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Solution of a Boundary Value Problem for Velocity-Linearized Navier–Stokes Equations in the Case of a Heated Spherical Solid Particle Settling in Fluid

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Abstract—Assuming that the fluid viscosity is an exponential-power function of temperature, a boundary value problem for the Navier–Stokes equations linearized with respect to velocity is solved and the uniqueness of the solution is proved. The problem of a nonuniformly heated spherical solid particle settling in fluid is considered as an application.

Keywords: Navier–Stokes equation linearized with respect to velocity, boundary value problem for a viscous incompressible nonisothermal fluid.

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INTRODUCTION

The stationary system of equations of a viscous nonisothermal incompressible fluid consists of the Navier–Stokes equations, the continuity equation, and the heat transfer equation, which are mathematical representations of the momentum, mass, and energy conservation laws. This system of equations is of major importance in fluid dynamics and is used to mathematically simulate and study numerous natural phenomena and engineering applications, for example, sedimentation. Particles in coarse dispersions deposit in the field of gravity. Colloidal particles and macromolecules can deposit in a centrifugal force field (centrifugation). Sedimentation is used industrially in mineral processing, various chemical and petrochemical technologies, water purification, etc. Sedimentation in centrifuges and ultracentrifuges and also in a gravitational field underlies sedimentation analysis.

A major difficulty of this system is that the Navier–Stokes equations are nonlinear. In view of this factor, approximate methods that simplify the system of equations to a certain extent and adjust it to particular types of physical problems have been developed in fluid dynamics. There is a broad class of hydrodynamic flows in which the nonlinear term in the Navier–Stokes equations can be neglected. In the scientific literature, such equations are called the Navier–Stokes equations linearized with respect to velocity (or velocity-linearized). Examples are the viscous incompressible steady flow between two planes (Couette flow), the plane and cylindrical Poiseuille flows, the motion of a viscous fluid between two rotating cylinders, etc. This subject is of current interest. For example, the influence exerted by the mass flux surface density and thermophoresis on the flow over a heated permeable vertical plate was studied in [1, 2]. The effect of thermophoresis on single-stranded DNA moved along a microscopic temperature gradient was considered in [3]. The influence exerted by thermophoresis and variable viscosity on MHD mixed convective heat and mass transfer of a viscous incompressible electrically conducting fluid passing through a porous wedge in the presence of a chemical reaction was analyzed in [4].

In studying the stationary system of equations of a viscous nonisothermal incompressible fluid, the term "relative temperature drop" is used. By the relative temperature drop, we mean the ratio of the difference between the mean surface temperature T_S of the particle and the temperature T_∞ away from it to T_∞. A relative temperature drop is said to be small if $(T_S - T_\infty)/T_\infty \ll 1$ and large otherwise. For small temperature drops, the fluid viscosity can be treated as a constant and the fluid is called isothermal. In this case, the velocity-linearized system of equations of a viscous isothermal fluid is simplified considerably. More specifically, it is divided into two independent systems, namely, the velocity-linearized Navier–Stokes equations and heat transfer equations, which are coupled by boundary conditions. These systems of equations have been extensively studied. For example, the Navier–Stokes equations linearized with respect to velocity were investigated in [5–13].

If the temperature drop is large, then the fluid transport coefficients have to be treated as functions of temperature. This complicates the search for solutions of such a system of equations, and the viscous medium in this case is called nonisothermal. In this paper, a solution of the Navier–Stokes equations linearized with respect to velocity is constructed assuming that the dynamic viscosity of the fluid is an exponential-power function of temperature. Since out of all fluid transport coefficients (viscosity, thermal conductivity, diffusivity) only viscosity depends on temperature, the resulting solution is not only of practical interest, but also of theoretical importance, providing a mathematical apparatus for studying the full Navier–Stokes equations. Moreover, the method developed for solving the velocity-linearized Navier–Stokes equations in the case of a particular physical problem can be used to address a broader class of physical problems, for example, in the case of forces that are not gravitational, but rather electromagnetic, thermophoretic, or of another nature.

Out of all fluid transport parameters, the viscosity coefficient has the strongest dependence on temperature (see [14, 15]). It was shown in [14, 15] that the fluid viscosity decays exponentially with increasing temperature. An analysis of the semiempirical formulas and experimental data presented in [14–16] has shown the temperature dependence of the viscosity in a wide range of temperatures can best be fitted with any prescribed accuracy by the formula

$$\mu_e(T_e) = \mu_\infty\left(1+\sum_{n=1}^{N}F_n\left(\frac{T_e}{T_\infty}-1\right)^n\right)\exp\left(-A\left(\frac{T_e}{T_\infty}-1\right)\right), \tag{1}$$

where $\mu_\infty = \mu_e(T_\infty)$, A, and F_n are a given finite set of constants its own for each particular fluid. Here and below, the indices e and p refer to the viscous fluid and the heated particle, respectively, the index ∞ denotes the free-stream fluid parameters, and the index s stands for physical quantities at the mean surface temperature.

For example, for water, at temperatures ranging from 273 to 363 K, we have $A = 5.779$, $F_1 = -2.318$, $F_2 = 9.118$, $F_3 = 0.00003$, $F_4 = 0.000002$ (at $T_\infty = 273$ K), etc., with a relative error of at most 2.5%. Therefore, for water within a relative error of at most 2.5%, which is frequently sufficient for applications, we can retain only two first terms in formula (1); moreover, $F_1 = -2.318$ and $F_2 = 9.118$. Numerical computations have shown that a similar situation occurs for other Newtonian fluids, such as glycerol, castor oil, and isopropyl alcohol. Thus, for most fluids, numerical computations based on formula (1) can be restricted to the case $N = 2$ and the other coefficients F_n, $n \geq 3$, can be assumed to vanish.

The studies performed in [17–19] showed that, under physically admissible simplifications, assuming that the solutions (components of the mass velocity) have a certain form and the fluid viscosity is a certain function of temperature, the boundary value problem for the velocity-linearized Navier–Stokes equations can be reduced to a boundary value problem for a fourth-order homogeneous ordinary differential equation with an isolated singular point. The solution of the latter problem is sought with the help of generalized power series.

1. FORMULATION OF THE PROBLEM: BASIC EQUATIONS AND BOUNDARY CONDITIONS

Consider the classical problem of the axisymmetric flow of a viscous incompressible nonisothermal fluid with a free-stream velocity $\mathbf{U}_\infty$ ($\mathbf{U}_\infty \| Oz$) over a heated spherical solid particle of radius R with heat sources of density q_p distributed nonuniformly inside the particle. The heated surface of the particle has a large effect on the thermophysical characteristics of the ambient fluid and, hence, on the velocity and pressure fields near the particle and, eventually, on the sedimentation velocity. As was noted in the Introduction, out of all fluid transport parameters, the viscosity has the strongest dependence on temperature. Accordingly, when solving the stationary system of equations for a nonisothermal viscous fluid, we use formula (1). The other parameters of the fluid motion are assumed to be constants.

The flow is described in a spherical coordinate system comoving with the center of mass of the particle. In this paper, we obtain an axisymmetric solution of the boundary value problem for the stationary velocity-linearized system of equations describing the vector mass velocity field $\mathbf{U}_e(x) = (U_1(x), U_2(x), U_3(x))$, the pressure $P_e(x)$, and the temperature $T_e(x)$ (see Eqs. (2), (3) below) in the exterior domain $x \in \Omega_e = \mathbb{R}^3 \backslash \Omega_p$, where Ω_p is the interior spherical domain centered at the origin, and also the temperature field inside the particle, $T_p(x)$, $x \in \Omega_p$ (see Eq. (4)):

$$\nabla P_e = \mu_e \nabla^2 \mathbf{U_e} + 2(\nabla\mu_e)\cdot\nabla\mathbf{U_e} + (\nabla\mu_e)\times(\nabla\times\mathbf{U_e}), \quad \operatorname{div}\mathbf{U_e} = 0, \tag{2}$$

$$\Delta T_e = 0, \tag{3}$$

$$\Delta T_p = -\frac{q_p}{\lambda_p}, \tag{4}$$

where λ_p is the thermal conductivity of the particle.

System (2)–(4) is solved in the spherical coordinate system ($y = r/R, \varphi, \theta$) with boundary conditions

$$\lim_{y\to 1} U_r(y,\theta) = 0, \quad \lim_{y\to 1} U_\theta(y,\theta) = 0, \tag{5}$$

$$\lim_{y\to 1} T_e(y,\theta) = \lim_{y\to 1} T_p(y,\theta), \quad \lim_{y\to 1}\left(\lambda_e \frac{\partial T_e(y,\theta)}{\partial y}\right) = \lim_{y\to 1}\left(\lambda_p \frac{\partial T_p(y,\theta)}{\partial y}\right), \tag{6}$$

$$\lim_{y\to\infty} U_r(y,\theta) = U_\infty\cos\theta, \quad \lim_{y\to\infty} U_\theta(y,\theta) = -U_\infty\sin\theta, \quad \lim_{y\to\infty} P_e = P_\infty, \quad \lim_{y\to\infty} T_e = T_\infty, \tag{7}$$

$$\lim_{y\to 0}\left|T_p\right| < \infty, \tag{8}$$

which represent the no-slip condition on the particle surface $y = 1$ for the normal $U_r(y,\theta)$ and tangent $U_\theta(y,\theta)$ components of $\mathbf{U_e}$ (see (5)), the equality of the temperatures and the continuity of the radial heat fluxes (see (6)), and standard conditions (see [20]) away from the particle, as $y \to \infty$ (see (7)) and inside the particle, as $y \to 0$ (see (8)).

The boundary conditions (7) for the mass velocity components away from the particle imply that a solution for $U_r(y,\theta)$, $U_\theta(y,\theta)$, and $P_e(y,\theta)$ can be sought in the form of expansions in terms of Legendre polynomials $P_n(x)$ and Gegenbauer polynomials $C_n^{-1/2}(x)$, $x = \cos\theta$.

Finally, the force acting on the particle is found by integrating the stress tensor over the particle surface (see [21]):

$$\lim_{r\to R} F_z(r) = \lim_{r\to R}\int_S \left(-P_e(r/R,\theta)\cos\theta + \sigma_{rr}\cos\theta - \sigma_{r\theta}\sin\theta\right) r^2 \sin\theta d\theta d\varphi. \tag{9}$$

It was shown in [20] that, since the Legendre and Gegenbauer polynomials are orthogonal, the indicated force is determined only by the first terms of these expansions. Accordingly, we assume that

$$U_r(y,\theta) = U_\infty G(y)\cos\theta, \quad U_\theta(y,\theta) = -U_\infty g(y)\sin\theta, \tag{10}$$

where $G(y)$ and $g(y)$ are the functions of radial coordinate to be determined.

2. SOLUTION OF THE HEAT TRANSFER EQUATIONS AND THE VELOCITY-LINEARIZED NAVIER–STOKES EQUATION

In spherical coordinates, the system of equations of a viscous incompressible nonisothermal fluid describing the velocity and pressure distributions outside the particle has the following form [21]:

$$\frac{\partial P_e}{\partial y} = \frac{\partial \sigma_{rr}}{\partial y} + \frac{2}{y}\sigma_{rr} + \frac{1}{y}\frac{\partial\sigma_{r\theta}}{\partial\theta} + \frac{\cot\theta}{y}\sigma_{r\theta} - \frac{\sigma_{\theta\theta}-\sigma_{\varphi\varphi}}{y}, \tag{11}$$

$$\frac{\partial P_e}{\partial\theta} = y\frac{\partial\sigma_{r\theta}}{\partial y} + 3\sigma_{r\theta} + \frac{\partial\sigma_{\theta\theta}}{\partial\theta} + \cot\theta(\sigma_{\theta\theta} - \sigma_{\varphi\varphi}), \tag{12}$$

$$\frac{1}{y}\frac{\partial}{\partial y}(y^2 U_r) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta U_\theta) = 0, \tag{13}$$

and the heat equations describing the temperature distributions outside ($t_e(y,\theta) = T_e(y,\theta)/T_\infty$) and inside the particle ($t_p(y,\theta) = T_p(y,\theta)/T_\infty$) become

$$\frac{\partial}{\partial y}\left(y^2\frac{\partial t_e}{\partial y}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t_e}{\partial\theta}\right) = 0. \tag{14}$$

$$\frac{1}{y^2}\frac{\partial}{\partial y}\left(y^2\frac{\partial t_p}{\partial y}\right) + \frac{1}{y^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t_p}{\partial\theta}\right) = -\frac{R^2}{\lambda_p T_\infty}q_p. \tag{15}$$

where σ_{rr}, $\sigma_{r\theta}$, $\sigma_{\theta\theta}$, and $\sigma_{\varphi\varphi}$ are the stress tensor components in the spherical coordinate system, which are given by the relations [21]

$$\sigma_{rr} = 2\mu_e\frac{\partial U_r}{\partial y}, \quad \sigma_{\theta\theta} = \mu_e\left(\frac{2}{y}\frac{\partial U_\theta}{\partial\theta} + \frac{2}{y}U_r\right),$$

$$\sigma_{\varphi\varphi} = \mu_e\left(\frac{2}{y}U_r + \frac{2}{y}\cot\theta\, U_\theta\right), \quad \sigma_{r\theta} = \mu_e\left(\frac{\partial U_\theta}{\partial y} + \frac{1}{y}\frac{\partial U_r}{\partial\theta} - \frac{U_\theta}{y}\right).$$

Let us find the temperature fields outside and inside the particle. Following [20], Eqs. (14) and (15) are solved by the method of separation of variables. The boundary conditions (5) show that the velocity field is coupled to the temperature field only via the temperature dependence of the viscosity (see formulas (1), (11), (12)). Since the thermal conductivity of the particle is much greater than that of the fluid ($\lambda_p \gg \lambda_e$), which takes place for most actual fluids, the angular dependence of the viscosity in the particle–fluid system can be neglected (weak angular asymmetry of the temperature distribution is assumed) and we can assume that $\mu_e(t_e(y,\theta)) \approx \mu_e(t_{e0}(y))$. Thus, we search for θ-independent solutions for $t_e(y,\theta) = t_{e0}(y)$, $t_p(y,\theta) = t_0(y)$. The functions $t_{e0}(y)$ are found using the equation

$$\frac{d}{dy}\left(y^2\frac{dt_{e0}}{dy}\right) = 0,$$

with the following boundary condition away from the particle as $y \to \infty$:

$$\frac{T_e(y)}{T_\infty} = 1 + \frac{\gamma}{y}, \tag{16}$$

where γ is a constant determined by the boundary conditions on the particle surface.

The temperature field inside the particle is described by the equation

$$\frac{1}{y^2}\frac{\partial}{\partial y}\left(y^2\frac{\partial t_p(y,\theta)}{\partial y}\right) + \frac{1}{y^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial t_p(y,\theta)}{\partial\theta}\right) = -\frac{R^2}{\lambda_p T_\infty}q_p(y,\theta). \tag{17}$$

To find this field, the right-hand side of Eq. (17) is decomposed in a series in terms of the Legendre polynomials $P_n(x)$, $x = \cos\theta$. As a result, we have

$$-\frac{R^2}{\lambda_p T_\infty}q_p(y,\theta) = \sum_{n=0}^{\infty} q_n(y)P_n(\cos\theta),$$

where

$$q_n(y) = -\frac{R^2}{\lambda_p T_\infty}\frac{2n+1}{2}\int_{-1}^{+1} q_p(y, \arccos x)P_n(x)dx.$$

The function $t_p(y,\theta)$ is sought in the form

$$t_p(y,\theta) = \sum_{n=0}^{\infty} t_n(y)P_n(\cos\theta).$$

Substituting this expression into (17) yields an inhomogeneous ordinary differential equation for determining $t_n(y)$:

$$\frac{d^2t_n(y)}{dy^2}+\frac{2}{y}\frac{dt_n(y)}{dy}-\frac{n(n+1)}{y^2}t_n(y)=q_n(y), \tag{18}$$

Its solution satisfying the condition that the temperature field is finite as $y\to 0$ has the form

$$t_n(y)=B_ny^n+\frac{1}{(2n+1)y^{n+1}}\int_0^1\psi_n(y)y^ndy+\frac{1}{2n+1}\left(y^n\int_1^y\frac{\psi_n(y)}{y^{n+1}}dy-\frac{1}{y^{n+1}}\int_1^y\psi_n(y)y^ndy\right), \tag{19}$$

where

$$\psi_n(y)=y^2q_n(y)=-\frac{(2n+1)R^2}{2\lambda_pT_\infty}y^2\int_{-1}^{+1}q_p(y,\arccos x)P_n(x)dx.$$

Under the above assumption, the temperature field inside the particle is expressed as

$$\frac{T_p(y)}{T_\infty}=B+\frac{1}{4\pi R\lambda_pT_\infty y}\int_V q_p(y,\theta)dV+\int_1^y\frac{\psi_0(y)}{y}dy-\frac{1}{y}\int_1^y\psi_0(y)dy; \tag{20}$$

here, we used the equality

$$\int_0^1\psi_0(y)dy=\frac{1}{4\pi R\lambda_pT_\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}q_p(r,\theta)r^2\sin\theta drd\theta d\varphi=\int_V q_p(r,\theta)dV.$$

The constants B and γ are determined by the boundary conditions on the particle surface and have the form

$$\gamma=t_S-1,\qquad B=\left(1-\frac{\lambda_e}{\lambda_p}\right)(t_S-1),$$

where λ_e is the thermal conductivity of the fluid, $\gamma=t_S-1$ is a dimensionless parameter characterizing the heating of the particle surface, $t_S=T_S/T_\infty$, T_S is the mean surface temperature of the particle determined by the relation

$$T_S=T_\infty+\frac{1}{4\pi R\lambda_e}\int_V q_p(y,\theta)dV; \tag{21}$$

here, the integral is taken over the entire volume of the particle.

It follows from formula (21) that the heat source density q_p has a large effect on the mean surface temperature of the particle.

In view of formula (16), expression (1) becomes

$$\mu_e(y)=\mu_e(t_{e0}(y))=\mu_\infty\left(1+\sum_{n=1}^{N}F_n\frac{\gamma^n}{y^n}\right)\exp\left(-\frac{A\gamma}{y}\right). \tag{22}$$

The relation between $G(y)$ and $g(y)$ is determined by the continuity equation (13). Using representations (10) yields

$$g(y)=G(y)+\frac{y}{2}\frac{dG(y)}{dy}. \tag{23}$$

A differential equation for $G(y)$ is derived as follows. Equation (11) is differentiated with respect to θ and (12), with respect to y; the results are subtracted from each other; and expressions (22) and (23) are taken into account. As a result, on the interval $y\in[1,\infty)$, we obtain the fourth-order homogeneous differential equation

$$y^3\sum_{n=0}^{N}\alpha_n^{(1)}\frac{\gamma^n}{y^n}\frac{d^4G(y)}{dy^4}+y^2\sum_{n=0}^{N+1}\alpha_n^{(2)}\frac{\gamma^n}{y^n}\frac{d^3G(y)}{dy^3}+y\sum_{n=0}^{N+2}\alpha_n^{(3)}\frac{\gamma^n}{y^n}\frac{d^2G(y)}{dy^2}+\sum_{n=0}^{N+2}\alpha_n^{(4)}\frac{\gamma^n}{y^n}\frac{dG(y)}{dy}=0, \tag{24}$$

where $\alpha_0^{(1)} = F_0 = 1$, $\alpha_0^{(2)} = 8$, $\alpha_0^{(3)} = 8$, $\alpha_0^{(4)} = -8$, the coefficients F_n vanish for $n < 0$ and $n \geq N + 1$ (the number N is determined by (1)), and, for $n \geq 1$,

$$\alpha_n^{(1)} = F_n, \quad \alpha_n^{(2)} = 2(4 - n)F_n + 2AF_{n-1},$$

$$\alpha_n^{(3)} = (n^2 - 9n + 8)F_n + 2A(10 - 2n)F_{n-1} + A^2F_{n-2},$$

$$\alpha_n^{(4)} = 2(n^2 - 4)F_n + 2A(1 - 2n)F_{n-1} + 2A^2F_{n-2}.$$

From boundary conditions (5)–(8), we need to remove four boundary conditions on T_e and T_p. The remaining conditions take the form

$$G(1) = G'(1) = 0, \quad \lim_{y\to\infty} G(y) = 1. \tag{25}$$

Note also that a solution of Eq. (24) is the function

$$G_0(y) = \text{const}. \tag{26}$$

For Eq. (24), $y = 0$ is a regular singular point; therefore, its solution is sought in the form of a generalized power series

$$G(y) = y^{\rho}\sum_{n=0}^{\infty}\frac{\Delta_n}{y^n}, \quad \Delta_0 \neq 0. \tag{27}$$

Below, we will need the following derivatives of $G(y)$:

$$G'(y) = \sum_{n=0}^{\infty}(-n+\rho)\Delta_n y^{-n+\rho-1}, \quad G''(y) = \sum_{n=0}^{\infty}(-n+\rho)(-n+\rho-1)\Delta_n y^{-n+\rho-2}, \tag{28}$$

$$G^{(3)}(y) = \sum_{n=0}^{\infty}(-n+\rho)(-n+\rho-1)(-n+\rho-2)\Delta_n y^{-n+\rho-3}, \tag{29}$$

$$G^{(4)}(y) = \sum_{n=0}^{\infty}(-n+\rho)(-n+\rho-1)(-n+\rho-2)(-n+\rho-3)\Delta_n y^{-n+\rho-4}. \tag{30}$$

Substituting (27)–(30) into (24) yields the following governing equation for ρ:

$$\rho^4 + 2\rho^3 - 5\rho^2 - 6\rho = 0. \tag{31}$$

Its roots are $\rho_1 = -3$, $\rho_2 = -1$, $\rho_3 = 0$, and $\rho_4 = 2$.

Using these roots of Eq. (31), we write the form of solutions [22]. The largest (in absolute value) root $\rho_1 = -3$ is associated with the solution

$$G_1(y) = \frac{1}{y^3}\sum_{n=0}^{\infty}\frac{\Delta_n^{(1)}}{y^n}, \quad \Delta_0^{(1)} \neq 0, \tag{32}$$

while the second and third solutions have the form

$$G_2(y) = \frac{1}{y}\sum_{n=0}^{\infty}\frac{\Delta_n^{(2)}}{y^n} + \beta_2 G_1(y)\ln y, \quad \Delta_0^{(2)} \neq 0, \tag{33}$$

$$G_3(y) = y^2\sum_{n=0}^{\infty}\frac{\Delta_n^{(3)}}{y^n} + \beta_3 G_1(y)\ln y, \quad \Delta_0^{(3)} \neq 0. \tag{34}$$

Substituting (32)–(34) into (24) and applying the method of undetermined coefficients, we derive recurrence formulas for the coefficients $\Delta_n^{(1)}$, $\Delta_n^{(2)}$, and $\Delta_n^{(3)}$, namely,

$$\Delta_0^{(1)} = 1, \quad \Delta_0^{(2)} = 1, \quad \Delta_1^{(2)} = \frac{\gamma}{8}(24\alpha_1^{(1)} - 6\alpha_1^{(2)} + 2\alpha_1^{(3)} - \alpha_1^{(4)}), \quad \Delta_2^{(2)} = 1,$$

$$\beta_2 = \frac{\gamma}{30}(2(60\alpha_1^{(1)} - 12\alpha_1^{(2)} + 3\alpha_1^{(3)} - \alpha_1^{(4)})\Delta_1^{(2)} + \gamma(24\alpha_2^{(1)} - 6\alpha_2^{(2)} + 2\alpha_2^{(3)} - \alpha_2^{(4)}));$$

for $n \geq 1$,

$$\Delta_n^{(1)} = -\frac{\gamma^n}{n(n+2)(n+3)(n+5)} \times \sum_{k=0}^{n-1}((k+4)(k+5)(k+6)\alpha_{n-k}^{(1)} - (k+4)(k+5)\alpha_{n-k}^{(2)} + (k+4)\alpha_{n-k}^{(3)} - \alpha_{n-k}^{(4)})\frac{(k+3)\Delta_k^{(1)}}{\gamma^k}; \tag{35}$$

for $n \geq 3$,

$$\Delta_n^{(2)} = -\frac{\gamma^n}{n(n+1)(n+3)(n-2)}\Bigg(\sum_{k=0}^{n-1}((k+2)(k+3)(k+4)\alpha_{n-k}^{(1)} - (k+2)(k+3)\alpha_{n-k}^{(2)} + (k+2)\alpha_{n-k}^{(3)} - \alpha_{n-k}^{(4)})\frac{(k+1)\Delta_k^{(2)}}{\gamma^k} - \beta_2\sum_{k=0}^{n-2}((4k^3 + 54k^2 + 238k + 342)\alpha_{n-k-2}^{(1)} - (3k^2 + 24k + 47)\alpha_{n-k-2}^{(2)} + (2k+7)\alpha_{n-k-2}^{(3)} - \alpha_{n-k-2}^{(4)})\frac{\Delta_k^{(1)}}{\gamma^{k+2}}\Bigg). \tag{36}$$

No recurrence formula for the coefficients $\Delta_n^{(3)}$ is presented, since function (34) does not satisfy boundary condition (25) as $y \to \infty$.

Note that the constants $\Delta_0^{(1)}$, $\Delta_0^{(2)}$, and $G_0(y)$ can be chosen to be arbitrary. However, in the limiting case of small temperature drops ($\gamma \to 0$), the desired solution for $G(y)$ has to pass into the Stokes solution for the flow past a fixed sphere [20]. Therefore, $\Delta_0^{(1)} = 1$, $\Delta_0^{(2)} = 1$, and $G_0(y) = 1$.

To show that the series determining $G_1(y)$, $G_2(y)$ in formulas (32) and (33) converge uniformly for $y \in [1, \infty)$, we establish an asymptotic estimate for the coefficients $\Delta_n^{(1)}$ and $\Delta_n^{(2)}$.

Note that the coefficient $\gamma = T_S/T_\infty - 1$ is always less than unity ($0 \leq \gamma < 1$), since T_S cannot exceed the boiling point of the corresponding fluid, while T_∞ is the fluid temperature under normal conditions. For example, for water, $T_S = 353°K$ (80°C), $T_\infty = 273°K$ (0°C), and $\gamma = 0.29$.

As $n \to \infty$, the leading term a_n in the asymptotic expansion of $\Delta_n^{(1)}$ is given by

$$a_n = \frac{\gamma^n}{n(n+2)(n+3)(n+5)}\sum_{k=0}^{n-1}(k+3)(k+4)(k+5)(k+6)\frac{\alpha_{n-k}^{(1)}a_k}{\gamma^k}. \tag{37}$$

Therefore, the sequence a_n obeying inequality (37) satisfy the estimates

$$a_0 = 1, \quad |a_1| = 5\left|\alpha_1^{(1)}\right|\gamma \leq C\gamma, \quad |a_2| \leq C(C+1)\gamma^2.$$

where $C = 5\max_{1 \leq k \leq N}\left|\alpha_k^{(1)}\right|$.

By induction, we prove the estimate

$$|a_n| \leq C(C+1)^{n-1}\gamma^n, \tag{38}$$

which implies the uniform convergence of the series in (32) for $y > \gamma(C+1)$.

If $\gamma(C+1) \geq 1$, then, since the coefficients of series (32) are analytic for $y \in [1, \infty)$, this series converges uniformly for $y \in [1, \infty)$. The uniform convergence of the series in (33) for $y \in [1, \infty)$ is established in a similar manner.

Thus, the solution of Eq. (24) satisfying boundary condition (25) as $y \to \infty$ has the form

$$G(y) = 1 + A_1G_1(y) + A_2G_2(y), \tag{39}$$

where the functions $G_1(y)$, $G_2(y)$ are given by (32) and (33).

Since the solutions $G_1(y)$, $G_2(y)$ are linearly independent, the constants A_1 and A_2 are uniquely determined by boundary conditions (25) as $y \to 1$. We have

$$\begin{cases} A_1G_1(1) + A_2G_2(1) = -1, \\ A_1G_1'(1) + A_2G_2'(1) = 0, \end{cases} \tag{40}$$

Therefore,

$$A_1 = -\frac{G_2'(1)}{G_2'(1)G_1(1) - G_1'(1)G_2(1)}, \quad A_2 = \frac{G_1'(1)}{G_2'(1)G_1(1) - G_1'(1)G_2(1)}. \tag{41}$$

As a result, we have proved the following theorem.

Theorem. *The function* $G(y) = 1 + A_1G_1(y) + A_2G_2(y)$, *where* $G_1(y)$, $G_2(y)$ *are given by* (32) *and* (33) *and the coefficients* A_1, A_2 *are given by* (41), *is a unique solution of Eq.* (24) *satisfying boundary conditions* (25).

Since $G(y)$ has been determined, using the relation between $G(y)$ and $g(y)$ and applying formulas (23) and (10), we derive the following expressions for the mass velocity components and pressure:

$$U_r(y,\theta) = U_\infty(1 + A_1G_1(y) + A_2G_2(y))\cos\theta, \tag{42}$$

$$U_\theta(y,\theta) = -U_\infty\left(1 + A_1\left(G_1(y) + \frac{y}{2}G_1'(y)\right) + A_2\left(G_2(y) + \frac{y}{2}G_2'(y)\right)\right)\sin\theta, \tag{43}$$

$$P_e(y,\theta) = P_\infty + \frac{\mu_e U_\infty}{R}\left(\frac{y^2}{2}\frac{d^3G(y)}{dy^3} + 3y\frac{d^2G(y)}{dy^2} + 2\frac{dG(y)}{dy} + \frac{1}{\mu_e}\frac{d\mu_e}{dy}\left(\frac{y^2}{2}\frac{d^2G(y)}{dy^2} + y\frac{dG(y)}{dy}\right)\right)\cos\theta. \tag{44}$$

Thus, the resulting expressions (42)–(44) for the mass velocity components and pressure are of general character, since they involve the integration constants A_1, A_2 determined by the boundary conditions on the particle surface. As a result, the above-developed method for solving the Navier–Stokes equations can be extended to a wide variety of applications (thermophoresis, photophoresis, diffusionphoresis, the motion of particles in a gravitational field, the motion of particles in different-temperature channels, etc.), where the force acting on spherical particles and the velocity of their ordered motion have to be estimated taking into account the temperature dependence of the fluid viscosity.

CONCLUSIONS

The boundary value problem for the velocity-linearized Navier–Stokes equation in spherical coordinates was solved assuming that the viscosity is an exponential-power function of temperature (see formula (1)). The uniqueness of the solution was proved, and expressions for the components of the mass velocity $\mathbf{U}_e$ and the pressure P_e were found. The results can be used to describe the sedimentation of particles in different-temperature channels, to develop methods for fine liquid purification, etc.

As an application of the above-developed theory, we consider the sedimentation of a nonuniformly heated spherical solid particle in a nonisothermal viscous fluid. As was noted in the Introduction, sedimentation is widely used in industry, agriculture, and medicine. The sedimentation velocity is a major characteristic of this process. It can be determined as follows. Let us introduce a coordinate system comoving with the center of mass of the particle. Then the problem is reduced to the classical one of the axisymmetric flow of a viscous incompressible nonisothermal fluid with velocity $\mathbf{U}_\infty$ ($\mathbf{U}_\infty \| Oz$) over a heated spherical solid particle with heat sources of intensity q_p distributed nonuniformly inside it. In such a coordinate system, the free-stream velocity of the fluid is opposite in sign to the sedimentation velocity, i.e., $\mathbf{U}_{\text{sed}} = -\mathbf{U}_\infty$.

The heated surface of the particle has a large effect on the thermophysical characteristics of the ambient fluid and, hence, on the velocity and pressure fields near the particle and, eventually, on the sedimentation velocity. Out of all fluid transport parameters, the viscosity has the strongest dependence on temperature. Accordingly, when solving the system of fluid dynamics equations, we use formula (1).

The force acting on the particle is found by integrating the stress tensor over the particle surface (see formula (9)). Substituting expressions (42)–(44) for the mass velocity components and pressure into (9) and integrating the result, we obtain

$$F_z = -4\pi R\mu_\infty U_\infty A_2,$$

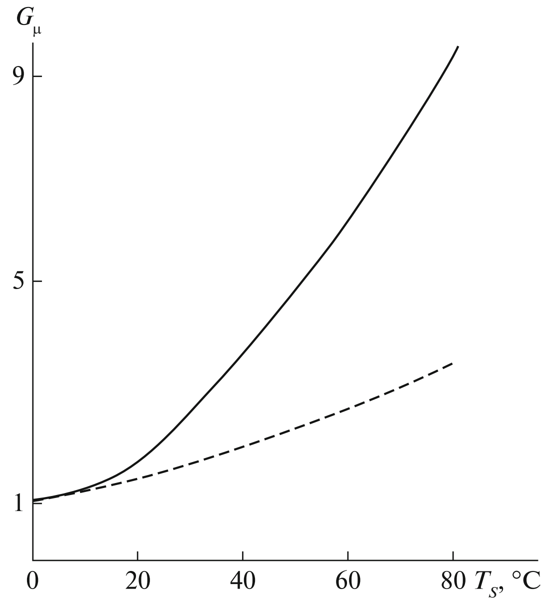

Fig. 1. Plot of G_μ as a function of the particle's mean surface temperature T_S.

which, in view of (41), yields

$$F_z = \frac{4\pi R\mu_\infty U_\infty G_1'(1)}{G_2(1)G_1'(1) - G_1(1)G_2'(1)}. \tag{45}$$

The particle settling under gravity (due to sedimentation) in the viscous fluid eventually moves with a constant velocity, at which the gravity is balance by the hydrodynamic forces. With allowance for buoyancy, the force of gravity acting on the particle is given by

$$F = \frac{4}{3}\pi R^3(\rho_p - \rho_e)g, \tag{46}$$

where g is the acceleration of gravity.

Equating forces (45) and (46) yields a formula for finding the sedimentation velocity:

$$\mathbf{U}_{\text{sed}} = G_\mu(T_S)\mathbf{n}_z, \quad G_\mu(T_S) = \frac{2}{9}R^2\frac{\rho_p(T_S) - \rho_e(T_S)}{\mu_\infty f_G}g, \quad f_G = \frac{2}{3}\frac{G_1(1)}{G_2(1)G_1'(1) - G_1(1)G_2'(1)}, \tag{47}$$

where $\mathbf{n}_z$ is the unit vector in the $0z$ direction.

It follows from (47) that the heating of the particle surface has a large effect on the sedimentation velocity. The coefficient $G_\mu(T_S)$ is expressed in terms of the functions $G_1(y)$ and $G_2(y)$, which depend on the exponential-power representation of viscosity (see (1)) and on the particle's mean surface temperature (see (21)). To illustrate the dependence of $U_{\text{sed}}(T_S)$ on T_S, Fig. 1 depicts the function $G_\mu(T_S)$ for large coal particles of radius $R = 10$ μm moving in water (solid curve). The broken curve corresponds to small relative temperature drops ($\gamma \to 0$), but the viscosity is taken at the particle's mean surface temperature. A comparison of these curves shows that the relative error grows as the particle's mean surface temperature increases. Thus, the dependence of viscosity on temperature has to be taken into account when we describe the sedimentation of a heated spherical solid particle in a viscous nonisothermal fluid.

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