat{\boldsymbol{U}} \widehat{\chi}, \qquad \widehat{\Pi} = (1 - \zeta)\Pi + \zeta \Pi^0, \qquad \widehat{\Pi}_f = \widehat{\Pi} \widehat{\chi} \quad \text{and} \quad \widehat{\Pi}_s = \widehat{\Pi}(1 - \widehat{\chi})$$

such that the subsequences chosen are two-scale convergent in $\mathbb{L}_2\big(Q\times(0,T)\big)$ to the functions (respectively)

$$\widehat{\boldsymbol{W}},\quad \widehat{\boldsymbol{W}}_f,\quad \widehat{\boldsymbol{U}},\quad \widehat{\boldsymbol{U}}_f,\quad \widehat{\Pi},\quad \widehat{\Pi}_f,\quad \widehat{\Pi}_s,$$
$$\boldsymbol{u}_s,\quad \boldsymbol{w}_s,\quad \mathbb{D}(x,\boldsymbol{u}_s)+\mathbb{D}(y,\widehat{\boldsymbol{U}}_s)\quad \text{and}\quad \mathbb{D}(x,\boldsymbol{w}_s)+\mathbb{D}(y,\widehat{\boldsymbol{W}}_s).$$

Moreover, $\widehat{\boldsymbol{W}}=\widehat{\boldsymbol{W}}_f+\widehat{\boldsymbol{W}}_s$, $\widehat{\boldsymbol{U}}=\widehat{\boldsymbol{U}}_f+\widehat{\boldsymbol{U}}_s$, $\widehat{\Pi}=\widehat{\Pi}_f+\widehat{\Pi}_s$, and the sequences $\{\varepsilon\widehat{\chi}^\varepsilon\mathbb{D}(x,\boldsymbol{u}^\varepsilon)\}$ and $\{\varepsilon\widehat{\chi}^\varepsilon\mathbb{D}(x,\boldsymbol{w}^\varepsilon)\}$ are two-scale convergent in $\mathbb{L}_2\big(Q\times(0,T)\big)$ to the functions $\mathbb{D}(y,\widehat{\boldsymbol{U}}_f)$ and $\mathbb{D}(y,\widehat{\boldsymbol{W}}_f)$, where

$$\widehat{\boldsymbol{W}}_f=\frac{\partial\widehat{\boldsymbol{U}}_f}{\partial t},\qquad \varrho=\lim_{\varepsilon\to0}\varrho^\varepsilon=m\varrho_f+(1-m)\varrho_s,$$
$$\varrho^0=\lim_{\varepsilon\to0}\varrho^{0,\varepsilon}=m^0\varrho_f+(1-m^0)\varrho_s,\qquad \widehat{\varrho}=(1-\zeta)\varrho+\zeta\,\varrho^0.$$

Finally, the sequences $\{\mathbb{D}(x,\boldsymbol{u}_s^\varepsilon)\}$ and $\{\mathbb{D}(x,\boldsymbol{w}_s^\varepsilon)\}$ are two-scale convergent in $\mathbb{L}_2\big(Q\times(0,T)\big)$ to $\mathbb{D}(x,\boldsymbol{u}_s)+\mathbb{D}(y,\widehat{\boldsymbol{U}}_s)$ and $\mathbb{D}(x,\boldsymbol{w}_s)+\mathbb{D}(y,\widehat{\boldsymbol{W}}_s)$ respectively.

Lemma 7.1. *Under the hypotheses of Theorem 4.2 we have*

$$\nabla_y\cdot\widehat{\boldsymbol{W}}=0,\qquad \boldsymbol{y}\in\widehat{Y}. \tag{7.1}$$

Proof. Letting $\varepsilon\to0$ in the identity (4.1) with $t_0=T$ and with test functions $\psi=\varepsilon\psi_0(\boldsymbol{x},t)\varphi(\boldsymbol{x}/\varepsilon)$, we obtain

$$0=\int_0^T\int_Q\boldsymbol{w}^\varepsilon\cdot\nabla\psi\,dx\,dt=\int_0^T\int_Q\boldsymbol{w}^\varepsilon\cdot\big(\varepsilon\nabla\psi_0\varphi+\psi_0\nabla_y\varphi\big)\,dx\,dt$$
$$\to\int_0^T\int_Q\psi_0(\boldsymbol{x},t)\Big(\int_{\widehat{Y}}\widehat{\boldsymbol{W}}\cdot\nabla_y\varphi(\boldsymbol{y})\,dy\Big)\,dx\,dt=0.$$

This is equivalent to (7.1). □

Lemma 7.2. *Under the hypotheses of Theorem 4.2 we have*

$$\Pi_f(\boldsymbol{x},t,\boldsymbol{y})=\pi_f(\boldsymbol{x},t)\widehat{\chi}(\boldsymbol{x},\boldsymbol{y}),\qquad \boldsymbol{y}\in Y. \tag{7.2}$$

Proof. Letting $\varepsilon\to0$ in the identity (6.13) with $t_0=T$ and with test functions $\boldsymbol{\psi}=\varepsilon\psi_0(\boldsymbol{x},t)\boldsymbol{\psi}_1(\boldsymbol{x}/\varepsilon)$, where the functions $\psi_0(\boldsymbol{x},t)$ and $\boldsymbol{\psi}_1(\boldsymbol{y})$ are smooth and compactly supported in Q and $\widehat{Y}_f$ respectively, we arrive at the identity

$$\int_0^T\int_Q\psi_0(\boldsymbol{x},t)\Big(\int_{\widehat{Y}_f}\Pi_f\nabla_y\cdot\boldsymbol{\psi}_1(\boldsymbol{y})\,dy\Big)\,dx\,dt=0,$$

whence (7.2) follows. □

Lemma 7.3. *Under the hypotheses of Theorem 4.2*,

$$\nabla\pi_f\in\mathbb{L}_2\big(Q\times(0,T)\big). \tag{7.3}$$

Proof. To prove (7.3), we consider the identity (6.13) and put

$$\boldsymbol{\psi} = \psi_0(\boldsymbol{x}, t)\boldsymbol{\varphi}\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big), \qquad \psi_0 \in \mathring{\mathbb{C}}^\infty\big(Q \times (0,T)\big), \quad \boldsymbol{\varphi} \in \mathring{\mathbb{C}}^1(\widehat{Y}_f),$$
$$\nabla_y \cdot \boldsymbol{\varphi}(\boldsymbol{y}) = 0, \qquad \boldsymbol{y} \in \widehat{Y}_f.$$

Letting $\varepsilon \to 0$, we obtain that

$$\int_0^T \int_Q (\widehat{A}\psi_0 + \widehat{B}\nabla\psi_0)\, dx\, dt = 0 \tag{7.4}$$

for arbitrary functions $\psi_0 \in \mathring{\mathbb{C}}^\infty\big(Q \times (0,T)\big)$. In (7.4) we use the notation

$$\widehat{A} = \int_{\widehat{Y}_f} \mu_1 \mathbb{D}(y, \widehat{\boldsymbol{W}}_f) : \mathbb{D}(y, \boldsymbol{\varphi})\, dy - \widehat{\varrho}\boldsymbol{\Phi} \cdot \int_{\widehat{Y}_f} \boldsymbol{\varphi}\, dy, \qquad \widehat{B} = -\pi_f \int_{\widehat{Y}_f} \boldsymbol{\varphi}\, dy.$$

We now use Lemma 5.2 to find, for every unit vector of the basis $\boldsymbol{e}_i$, $i = 1, 2, 3$, a solenoidal function $\boldsymbol{\psi}_i$ such that

$$\int_{\widehat{Y}_f} \boldsymbol{\psi}_i(\boldsymbol{y})\, dy = \boldsymbol{e}_i, \qquad i = 1, 2, 3. \tag{7.5}$$

Substituting these functions for the test function $\boldsymbol{\varphi}$ in (7.4), we obtain

$$\int_0^T \int_Q \Big(\widehat{A}_i\psi_0 - \pi_f \frac{\partial \psi_0}{\partial x_i}\Big)\, dx\, dt = 0, \qquad i = 1, 2, 3, \tag{7.6}$$

where

$$\widehat{A}_i = \mu_1 \mathbb{D}(y, \widehat{\boldsymbol{W}}_f) : \mathbb{D}(y, \boldsymbol{\psi}_i)\, dy - \widehat{\varrho}\boldsymbol{\Phi} \cdot \int_{\widehat{Y}_f} \boldsymbol{\psi}_i\, dy \in \mathbb{L}_2\big(Q \times (0,T)\big).$$

The identity (7.6) means that

$$-\frac{\partial \pi_f}{\partial x_i} = \widehat{A}_i \in \mathbb{L}_2\big(Q \times (0,T)\big) \quad \text{or} \quad \nabla\pi \in \mathbb{L}_2\big(Q \times (0,T)\big). \qquad \square$$

Lemma 7.4. *Under the hypotheses of Theorem 4.2, the limiting functions* $\widehat{\boldsymbol{W}}_f$, π_f, $\boldsymbol{w}_s$ *satisfy the following boundary-value problem in the domain* $\widehat{Y}_f$:

$$\begin{gathered}
\mu_1 \nabla_y \cdot \big(\mathbb{D}(y, \widehat{\boldsymbol{W}}_f)\big) - \nabla_y \Pi_f - \nabla\pi_f = \widehat{\varrho}\boldsymbol{\Phi}, \quad \nabla_y \cdot \widehat{\boldsymbol{W}}_f = 0, \qquad \boldsymbol{y} \in \widehat{Y}_f, \\
\lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega^0}} \widehat{\boldsymbol{W}}_f(\boldsymbol{x}, t, \boldsymbol{y}) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega}} \widehat{\boldsymbol{W}}_f(\boldsymbol{x}, t, \boldsymbol{y}), \\
\lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega^0}} \mathbb{D}\big(x, \widehat{\boldsymbol{W}}_f(\boldsymbol{x}, t, \boldsymbol{y})\big) = \lim_{\substack{\boldsymbol{x} \to \boldsymbol{x}^0 \\ \boldsymbol{x} \in \Omega}} \widehat{\boldsymbol{W}}_f(\boldsymbol{x}, t, \boldsymbol{y}), \\
\widehat{\boldsymbol{W}}_f(\boldsymbol{x}, t, \boldsymbol{y}) = \boldsymbol{w}_s(\boldsymbol{x}, t), \qquad \boldsymbol{y} \in \widehat{\gamma}
\end{gathered} \tag{7.7}$$

(*understood in the generalized sense as a solution of the appropriate integral identities*).

Proof. The first (dynamical) equation in (7.7) follows from the analogue of (7.4) with arbitrary (non-solenoidal) test functions, and from (7.3). The term $\nabla_y \Pi_f$ arises because the test functions are not solenoidal.

The continuity equation for $\widehat{\boldsymbol{W}}$ on $\widehat{Y}$ was deduced in Lemma 7.1. Also, the equation (7.1) obviously holds for $\widehat{\boldsymbol{W}}_f$ on $\widehat{Y}_f$.

To prove the boundary condition on the boundary $\widehat{\gamma}$, consider the equality

$$\boldsymbol{w}^\varepsilon = \widehat{\chi}^\varepsilon \boldsymbol{w}^\varepsilon + (1 - \widehat{\chi}^\varepsilon)\boldsymbol{w}_s^\varepsilon$$

and perform two-scale convergence:

$$\begin{gathered} \boldsymbol{w}^\varepsilon \xrightarrow{\text{t.-s.}} \boldsymbol{W}, \\ \widehat{\chi}^\varepsilon \boldsymbol{w}^\varepsilon + (1 - \widehat{\chi}^\varepsilon)\boldsymbol{w}_s^\varepsilon \xrightarrow{\text{t.-s.}} \widehat{\chi}(\boldsymbol{y})\boldsymbol{W} + \big(1 - \widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\boldsymbol{w}_s, \\ \boldsymbol{W} = \boldsymbol{W}_f + \big(1 - \widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\boldsymbol{w}_s. \end{gathered} \tag{7.8}$$

Since

$$\widehat{\boldsymbol{W}}, \widehat{\boldsymbol{W}}_f + \big(1 - \widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\boldsymbol{w}_s \in \mathbb{W}_2^{1,0}\big(Y \times (0,T)\big),$$

the equality in (7.8) yields the boundary condition on $\widehat{\gamma}$. □

Lemma 7.5. *Under the hypotheses of Theorem 4.2, the limiting functions $\boldsymbol{w}_f$ and $\boldsymbol{w}_s$ satisfy the following continuity equation in Q for $t > 0$:*

$$\boldsymbol{w}_f + (1 - \widehat{m})\boldsymbol{w}_s = 0. \tag{7.9}$$

Proof. To prove this continuity equation, we write the integral identity (4.1) in the form

$$\int_0^T \int_Q \big(\widehat{\chi}^\varepsilon \boldsymbol{w}^\varepsilon + (1 - \widehat{\chi}^\varepsilon)\boldsymbol{w}_s^\varepsilon\big) \cdot \nabla\psi \, dx\, dt = 0$$

with test functions $\psi = \psi(\boldsymbol{x}, t)$ and let $\varepsilon \to 0$:

$$\int_0^T \int_Q \big(\boldsymbol{w}_f + (1 - \widehat{m})\boldsymbol{w}_s\big) \cdot \nabla\psi \, dx\, dt = 0.$$

Since ψ may be chosen arbitrarily, this identity is equivalent to (7.9). □

Lemma 7.6. *Under the hypotheses of Theorem 4.2, the limiting functions $\boldsymbol{u}_s(\boldsymbol{x}, t)$, $\widehat{\boldsymbol{U}}_s(\boldsymbol{x}, t, \boldsymbol{y})$ and $\widehat{\Pi}(\boldsymbol{x}, t, \boldsymbol{y})$ satisfy the boundary-value problem in the domain $\widehat{Y}$ for the following system of microscopic equations:*

$$\begin{gathered} \nabla_y \cdot \Big(\widehat{\lambda}(\boldsymbol{x})\big(1 - \widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\big(\mathbb{D}(x, \boldsymbol{u}_s) + \mathbb{D}(y, \widehat{\boldsymbol{U}}_s) - \widehat{\Pi}\,\mathbb{I}\big)\Big) = 0, \\ \big(1 - \widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\big(\nabla \cdot \boldsymbol{u}_s + \nabla_y \cdot \widehat{\boldsymbol{U}}_s\big) = 0, \end{gathered} \tag{7.10}$$

$$\langle \widehat{\boldsymbol{U}}_s \rangle_{\widehat{Y}_s} = 0, \qquad \langle \widehat{\Pi} \rangle_{\widehat{Y}_s} = 0, \tag{7.11}$$

where

$$\widehat{\boldsymbol{U}}_s(\boldsymbol{x}, t, \boldsymbol{y}) = \int_0^t \widehat{\boldsymbol{W}}_s(\boldsymbol{x}, \tau, \boldsymbol{y})\, d\tau.$$

Proof. The equality (7.10) follows from (6.13) if we put

$$\boldsymbol{\psi} = h(\boldsymbol{x},t)\boldsymbol{\psi}_0\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big), \qquad h \in \mathring{\mathbb{W}}^1_2(Q),$$

and pass to a limit as $\varepsilon \to 0$.

The equality (7.11) is obtained from the continuity equation (4.1) in the form

$$\int_0^T \int_Q \nabla \cdot \boldsymbol{u}^\varepsilon \, \psi \, dx \, dt = 0$$

if we choose

$$\psi = h(\boldsymbol{x},t)\Big(1 - \widehat{\chi}\Big(\boldsymbol{x}, \frac{\boldsymbol{x}}{\varepsilon}\Big)\Big)$$

for the test functions and let $\varepsilon \to 0$. □

Lemma 7.7. *Under the hypotheses of Theorem 4.2, the limiting functions* $\boldsymbol{u}_s(\boldsymbol{x},t)$, $\widehat{\boldsymbol{U}}_s(\boldsymbol{x},t,\boldsymbol{y})$ *and* π *satisfy the following dynamical macroscopic equation in* $\widehat{Y}$:

$$\nabla_x \cdot \big(\lambda(\boldsymbol{x})\big((1-\widehat{m})\mathbb{D}(x,\boldsymbol{u}_s) + \langle \mathbb{D}(y,\widehat{\boldsymbol{U}}_s)\rangle_{Y_s}\big) - \pi\,\mathbb{I}\big) + \widehat{\varrho}\boldsymbol{\Phi} = 0. \tag{7.12}$$

Proof. The equality (7.12) follows from (6.13) if we let $\varepsilon \to 0$ with test functions of the form $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{x},t)$. □

Lemma 7.8. *Under the hypotheses of Theorem 4.2, the limiting functions* $\boldsymbol{w}_f(\boldsymbol{x},t)$ *and* π_f *satisfy the Darcy law*

$$\boldsymbol{w}_f = \widehat{m}\int_0^t \boldsymbol{w}_s(\boldsymbol{x},\tau)\,d\tau + \frac{1}{\mu_1}\widehat{\mathbb{B}}(-\nabla\pi_f + \widehat{\varrho}\boldsymbol{\Phi}) \tag{7.13}$$

in Q *for* $t > 0$, *where the symmetric matrix* $\widehat{\mathbb{B}}$ *is strictly positive definite.*

Proof. To derive the Darcy law, we consider a boundary-value problem consisting of the differential equation, the boundary condition in (7.7) and the continuity equation (7.1):

$$\begin{gathered}
\mu_1\nabla_y \cdot \big(\mathbb{D}(y,\widehat{\boldsymbol{W}}_f)\big) - \nabla_y\widehat{\Pi}_f - \nabla\pi_f + \widehat{\varrho}\boldsymbol{\Phi}_f = 0, \qquad \boldsymbol{y} \in \widehat{Y}_f,\\
\nabla_y \cdot \widehat{\boldsymbol{W}}_f = 0, \qquad \boldsymbol{y} \in \widehat{Y}_f,\\
\widehat{\boldsymbol{W}}_f(\boldsymbol{x},t,\boldsymbol{y}) = \boldsymbol{w}_s(\boldsymbol{x},t), \qquad \boldsymbol{y} \in \widehat{\gamma}.
\end{gathered} \tag{7.14}$$

Put

$$\begin{gathered}
\frac{1}{\mu_1}(-\nabla\pi_f + \widehat{\varrho}\boldsymbol{\Phi}) = \sum_{i=1}^3 z_i\boldsymbol{e}^i,\\
\widehat{\boldsymbol{W}}_f(\boldsymbol{x},t,\boldsymbol{y}) = \boldsymbol{w}_s(\boldsymbol{x},t) + \sum_{i=1}^3 z_i(\boldsymbol{x},t)\widehat{\boldsymbol{W}}^i_f(\boldsymbol{y}),\\
\Pi_f(\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i=1}^3 z_i(\boldsymbol{x},t)\Pi^i_f(\boldsymbol{y}).
\end{gathered} \tag{7.15}$$

Substituting the expression (7.15) into the system of equations (7.14), we obtain

$$
\begin{aligned}
\nabla_y \cdot \big(\mathbb{D}(y, \widehat{\boldsymbol{W}}{}^i_f)\big) - \nabla_y \Pi^i_f + \boldsymbol{e}^i = 0, &\qquad \boldsymbol{y} \in \widehat{Y}_f, \\
\nabla_y \cdot \widehat{\boldsymbol{W}}{}^i_f = 0, &\qquad \boldsymbol{y} \in \widehat{Y}_f, \\
\widehat{\boldsymbol{W}}{}^i_f(\boldsymbol{x}, t, \boldsymbol{y}) = 0, &\qquad \boldsymbol{y} \in \widehat{\gamma}.
\end{aligned}
\tag{7.16}
$$

The proof of the existence and uniqueness of a generalized solution of (7.16) is standard [20]. Moreover, since the boundary $\widehat{\gamma}$ is infinitely smooth, this solution will be infinitely differentiable.

Thus,

$$
\begin{aligned}
\boldsymbol{w}_f = \langle \widehat{\boldsymbol{W}}_f \rangle_{\widehat{Y}_f} &= \widehat{m}\boldsymbol{w}_s + \frac{1}{\mu_1}\Big(\sum_{i=1}^{3} \langle \widehat{\boldsymbol{W}}{}^i_f \rangle_{\widehat{Y}_f} \otimes \boldsymbol{e}^i\Big)(-\nabla \pi_f + \widehat{\varrho}\boldsymbol{\Phi}) \\
&= \widehat{m}\boldsymbol{w}_s + \frac{1}{\mu_1}\widehat{\mathbb{B}}(-\nabla \pi_f + \widehat{\varrho}\boldsymbol{\Phi}),
\end{aligned}
\tag{7.17}
$$

$$
\widehat{\mathbb{B}} = \sum_{i=1}^{3} \langle \widehat{\boldsymbol{W}}{}^i_f \rangle_{\widehat{Y}_f} \otimes \boldsymbol{e}^i. \tag{7.18}
$$

We claim that the matrix $\widehat{\mathbb{B}} = (\widehat{B}^{ij})$ is symmetric and strictly positive definite:

$$
\sum_{i,j=1}^{3} \widehat{B}^{ij}\xi^i\xi^j \geqslant b_0 \sum_{i=1}^{3} |\xi^i|^2, \tag{7.19}
$$

where $b_0 = \mathrm{const} > 0$.

First, we multiply the first equation in (7.16) by $\widehat{\boldsymbol{W}}{}^o_f$ and integrate the resulting equality by parts over the domain $\widehat{Y}_f$:

$$
\int_{\widehat{Y}_f} \mathbb{D}(y, \widehat{\boldsymbol{W}}{}^i_f) : \mathbb{D}(y, \widehat{\boldsymbol{W}}{}^j_f)\, dy = \int_{\widehat{Y}_f} \widehat{\boldsymbol{W}}{}^j_f \cdot \boldsymbol{e}^i\, dy. \tag{7.20}
$$

For arbitrary $(\zeta_1, \zeta_2, \zeta_3) = \boldsymbol{\zeta}$ and $(\eta_1, \eta_2, \eta_3) = \boldsymbol{\eta} \in \mathbb{R}^3$ we put

$$
\widehat{\boldsymbol{z}}_\zeta = \sum_{i=1}^{3} \widehat{\boldsymbol{W}}{}^i_f \zeta^i, \qquad \widehat{\boldsymbol{z}}_\eta = \sum_{i=1}^{3} \widehat{\boldsymbol{W}}{}^i_f \eta^i, \qquad \widehat{Q}_\zeta = \sum_{i=1}^{3} \Pi^i_f \zeta^i, \qquad \widehat{Q}_\eta = \sum_{i=1}^{3} \Pi^i_f \eta^i.
$$

Then the following relations hold in accordance with (7.16):

$$
\begin{aligned}
\nabla_y \cdot \big(\mathbb{D}(y, \widehat{\boldsymbol{z}}_\zeta)\big) - \nabla_y \widehat{Q}_\zeta + \boldsymbol{\zeta} = 0, &\qquad \boldsymbol{y} \in \widehat{Y}_f, \\
\nabla_y \cdot \widehat{\boldsymbol{z}}_\zeta = 0, &\qquad \boldsymbol{y} \in \widehat{Y}_f, \\
\widehat{\boldsymbol{z}}_\zeta(\boldsymbol{x}, t, \boldsymbol{y}) = 0, \quad \boldsymbol{y} \in \widehat{\gamma}, &\qquad \int_{\widehat{Y}_f} \widehat{Q}_\zeta\, dy = 0.
\end{aligned}
\tag{7.21}
$$

Multiplying the first equation in (7.21) by $\widehat{\boldsymbol{z}}_\eta$ and integrating by parts, we obtain the identity

$$
\langle \mathbb{D}(y, \widehat{\boldsymbol{z}}_\zeta) : \mathbb{D}(y, \widehat{\boldsymbol{z}}_\eta) \rangle_{\widehat{Y}_f} = \langle \boldsymbol{\zeta} \cdot \widehat{\boldsymbol{z}}_\eta \rangle_{\widehat{Y}_f} \tag{7.22}
$$

for all $\boldsymbol{\zeta}, \boldsymbol{\eta} \in \mathbb{R}^3$.

On the other hand, multiplying (7.20) by $\zeta^i\,\eta^j$ and summing over i and j, we obtain the identity

$$\boldsymbol{\eta}\cdot\big(\widehat{\mathbb{B}}(\boldsymbol{\zeta})\big)=\langle\mathbb{D}(y,\widehat{\boldsymbol{z}}_{\zeta}):\mathbb{D}(y,\widehat{\boldsymbol{z}}_{\eta})\rangle_{\widehat{Y}_f}$$

for arbitrary $\boldsymbol{\zeta}$ and $\boldsymbol{\eta}$. In particular, it follows that the matrix $\widehat{\mathbb{B}}$ is symmetric. Moreover,

$$\boldsymbol{\eta}\cdot\big(\widehat{\mathbb{B}}(\boldsymbol{\zeta})\big)=\langle\mathbb{D}(y,\widehat{\boldsymbol{z}}_{\zeta}):\mathbb{D}(y,\widehat{\boldsymbol{z}}_{\eta})\rangle_{\widehat{Y}_f}\geqslant\alpha_0 \tag{7.23}$$

for arbitrary $\boldsymbol{\zeta}$ with $|\boldsymbol{\zeta}|=1$ and some constant $\alpha_0>0$.

Indeed, assuming the opposite, we arrive at the equality $\mathbb{D}(y,\widehat{\boldsymbol{z}}_{\zeta})=0$, which is possible only when $\widehat{\boldsymbol{z}}_{\zeta}$ is a linear function of $\boldsymbol{y}$. However, a linear function which is infinitely differentiable and periodic with respect to $\boldsymbol{y}$ and satisfies the homogeneous boundary condition in (7.21), must necessarily be identically equal to zero.

Again considering (7.21), we see that $\nabla_y\widehat{Q}_{\zeta}=-\boldsymbol{\zeta}$, or $\widehat{Q}_{\zeta}=-\boldsymbol{\zeta}\cdot\boldsymbol{y}+\text{const}$. This is impossible because $\widehat{Q}_{\zeta}$ is a periodic function of $\boldsymbol{y}$. □

Lemma 7.9. *Under the hypotheses of Theorem 4.2, the limiting functions $\boldsymbol{w}_s(\boldsymbol{x},t)$, $\boldsymbol{w}_f(\boldsymbol{x},t)$ and π_f satisfy the Lamé system of equations*

$$\nabla_x\cdot\Big(\lambda\big(\boldsymbol{x}\big)\big(\widehat{\mathfrak{N}}^s:\mathbb{D}(x,\boldsymbol{u}_s)-\pi_f\,\mathbb{I}\big)\Big)+\widehat{\varrho}\boldsymbol{\Phi}=0 \tag{7.24}$$

in the domain Q, where the tensor $\widehat{\mathfrak{N}}^s(\boldsymbol{x})$ of rank four is defined by (7.28).

Proof. The system of equations (7.24) can be deduced from the system of equations (7.12) once we have calculated the functions $\langle\mathbb{D}(y,\widehat{\boldsymbol{U}}_s)\rangle_{Y_s}$ and π.

To do this, we rewrite the system of equations (7.10) in the form

$$\begin{gathered}\nabla_y\cdot\left(\widehat{\lambda}(\boldsymbol{x})\big(1-\widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\left(\mathbb{D}(x,\boldsymbol{u}_s)+\mathbb{D}(y,\widehat{\boldsymbol{U}}_s)-\frac{1}{\widehat{\lambda}(\boldsymbol{x})}(\widehat{\Pi}_s-\pi_f)\mathbb{I}\right)\right)=0,\\ \big(1-\widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)(\nabla\cdot\boldsymbol{u}_s+\nabla_y\cdot\widehat{\boldsymbol{U}}_s)=0\end{gathered} \tag{7.25}$$

and put

$$\boldsymbol{u}_s=(u_s^1,u_s^2,u_s^3),\qquad D^{ij}(\boldsymbol{x},t)=\frac{1}{2}\left(\frac{\partial u_s^i}{\partial x_j}(\boldsymbol{x},t)+\frac{\partial u_s^j}{\partial x_i}(\boldsymbol{x},t)\right),$$
$$D^0(\boldsymbol{x},t)=\nabla\cdot\boldsymbol{u}_s(\boldsymbol{x},t).$$

We seek a solution of the system (7.10), (7.11) in the form

$$\widehat{\boldsymbol{U}}_s(\boldsymbol{x},t,\boldsymbol{y})=\sum_{i,j=1}^{3}D^{ij}(\boldsymbol{x},t)\widehat{\boldsymbol{U}}_s^{ij}(\boldsymbol{x},\boldsymbol{y})+D^0(\boldsymbol{x},t)\widehat{\boldsymbol{U}}_s^{0}(\boldsymbol{x},\boldsymbol{y}),$$

$$\widehat{\Pi}_s-\pi_f=\widehat{\lambda}(\boldsymbol{x})\left(\sum_{i,j=1}^{3}D^{ij}(\boldsymbol{x},t)\widehat{\Pi}_s^{ij}(\boldsymbol{x},\boldsymbol{y})+D^0(\boldsymbol{x},t)\widehat{\Pi}_s^0(\boldsymbol{x},\boldsymbol{y})\right).$$

We have

$$\begin{gathered}\nabla_y\cdot\big(\big(1-\widehat{\chi}(\boldsymbol{x},\boldsymbol{y})\big)\big(\mathbb{D}(y,\widehat{\boldsymbol{U}}_s^{ij})+\mathbb{J}^{ij}-\widehat{\Pi}_s^{ij}\mathbb{I}\big)\big)=0,\\ (1-\widehat{\chi}(\boldsymbol{x},\boldsymbol{y}))(\nabla_y\cdot\widehat{\boldsymbol{U}}_s^{ij})=0,\qquad \boldsymbol{y}\in\widehat{Y},\\ \langle\widehat{\boldsymbol{U}}_s^{ij}\rangle_{\widehat{Y}_s}=0,\qquad\langle\widehat{\Pi}_s^{ij}\rangle_{\widehat{Y}_s}=0,\end{gathered} \tag{7.26}$$

$$\nabla_y \cdot \big(\big(1 - \widehat{\chi}(\boldsymbol{x}, \boldsymbol{y})\big)\big(\mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) - \widehat{\Pi}_s^0 \mathbb{I}\big)\big) = 0,$$
$$(1 - \widehat{\chi}(\boldsymbol{x}, \boldsymbol{y}))\big(\nabla_y \cdot (\widehat{\boldsymbol{U}}_s^0 + 1)\big) = 0, \qquad \boldsymbol{y} \in \widehat{Y}, \tag{7.27}$$
$$\langle \widehat{\boldsymbol{U}}_s^0 \rangle_{\widehat{Y}_s} = 0, \qquad \langle \widehat{\Pi}_s^0 \rangle_{\widehat{Y}_s} = 0.$$

The proof of the existence and uniqueness of periodic generalized solutions of the linear systems (7.26) and (7.27) of differential equations is standard (Galerkin's method and *a priori* bounds).

The equation (7.24) follows from (7.12) after calculating the expressions

$$\widehat{\mathbb{P}}^s = \widehat{\lambda}(1 - \widehat{m})\mathbb{D}(x, \boldsymbol{u}_s) + \boldsymbol{S}, \qquad \boldsymbol{S} = \widehat{\lambda}\langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s) \rangle_{\widehat{Y}_s} - \pi \mathbb{I},$$

$$\langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s) \rangle_{\widehat{Y}_s} = \sum_{i,j=1}^{3} \langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{ij}) \rangle_{\widehat{Y}_s} D^{ij} + \langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) \rangle_{\widehat{Y}_s} D^0$$
$$= \left(\sum_{i,j=1}^{3} \langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{ij}) \otimes \mathbb{J}^{ij} + \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) \otimes \mathbb{I} \rangle_{\widehat{Y}_s} \right) : \mathbb{D}(x, \boldsymbol{u}_s),$$

$$\pi = \langle \Pi \rangle_{\widehat{Y}} = \langle \widehat{\chi}\pi_f + (1 - \widehat{\chi})\widehat{\Pi}_s \rangle_{\widehat{Y}} = \langle \pi_f + (1 - \widehat{\chi})(\widehat{\Pi}_s - \pi_f) \rangle_{\widehat{Y}}$$
$$= \pi_f + \widehat{\lambda}\left\langle \sum_{i,j=1}^{3} \widehat{\Pi}_s^{ij} \right\rangle_{\widehat{Y}_s} D^{ij} + \widehat{\lambda}\langle \widehat{\Pi}_s^0 \rangle_{\widehat{Y}_s} D^0$$
$$= \pi_f + \widehat{\lambda}\left(\sum_{i,j=1}^{3} \langle (\widehat{\Pi}_s^{ij})\mathbb{I} \otimes \mathbb{J}^{ij} + (\widehat{\Pi}_s^0)\mathbb{I} \otimes \mathbb{I} \rangle_{\widehat{Y}_s} \right) : \mathbb{D}(x, \boldsymbol{u}_s),$$

$$\widehat{\mathbb{P}}^s = \widehat{\lambda}\widehat{\mathfrak{N}}^s : \mathbb{D}(x, \boldsymbol{u}_s) - \pi_f \mathbb{I},$$

$$\widehat{\mathfrak{N}}^s = \widehat{\lambda}\bigg((1 - \widehat{m})\mathbb{D}(x, \boldsymbol{u}_s) + \sum_{i,j=1}^{3} \langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{ij}) \rangle_{\widehat{Y}_s} \otimes \mathbb{J}^{ij} + \langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) \rangle_{\widehat{Y}_s} \otimes \mathbb{I}$$
$$- \sum_{i,j=1}^{3} \langle \widehat{\Pi}_s^{ij} \rangle_{\widehat{Y}_s} \mathbb{I} \otimes \mathbb{J}^{ij} - \langle \widehat{\Pi}_s^0 \rangle_{\widehat{Y}_s} \mathbb{I} \otimes \mathbb{I} \bigg). \qquad \square \tag{7.28}$$

Lemma 7.10. *Under the hypotheses of Theorem 4.2, the tensor $\widehat{\mathfrak{N}}^s$ of rank four is symmetric and strictly positive definite.*

Proof. All the properties of $\widehat{\mathfrak{N}}^s$ follow from the equalities

$$\langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{ij}) : \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{kl}) \rangle_{\widehat{Y}_s} + \langle \mathbb{J}^{ij} : \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{kl}) \rangle_{\widehat{Y}_s} = 0, \tag{7.29}$$
$$\langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^{ij}) : \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) \rangle_{\widehat{Y}_s} = 0, \tag{7.30}$$
$$\langle \widehat{\Pi}_s^{ij} \rangle_{\widehat{Y}_s} = -\mathbb{J}^{ij} : \langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) \rangle_{\widehat{Y}_s}, \tag{7.31}$$
$$\langle \widehat{\Pi}_s^0 \rangle_{\widehat{Y}_s} = -\langle \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) : \mathbb{D}(y, \widehat{\boldsymbol{U}}_s^0) \rangle_{\widehat{Y}_s} \tag{7.32}$$

for $i, j, k, l = 1, 2, 3$.

These equalities can be obtained from (7.26) and (7.27) if we multiply (7.26) (resp. (7.27)) by $\widehat{\boldsymbol{U}}_s^{kl}$ and $\widehat{\boldsymbol{U}}_s^{0}$ (resp. $\widehat{\boldsymbol{U}}_s^{0}$ and $\widehat{\boldsymbol{U}}_s^{ij}$) and integrate by parts over the domain $\widehat{Y}_s$.

Indeed, multiplying (7.26) by $\widehat{\boldsymbol{U}}_s^{kl}$ and integrating by parts over $\widehat{Y}_s$, we obtain (7.29) since $\nabla\cdot\widehat{\boldsymbol{U}}_s^{kl} = 0$. Multiplying (7.27) by $\widehat{\boldsymbol{U}}_s^{ij}$ and integrating by parts over $\widehat{Y}_s$, we obtain (7.30) by the previous argument. Multiplying (7.26) by $\widehat{\boldsymbol{U}}_s^{0}$ and integrating by parts over $\widehat{Y}_s$, we obtain (7.31) in view of (7.30). Finally, multiplying (7.27) by $\widehat{\boldsymbol{U}}_s^{0}$ and integrating by parts over $\widehat{Y}_s$, we obtain (7.32).

Let $\zeta=(\zeta_{ij})$ and $\eta=(\eta_{ij})$ be arbitrary symmetric matrices. We put

$$\widehat{\boldsymbol{Z}}_\zeta=\sum_{i,j=1}^{3}\widehat{\boldsymbol{U}}_s^{ij}\zeta_{ij},\qquad \widehat{\boldsymbol{Z}}_\eta=\sum_{i,j=1}^{3}\widehat{\boldsymbol{U}}_s^{ij}\eta_{ij},\qquad \widehat{\boldsymbol{Z}}_\zeta^{0}=\widehat{\boldsymbol{U}}_s^{0}\,\mathrm{tr}(\zeta),\qquad \widehat{\boldsymbol{Z}}_\eta^{0}=\widehat{\boldsymbol{U}}_s^{0}\,\mathrm{tr}(\eta),$$

where $\mathrm{tr}(\zeta)=\sum_{i=1}^{3}\zeta_{ii}$.

Then the expressions (7.29)–(7.32) take the form

$$\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta):\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta)\rangle_{\widehat{Y}_s}+\zeta:\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta)\rangle_{\widehat{Y}_s}=0, \tag{7.33}$$

$$\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta):\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta^{0})\rangle_{\widehat{Y}_s}=0, \tag{7.34}$$

$$\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta^{0})\rangle_{\widehat{Y}_s}:\zeta=-\big((\langle\widehat{\Pi}_s^{ij}\rangle_{\widehat{Y}_s}\mathbb{I}:\mathbb{J}^{ij}):\zeta\big):\eta, \tag{7.35}$$

$$\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta^{0}):\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta^{0})\rangle_{\widehat{Y}_s}=-\langle\widehat{\Pi}_s^{0}\rangle_{\widehat{Y}_s}\big((\mathbb{I}\otimes\mathbb{I}):\zeta\big):\eta. \tag{7.36}$$

Thus,

$$(\widehat{\mathfrak{N}^s}:\zeta):\eta=(1-\widehat{m})\zeta:\eta+\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta)\rangle_{\widehat{Y}_s}:\eta+\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta^{0})\rangle_{\widehat{Y}_s}:\eta$$
$$+\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta^{0})\rangle_{\widehat{Y}_s}:\zeta+\langle\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta^{0}):\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta^{0})\rangle_{\widehat{Y}_s}.$$

Using (7.33) and (7.34), we finally obtain

$$(\widehat{\mathfrak{N}^s}:\zeta):\eta=\langle\big(\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\zeta+\widehat{\boldsymbol{Z}}_\zeta^{0})+\zeta\big):\big(\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta+\widehat{\boldsymbol{Z}}_\eta^{0})+\eta\big)\rangle_{\widehat{Y}_s}. \tag{7.37}$$

This expression shows that the tensor $\widehat{\mathfrak{N}^s}$ is symmetric. In particular,

$$(\widehat{\mathfrak{N}^s}:\eta):\eta=\big\langle\big(\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta+\widehat{\boldsymbol{Z}}_\eta^{0})+\eta\big):\big(\mathbb{D}(y,\widehat{\boldsymbol{Z}}_\eta+\widehat{\boldsymbol{Z}}_\eta^{0})+\eta\big)\big\rangle_{\widehat{Y}_s}>a_0=\mathrm{const}>0$$

for all η with $\sum_{i,j=1}^{3}|\eta_{ij}|^2=1$.

Indeed, assume the opposite: there is a constant matrix $\widetilde{\eta}$ with $\sum_{i,j=1}^{3}|\widetilde{\eta}_{ij}|^2=1$ such that

$$\mathbb{D}(y,\widehat{\boldsymbol{V}}_{\widetilde{\eta}})+\widetilde{\eta}=0,\qquad \widehat{\boldsymbol{V}}_{\widetilde{\eta}}=\widehat{\boldsymbol{Z}}_{\widetilde{\eta}}+\widehat{\boldsymbol{Z}}_{\widetilde{\eta}}^{0}, \tag{7.38}$$

that is, the function $\widehat{\boldsymbol{V}}_{\widetilde{\eta}}$ is linear. By (7.26) and (7.27) we have

$$\begin{gathered}\nabla_y\cdot\big((1-\widehat{\chi}(\boldsymbol{x},\boldsymbol{y}))\big(\mathbb{D}(y,\widehat{\boldsymbol{V}}_{\widetilde{\eta}})+\widetilde{\eta}-\widehat{Q}_{\widetilde{\eta}}\mathbb{I}\big)\big)=0,\\ (1-\widehat{\chi}(\boldsymbol{x},\boldsymbol{y}))\big(\nabla_y\cdot(\widehat{\boldsymbol{V}}_{\widetilde{\eta}}+1)\big)=0,\qquad \boldsymbol{y}\in\widehat{Y},\\ \langle\widehat{\boldsymbol{V}}_{\widetilde{\eta}}\rangle_{\widehat{Y}_s}=0,\qquad \langle\widehat{Q}_{\widetilde{\eta}}\rangle_{\widehat{Y}_s}=0.\end{gathered} \tag{7.39}$$

Since this system has a unique periodic solution $\widehat{\boldsymbol{V}}_{\widetilde{\eta}} \in \mathbb{C}^\infty(\widehat{Y})$ and every periodic linear infinitely differentiable function is constant, we obtain that $\widehat{\boldsymbol{V}}_{\widetilde{\eta}} \in \mathbb{C}^\infty(\widehat{Y}) = 0$ and $\eta = 0$. Thus the symmetric tensor $\widehat{\mathfrak{N}}^s$ is strictly positive definite. □

Lemma 7.11. *Under the hypotheses of Theorem 4.2, if the common porous space is connected, then the boundary condition (4.14) holds on the common boundary S^0. In the opposite case, the boundary condition (4.15) holds on the common boundary S^0.*

Proof. When the common porous space is disconnected, we consider a ball B_δ with centre $\boldsymbol{x}_0 \in S^0$ and radius δ and put $\Gamma^0_\delta = \partial B_\delta \bigcap S^0$ and $B^+_\delta = \{\boldsymbol{x} \in B_\delta(\boldsymbol{x}_0) : x_3 > 0\}$.

We define $\widetilde{\boldsymbol{w}}^\varepsilon = \boldsymbol{w}^\varepsilon - \boldsymbol{w}^\varepsilon_s$. Then $\widetilde{\boldsymbol{w}}^\varepsilon = 0$ in Q^ε_s. By assumption, $S^0 \subset \overline{Q^\varepsilon_s}$ (the common porous space is disconnected). Hence, $\widetilde{\boldsymbol{w}}^\varepsilon = 0$ on S^0.

We now use the continuity equation (3.1). For an arbitrary smooth compactly supported function $\xi(\boldsymbol{x}, t)$ on B_δ, we have

$$\begin{aligned}\int_0^T \int_{B^+_\delta} \xi\, \nabla \cdot \boldsymbol{w}^\varepsilon_s \, dx\, dt &= -\int_0^T \int_{B^+_\delta} \xi\, \nabla \cdot \widetilde{\boldsymbol{w}}^\varepsilon \, dx\, dt \\ &= \int_0^T \int_{B^+_\delta} \nabla\xi \cdot \widetilde{\boldsymbol{w}}^\varepsilon \, dx\, dt = \int_0^T \int_{B^+_\delta} \chi^\varepsilon \nabla\xi \cdot \widetilde{\boldsymbol{w}}^\varepsilon \, dx\, dt \\ &= \int_0^T \int_{B^+_\delta} \chi^\varepsilon \nabla\xi \cdot (\boldsymbol{w}^\varepsilon - \boldsymbol{w}^\varepsilon_s) \, dx\, dt = \int_0^T \int_{B^+_\delta} \nabla\xi \cdot (\boldsymbol{w}^\varepsilon_f - \chi^\varepsilon \boldsymbol{w}^\varepsilon_s) \, dx\, dt\end{aligned}$$

or

$$\int_0^T \int_{B^+_\delta} \xi\, \nabla \cdot \boldsymbol{w}^\varepsilon_s \, dx\, dt = \int_0^T \int_{B^+_\delta} \nabla\xi \cdot (\boldsymbol{w}^\varepsilon_f - \chi^\varepsilon \boldsymbol{w}^\varepsilon_s) \, dx\, dt.$$

Letting $\varepsilon \to 0$ in the last identity, we obtain that

$$\int_0^T \int_{B^+_\delta} \xi\, \big(m\, \nabla \cdot \boldsymbol{w}^\varepsilon_s + \langle \mathbb{D}(y, \widehat{\boldsymbol{W}}_s)\rangle_{\widehat{Y}_f} \, dx\, dt = \int_0^T \int_{B^+_\delta} \nabla\xi \cdot (\boldsymbol{w}_f - m\boldsymbol{w}_s) \, dx\, dt.$$

Integrating by parts once again, we arrive at the equality

$$\begin{aligned}\int_0^T \int_{B^+_\delta} \xi\big(m\nabla \cdot \boldsymbol{w}^\varepsilon_s + \langle \mathbb{D}(y, \widehat{\boldsymbol{W}}_s)\rangle_{\widehat{Y}_f} + \nabla \cdot (\boldsymbol{w}_f - m\boldsymbol{w}_s)\big) \, dx\, dt \\ = \int_0^T \int_{\Gamma^0_\delta} \xi \boldsymbol{n} \cdot (\boldsymbol{w}_f - m\boldsymbol{w}_s) \, d\sigma\, dt.\end{aligned} \tag{7.40}$$

For compactly supported functions ξ on B^+_δ, (7.40) yields that

$$m\nabla \cdot \boldsymbol{w}^\varepsilon_s + \langle \mathbb{D}(y, \widehat{\boldsymbol{W}}_s)\rangle_{\widehat{Y}_f} + \nabla \cdot (\boldsymbol{w}_f - m\boldsymbol{w}_s) = 0.$$

In view of this equality, the identity (7.40) takes the following form for arbitrary functions ξ:

$$\int_0^T \int_{\Gamma^0_\delta} \xi \boldsymbol{n} \cdot (\boldsymbol{w}_f - m\boldsymbol{w}_s) \, d\sigma\, dt = 0.$$

This identity is equivalent to the boundary condition (4.15).

We now assume that the common porous space is connected. By definition of a connected porous space, $Y_f^0 \cap Y_f \neq \varnothing$. Consider the identity (6.13) with

$$t_0 = T, \qquad \boldsymbol{\psi} = h(\boldsymbol{x}, t)\boldsymbol{\psi}_i\Big(\frac{\boldsymbol{x}}{\varepsilon}\Big),$$

where

$$\operatorname{supp}(h) \subset B_\delta, \qquad \operatorname{supp}(\boldsymbol{\psi}_i) \subset Y_f^0 \cap Y_f, \qquad \langle \boldsymbol{\psi}_i \rangle_Y = \boldsymbol{e}_i, \quad i = 1, 2, 3,$$

and $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ is the Cartesian orthogonal basis, and pass to a limit as $\varepsilon \to 0$.

We have

$$\begin{aligned} 0 &= \int_0^T \int_Q \Big(\big(\mu_1\, \varepsilon^2 \widehat{\chi}^\varepsilon \mathbb{D}(x, \boldsymbol{w}^\varepsilon) + (1 - \widehat{\chi}^\varepsilon)\widehat{\lambda}(\boldsymbol{x})\mathbb{D}(x, \boldsymbol{u}^\varepsilon)\big) : \mathbb{D}(x, h\,\boldsymbol{\psi}_i) \\ &\qquad - \pi^\varepsilon\, \nabla \cdot \boldsymbol{\psi} - \widehat{\varrho}^\varepsilon \boldsymbol{\Phi}^\varepsilon \cdot \boldsymbol{\psi}_i\Big)\, dx\, dt \\ &= \int_0^T \int_Q \Big(h(\boldsymbol{x})\widehat{\chi}^\varepsilon \big(\mu_1\, \varepsilon \mathbb{D}(x, \boldsymbol{w}^\varepsilon) : \mathbb{D}(x, \boldsymbol{\psi}_i) - \pi^\varepsilon\, \nabla h(\boldsymbol{x}) \cdot \boldsymbol{\psi}_i\big)\Big)\, dx\, dt + o(\varepsilon) \\ &\to \int_0^T \int_Q \big(h(\boldsymbol{x})A_i - \pi_f\, \langle \boldsymbol{\psi}_i \rangle_{\widehat{Y}_f}\big)\, dx\, dt = 0. \end{aligned}$$

Here

$$A_i = \int_{\widehat{Y}_f} \mu_1 \mathbb{D}(y, \widehat{\boldsymbol{W}}) : \mathbb{D}(y, \boldsymbol{\psi}_i)\, dy \in \mathbb{L}_2\big(Q \times (0, T)\big), \qquad \pi_f \in \mathbb{L}_2\big(Q \times (0, T)\big).$$

Therefore,

$$\frac{\partial \pi_f}{\partial x_i} = -A_i \in \mathbb{L}_2\big(Q \times (0, T)\big) \quad \text{and} \quad \pi_f \in \mathbb{W}_2^{1,0}\big(Q \times (0, T)\big).$$

This guarantees the continuity (in the sense of $\mathbb{L}_2\big(Q \times (0, T)\big)$) of the function $\pi_f(\boldsymbol{x}, t)$ on Q and, in particular, on S^0. □

§ 8. Proof of Theorem 4.3

The proof of this theorem follows from Theorem 4.2 as $k \to \infty$ and from the bounds (4.6), which still hold for the solutions of the problem (4.7)–(4.15) after passage to the limit as $\varepsilon \to 0$.

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