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A Family of Singular Differential Equations

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Abstract—A family of singular differential equations with variable coefficients and parameter $k \in \mathbb{R}$ is introduced into the consideration. The properties inherent in all differential equations of this family are investigated and theorems on the solvability of a number of initial problems for the considered family are formulated.

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1. INTRODUCTION

In this paper we study a family of singular differential equations with variable coefficients, parameter $k \in \mathbb{R}$ and operator coefficient A. This family contains equations, each of which was studied earlier by different authors independently of other equations of the considered family. Appropriate references to these equations will be given in the course of the presentation (see Theorem 1 below). In Lemmas 1–6 we establish a number of properties related to the change of the parameter k and simultaneously inherent in all equations of the introduced family. These properties will be further applied in Theorems 2–4 to establish the solvability of a number of initial problems for this family. The equations will be considered in the Banach space E, although the established results can be extended to wider sets.

2. OPERATORS OF MOTION BY PARAMETER $k \in \mathbb{R}$

We will begin the study of equations by studying the properties of various differential expressions, which will be also called as differential operators.

For $t > 0$, we introduce the differential operator

$$L_{g,k}u(t) = u''(t) + kg''(t)\frac{du(t)}{dg(t)} + \frac{k^2 g^{(3)}(t)}{4g'(t)}u(t), \tag{1}$$

where the parameter $k \in \mathbb{R}$, $g(t)$, $t \geq 0$, is a monotone thrice continuously differentiable function and we use a convenient notation for the following

$$\frac{du(t)}{dg(t)} = \frac{1}{g'(t)}\frac{du(t)}{dt}.$$

We will also assume that $g'(0) = 0$, i.e., the differential operator $L_{g,k}u(t)$ has a singular feature in the coefficients. Note that the operator $L_{g,k}$ includes only derivatives of the function $g(t)$, but in the future the function itself will be used.

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Lemma 1. *Let the functions $u(t)$ and $g(t)$ be three and four times continuously differentiable, respectively. Then the equality*

$$L_{g,k+2}\frac{du(t)}{dg(t)}=\frac{d}{dg(t)}L_{g,k}u(t)-\frac{k^2\left(g'(t)g^{(4)}(t)-g''(t)g^{(3)}(t)\right)}{4(g'(t))^3}u(t),\quad t>0 \tag{2}$$

is true.

Proof. Direct computations lead to the equations

$$L_{g,k+2}\frac{du(t)}{dg(t)}=\frac{1}{g'(t)}u^{(3)}(t)+\frac{kg''(t)}{(g'(t))^2}u''(t)+\left(\frac{(k^2+4k)g^{(3)}(t)}{4(g'(t))^2}-\frac{k(g''(t))^2}{(g'(t))^3}\right)u'(t),$$

$$\frac{d}{dg(t)}L_{g,k}u(t)=\frac{1}{g'(t)}u^{(3)}(t)+\frac{kg''(t)}{(g'(t))^2}u''(t)+\left(\frac{(k^2+4k)g^{(3)}(t)}{4(g'(t))^2}-\frac{k(g''(t))^2}{(g'(t))^3}\right)u'(t)$$
$$+\frac{k^2\left(g'(t)g^{(4)}(t)-g''(t)g^{(3)}(t)\right)}{4(g'(t))^3}u(t),$$

from which the required statement follows. The lemma is proved.

Let us introduce the operators

$$\Phi_{g,k}u(t)=(g'(t))^{k-1}u(t),\quad \Psi_{g,k}u(t)=g'(t)u'(t)+(k-1)g''(t)u(t),$$

which also as in Lemma 1 will allow to change the second index of the operator $L_{g,k}$ after their application.

The following four statements are also established by a direct verification.

Lemma 2. *Let the functions $u(t)$ and $g(t)$ be twice and thrice continuously differentiable, respectively. Then the equality*

$$L_{g,2-k}\Phi_{g,k}u(t)=\Phi_{g,k}L_{g,k}u(t),\quad t>0 \tag{3}$$

is true.

Lemma 3. *Let the functions $u(t)$ and $g(t)$ be three and four times continuously differentiable, respectively. Then the equality*

$$L_{g,-k}\Phi_{g,k+1}u(t)=\Phi_{g,k+1}L_{g,k}u(t)-\frac{k^2}{4}(g'(t))^{k-2}\left(g'(t)g^{(4)}(t)-g''(t)g^{(3)}(t)\right)u(t),\quad t>0 \tag{4}$$

is true.

Lemma 4. *Let the functions $u(t)$ and $g(t)$ be three and four times continuously differentiable, respectively. Then the equality*

$$L_{g,k-2}\Psi_{g,k}u(t)=\Psi_{g,k}L_{g,k}u(t)-\frac{(k-2)^2\left(g'(t)g^{(4)}(t)-g''(t)g^{(3)}(t)\right)}{4g'(t)}u(t),\quad t>0 \tag{5}$$

is true.

Lemma 5. *Let the functions $u(t)$ and $g(t)$ be one and twice continuously differentiable, respectively. Then the equality*

$$\Psi_{g,k}u(t)=\Phi_{g,4-k}\frac{d}{dg(t)}\Phi_{g,k}u(t),\quad t>0$$

is true.

Thus, by applying the operator $\frac{d}{dg(t)}$ the second index of operator $L_{g,k}$ can be increased by two and therefore $\frac{d}{dg(t)}$ is the lift operator by parameter. The operator $\Psi_{g,k}$ reduces the second index by two, so $\Psi_{g,k}$ is the parameter descent operator. The operator $\Psi_{g,k}$ is the reflection operator by parameter, it reflects relative to the value $k=1$, and the operator $\Psi_{g,k+1}$ reflects relative to the value $k=0$. In this case, in the equations (4) and (5) there are additional terms containing the functions $u(t)$, $g(t)$.

In the future, we will need the concept of a fractional integral of a function $u(t)$ with respect to function $g(t)$

$$I_g^{\alpha} u(t) \equiv \left(\frac{d}{dg(t)}\right)^{-\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int\limits_0^t (g(t)-g(s))^{\alpha-1}\ g'(s)u(s)\,ds, \quad \alpha > 0,$$

as well as the definition of the fractional derivative of the function $g(t)$ (see Chapter 4, Sec. 18 [1]).

Also, the direct test establishes an analogue of Lemma 1 for the fractional integral $\left(\frac{d}{dg(t)}\right)^{-1}$, which is the operator of the "lifting" by the parameter by the number -2, i.e., the operator of the descent.

Lemma 6. *Let the functions $u(t)$ and $g(t)$ be twice and thrice continuously differentiable, respectively. Then the equality*

$$L_{g,k}\left(\frac{d}{dg(t)}\right)^{-1} u(t) = \left(\frac{d}{dg(t)}\right)^{-1} L_{g,k+2} u(t) + g''(0)u(0)$$

$$+\frac{k^2 g^{(3)}(t)}{4g'(t)} \int\limits_0^t g'(s)u(s)\,ds - \frac{k^2}{4}\int\limits_0^t g^{(3)}(s)u(s)\,ds, \quad t > 0 \tag{6}$$

is true.

In the equality (2), (4)–(6) there are terms containing the function $g(t)$, which do not allow us to state that the transformation operators are found, weaving the operators considered in them (differential expressions) and which will make it difficult to find the solution of the corresponding equation in the future. For more information about conversion operators, see, for example, [2] and Chapter 2 in [3]. We therefore impose a simplifying condition on the function $g(t)$, and accordingly the class of equations in question.

Condition 1. At $t \geq 0$, the function $g(t)$ is monotonic and four times continuously differentiable. We will also assume that $g'(0) = 0$ and

$$g'(t)g^{(4)}(t) - g''(t)g^{(3)}(t) = 0. \tag{7}$$

The set of functions $g(t)$ satisfying condition 1 can be explicitly specified. Indeed, equation (7) by replacing $g'(t) = h(t)$ reduces to the differential equation $h(t)h^{(3)}(t) - h'(t)h''(t) = 0$, whence

$$\left(\frac{h''(t)}{h(t)}\right)' = 0 \ \Leftrightarrow\ h''(t) - c_0 h(t) = 0, \quad c_0 \in \mathbb{R}.$$

Accounting the condition $g'(0) = 0$, for the function $g(t)$ we obtain the following representations.

If $c_0 = 0$, then $g(t) = c_1 t^2 + c_2,\ c_1, c_2 \in \mathbb{R}$.

If $c_0 = \mu^2, \mu \neq 0$, then $g(t) = c_1 \cosh \mu t + c_2,\ c_1, c_2 \in \mathbb{R}$.

If $c_0 = -\mu^2, \mu \neq 0$, then $g(t) = c_1 \cos \mu t + c_2,\ c_1, c_2 \in \mathbb{R}$.

Let the function $g(t)$ satisfy Condition 1, then the set of operators $L_{g,k}$ by their properties, naturally, be combined into one family. The constants $\mu, c_1, c_2 \in \mathbb{R}$ for the solution of the equation considered in the following do not have a fundamental value, and since the differential operator $L_{g,k}$ includes only derivatives of the function $g(t)$, then instead of Condition 1 we assume the following condition to be fulfilled.

Condition 2. The function $g(t)$ has one of the types: either (a) $g(t) = t^2/2$, or (b) $g(t) = \cosh t$, or (c) $g(t) = \cos t$.

We also note that the operator $L_{\mathrm{ch}\,t,k}$ by replacing $t = -i\tau$ reduces to the operator $L_{\cos\tau,k}$ with the differentiation operation by τ, and the operator $L_{t^2/2,k}$ is limiting to $L_{\mathrm{ch}\,t,k}$, since

$$\lim_{\mu\to 0} \frac{\cosh \mu t - 1}{\mu^2} = \frac{t^2}{2}.$$

If Condition 2 is satisfied and in Lemma 6 the function $u(t)$ additionally satisfies condition $u(0) = 0$, then the equalities (2), (4)–(6) accordingly have the form

$$L_{g,k+2}\frac{du(t)}{dg(t)} = \frac{dL_{g,k}u(t)}{dg(t)}, \tag{8}$$

$$L_{g,-k}\Phi_{g,k+1}u(t) = \Phi_{g,k+1}L_{g,k}u(t),$$

$$L_{g,k-2}\Psi_{g,k}u(t) = \Psi_{g,k}L_{g,k}u(t),$$

$$L_{g,k}\left(\frac{d}{dg(t)}\right)^{-1}u(t) = \left(\frac{d}{dg(t)}\right)^{-1}L_{g,k+2}u(t). \tag{9}$$

Theorem 1. *Let the function $u(t)$ be twice continuously differentiable and the function $g(t)$ satisfy Condition 2. Then at $t > 0$ the equality*

$$L_{g,k}\Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u(t)}{g'(t)} = \Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u''(t)}{g'(t)} + \frac{\Phi_{g,2-k}\left(g(t)-g(0)\right)^{k/2-1}}{\Gamma(k/2)}u'(0) \tag{10}$$

is true.

Proof. If $g(t) = t^2/2$, then the operator $L_{g,k}$ defined by equality (1) has the form

$$L_{g,k}u(t) = u''(t) + \frac{k}{t}u'(t)$$

and is the Bessel operator, and the validity of equality (10) is established earlier in [4], see also Chapter 1 in [5, 6–11].

If $g(t) = \cosh t$, then the operator $L_{g,k}$ has the form

$$L_{g,k}u(t) = u''(t) + k\coth t\, u'(t) + (k/2)^2\, u(t),$$

is the Jacobi equality and the equity equality (10) is established previously in [12], see [13–15].

If $g(t) = \cos t$, then the operator $L_{g,k}$ has the form

$$L_{g,k}u(t) = u''(t) + k\cot t\, u'(t) - (k/2)^2\, u(t),$$

and the equity equality (10) is established previously in [16]. The statement formulated in Theorem 1 is proved using the equalities (3), (8), (9). Really,

$$L_{g,k}\Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u(t)}{g'(t)} = \Phi_{g,2-k}L_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u(t)}{g'(t)}$$

$$= \Phi_{g,2-k}L_{g,2-k}\left(\frac{d}{dg(t)}\right)^{1-k/2}\int_0^t u(\tau)\,d\tau = \Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{1-k/2}\frac{d^2}{dt^2}\int_0^t u(\tau)\,d\tau$$

$$= \Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{1-k/2}\left(\int_0^t u''(\tau)\,d\tau + u'(0)\right)$$

$$= \Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u''(t)}{g'(t)} + \Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{1-k/2}u'(0). \tag{11}$$

Let us calculate further the expression included in (11) $\left(\frac{d}{dg(t)}\right)^{1-k/2}u'(0)$. If $0 < k < 2$, then, given the definition of the fractional derivative with respect to function $g(t)$, we obtain

$$\left(\frac{d}{dg(t)}\right)^{1-k/2}u'(0) = \frac{d}{dg(t)}\left(\frac{d}{dg(t)}\right)^{-k/2}u'(0)$$

$$= \frac{d}{dg(t)}\left(\frac{1}{\Gamma(k/2)}\int_0^t (g(t)-g(s))^{k/2-1} g'(s)\, u'(0)\, ds\right)$$

$$= \frac{1}{\Gamma(k/2+1)}\frac{d}{dg(t)}\left((g(t)-g(0))^{k/2}\right)u'(0) = \frac{1}{\Gamma(k/2)}(g(t)-g(0))^{k/2-1}u'(0).$$

If $k=2$ then $\left(\frac{d}{dg(t)}\right)^{1-k/2} u'(0) = u'(0)$. And finally, if $k>2$, then

$$\left(\frac{d}{dg(t)}\right)^{1-k/2} u'(0) = \frac{1}{\Gamma(k/2-1)}\int_0^t (g(t)-g(s))^{k/2-2} g'(s)\, u'(0)\, ds$$

$$= \frac{1}{\Gamma(k/2)}(g(t)-g(0))^{k/2-1}u'(0).$$

Thus, the equality (11) can be rewritten as

$$L_{g,k}\Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u(t)}{g'(t)} = \Phi_{g,2-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\frac{u''(t)}{g'(t)} + \frac{\Phi_{g,2-k}\,(g(t)-g(0))^{k/2-1}}{\Gamma(k/2)}u'(0),$$

thus the equality (10), and the theorem, are proved.

In the next section, Theorem 1 will be used to investigate the solvability of initial problems for the family of abstract singular equations under consideration.

3. SOLVABILITY OF INITIAL VALUE PROBLEMS

Let A be a linear closed operator in a Banach space E with the domain of definition $D(A)$ dense in E. For $k>0$, we will consider the equation

$$L_{g,k}u(t) \equiv u''(t) + kg''(t)\frac{du(t)}{dg(t)} + \frac{k^2 g^{(3)}(t)}{4g'(t)}u(t) = Au(t), \quad t>0. \tag{12}$$

As it will be indicated later, the correct statement of the initial conditions for equation (12) consists in setting the initial conditions at the point $t=0$

$$u(0)=u_0, \quad u'(0)=0. \tag{13}$$

The problem (12), (13) at $k=0$ is uniformly correct only when the operator A is the generator of cosine-operator-function $C(t)$. See Chapter 2.8 [17] and [18] for correctness of statements, terminology, and examples of generators. In this paper we will keep this assumption for $k>0$, although, as we know from [8, 15], the uniform correctness of the problem (12), (13) can be established for a wider class of operators A.

Theorem 2. *Let $k>0$, $u_0\in D(A)$, Condition 2 is satisfied and operator A is a generator of cosine-operator-function $C(t)$. Then the problem (12), (13) is uniformly correct and its solution is represented as*

$$u(t) = P_{g,k}C(t)u_0 \equiv \frac{2^{k/2}\,\Gamma(k/2+1/2)}{\sqrt{\pi}}\,(g'(t))^{1-k}\left(\frac{d}{dg(t)}\right)^{-k/2}\left(\frac{C(t)}{g'(t)}\right)u_0$$

$$= \frac{2^{k/2}\,\Gamma(k/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)}\,(g'(t))^{1-k}\int_0^t (g(t)-g(s))^{k/2-1}\,C(s)u_0\,ds. \tag{14}$$

Proof. Since $C'(0)=0$, the representation (14) for the solution of the problem (12), (13) follows from theorem 1, and the validity of the initial condition (13) and the uniqueness of the solution for each of the three possible values of the function $g(t)$ are set respectively in [4, 8, 16]. The theorem is proved.

Consequence 1. For $k > 0$ on the set of doubly differentiable functions satisfying the condition $u'(0) = 0$, the Poisson operator $P_{g,k}$ is a transformation operator weaving the operators $L_{g,k}$ and $L_{g,0}$, i.e.,

$$L_{g,k} P_{g,k} u(t) = P_{g,k} u''(t).$$

Consequence 2. Let $m > k > 0$ and the conditions of Theorem 2 are satisfied. Then the parameter shift formula is valid

$$P_{g,m} C(t) u_0 = \frac{2^{(m-k)/2}\,\Gamma(m/2+1/2)(g'(t))^{1-m}}{\Gamma(k/2+1/2)\Gamma(m/2-k/2)} \int_0^t (g(t)-g(s))^{(m-k)/2-1} \left(g'(s)\right)^k P_{g,k} C(s) u_0 \, ds.$$

Proof. Really,

$$\frac{2^{(m-k)/2}\,\Gamma(m/2+1/2)}{\Gamma(k/2+1/2)\Gamma(m/2-k/2)} (g'(t))^{1-m} \int_0^t (g(t)-g(s))^{(m-k)/2-1} \left(g'(s)\right)^k P_{g,k} C(s) u_0 \, ds$$

$$= \frac{2^{m/2}\,\Gamma(m/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)\Gamma(m/2-k/2)} (g'(t))^{1-m} \int_0^t (g(t)-g(s))^{(m-k)/2-1} g'(s)$$

$$\times \int_0^s (g(s)-g(\tau))^{k/2-1} C(\tau) u_0 \, d\tau ds = \frac{2^{m/2}\,\Gamma(m/2+1/2)}{\sqrt{\pi}\,\Gamma(k/2)\Gamma(m/2-k/2)} (g'(t))^{1-m}$$

$$\times \int_0^t C(\tau) u_0 \int_\tau^t g'(s) \, (g(t)-g(s))^{(m-k)/2-1} (g(s)-g(\tau))^{k/2-1} \, ds d\tau$$

$$= \frac{2^{m/2}\,\Gamma(m/2+1/2)(g'(t))^{1-m}}{\sqrt{\pi}\,\Gamma(k/2)\Gamma(m/2-k/2)} \int_0^t C(\tau) u_0 \int_{g(\tau)}^{g(t)} (g(t)-\xi)^{(m-k)/2-1} (\xi - g(\tau))^{k/2-1} \, d\xi d\tau$$

$$= \frac{2^{m/2}\,\Gamma(m/2+1/2)}{\sqrt{\pi}\,\Gamma(m/2)} (g'(t))^{1-m} \int_0^t (g(t)-g(\tau))^{m/2-1} C(\tau) u_0 \, d\tau = P_{g,m} C(t) u_0,$$

and the consequence is proved.

As in the paper [19], for the Poisson operator $P_{g,k}$, the inverse operator (the Sonin operator)

$$P_{g,k}^{-1} u(t) = \frac{\sqrt{\pi}}{2^{k/2}\,\Gamma(k/2+1/2)} g'(t) \left(\frac{d}{dg(t)}\right)^{k/2} (g'(t))^{k-1} u(t),$$

can be entered which has the following property on smooth functions

$$P_{g,k}^{-1} L_{g,k} u(t) = \frac{d^2}{dt^2} P_{g,k}^{-1} u(t).$$

Let us note one more possibility of solutions construction of differential equations. If in determining the solution by formula (14) instead of cosine-operator-function $C(t)$ use sine-operator-function

$$S(t) = \int_0^t C(\tau)\, d\tau, \quad S'(0) = I,$$

then, accounting the equality (10), we obtain the solution of the loaded equation

$$M_{g,k} v(t) \equiv v''(t) + \frac{k g''(t)}{g'(t)} \left(v'(t) - \frac{2^{k/2-1}\,(g(t)-g(0))^{k/2-1}}{(g'(t))^{k-2} g''(t)} v'(0) \right)$$

$$+\frac{k^2 g^{(3)}(t)}{4g'(t)}v(t) = Av(t), \quad t > 0 \tag{15}$$

satisfying the condition

$$v(0) = 0, \quad v'(0) = v_1. \tag{16}$$

Similarly to Theorem 2, the following statement is established.

Theorem 3. *Let $k > 0$, $v_1 \in D(A)$, Condition 2 is satisfied and operator A is the generator of cosine-operator-function $C(t)$. Then the problem (15), (16) is uniformly correct and its solution is represented as*

$$v(t) = Q_{g,k}S(t)v_1 \equiv k\, 2^{k/2-1}\, \Gamma(k/2)\, (g'(t))^{1-k} \left(\frac{d}{dg(t)}\right)^{-k/2} \left(\frac{S(t)}{g'(t)}\right) v_1$$

$$= k\, 2^{k/2-1}(g'(t))^{1-k} \int_0^t (g(t)-g(s))^{k/2-1}\, S(s)v_1\, ds.$$

The operator $Q_{g,k}$ differs from the Poisson operator $P_{g,k}$ only by a numerical multiplier. We will also note that if $g(t) = t^2/2$, then the operator $M_{g,k}$ defined by the equality (15) has the form

$$M_{g,k}v(t) = v''(t) + \frac{k}{t}\left(v'(t) - v'(0)\right)$$

and is the Bessel–Struve operator, and the solvability of the problem (15), (16) is set in [20].

If $g(t) = \operatorname{ch} t$, then the operator $M_{g,k}$ has the form

$$M_{g,k}v(t) = v''(t) + k\coth t\left(v'(t) - \frac{\operatorname{ch}^{2-k}(t/2)}{\operatorname{ch} t}v'(0)\right) + \frac{k^2}{4}v(t),$$

and the solvability of the problem (15), (16) is established in [21].

Consequence 3. Let $m > k > 0$ and the conditions of Theorem 3 are satisfied. Then the parameter shift formula is valid

$$Q_{g,m}S(t)v_1 = \frac{2^{(m-k)/2}\,\Gamma(m/2+1/2)(g'(t))^{1-m}}{\Gamma(k/2+1/2)\Gamma(m/2-k/2)} \int_0^t (g(t)-g(\xi))^{(m-k)/2-1} \left(g'(\xi)\right)^k Q_{g,k}S(\xi)v_1\, d\xi.$$

The proof is analogous to the proof of Consequence 2.

A theorem on the solvability of the Cauchy problem with two nonzero conditions follows from Theorems 2 and 3.

Theorem 4. *Let $k > 0$, v_0, $v_1 \in D(A)$, condition 2 is satisfied and operator A is the generator of cosine-operator-function $C(t)$. Then the function*

$$v(t) = P_{g,k}C(t)v_0 + Q_{g,k}S(t)v_1$$

is the unique solution of the equation (15) satisfying the conditions

$$v(0) = v_0, \quad v'(0) = v_1.$$

4. CONCLUSION

The operators $\Phi_{g,k}$ and $\dfrac{d}{dg(t)}$ were used in [4] and [15] to construct a solution of the problem (12), (13) at $k<0$ and smooth initial conditions, respectively in the cases $g(t)=t^2/2$ and $g(t)=\operatorname{ch} t$. Note that at $k<0$ there is a loss of uniqueness of this problem solution.

In this case, another transformation operator is used to construct the solution of the problem (12), (13) for $k=-2n+1$, $n\in\mathbb{N}$

$$\tilde P_{g,1}u(t)=\int\limits_0^t (g(t)-g(s))^{-1/2}\,\ln\frac{2\,(g(t)-g(s))}{g'(t)}\,u(s)\,ds,$$

which on the functions satisfying the condition $u'(0)=0$ has the property

$$L_{g,1}\tilde P_{g,1}u(t)=\tilde P_{g,1}u''(t).$$

We specify that the form of the operator $\tilde P_{g,1}$ is found by calculating the limit

$$\lim_{k\to 1}\frac{P_{g,k}u(t)-t^{1-k}P_{g,2-k}u(t)}{1-k}.$$

In order to find a unique solution of equation (12) at $k<0$, one should, as in [22], set the Cauchy weight problem

$$u(0)=0,\qquad \lim_{t\to 0+}(g'(t))^k u'(t)=u_1, \tag{17}$$

herewith the only solution of the problem (12), (17) has the form

$$u(t)=\frac{2^{1-k/2}\,\Gamma(3/2-k/2)}{(1-k)\sqrt{\pi}}\Phi_{g,2-k}P_{g,2-k}C(t)u_1=\frac{2^{1-k/2}\,\Gamma(3/2-k/2)}{(1-k)\sqrt{\pi}}\left(\frac{d}{dg(t)}\right)^{k/2-1}\left(\frac{C(t)}{g'(t)}\right)u_1$$

$$=\frac{2^{1-k/2}\,\Gamma(3/2-k/2)}{(1-k)\sqrt{\pi}\,\Gamma(1-k/2)}\int\limits_0^t (g(t)-g(s))^{-k/2}\,C(s)u_1\,ds.$$

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