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FRACTIONAL POWERS OF BESSEL OPERATOR AND ITS NUMERICAL CALCULATION

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The article discusses the fractional powers of the Bessel operator and their numerical implementation. An extensive literature is devoted to the study of fractional powers of the Laplace operator and their applications. Such degrees are used in the construction of functional spaces, in the natural generalization of the Schrödinger equation in the quantum theory, in the construction of the models of acoustic wave propagation in complex media (for example, biological tissues) and space-time models of anomalous (very slow or very fast) diffusion, in spectral theory etc. If we assume the radiality of the function on which the Laplace operator acts, then we receive the problem of constructing the fractional power of the Bessel operator. We propose to use a compositional method for constructing the operators mentioned earlier, which leads to constructions similar in their properties to the Riesz derivatives. The Hankel transform is considered as a basic integral transformation. On its basis, the compositional method proposed by V.V. Katrakhov and S.M. Sitnik, negative powers of the Bessel operator are constructed. The resulting operator contains the Gaussian hypergeometric function in the kernel. For further study, the generalized translation operator is considered in the article, and its properties are proved. For constructing a positive fractional power of the Bessel operator known methods of regularization of the integral are considered. Then, a scheme for the numerical calculation of fractional powers of the Bessel operator is proposed. This scheme is based on the Taylor — Delsarte formula obtained by B.M. Levitan. Examples containing the exact and approximate values of the positive and negative powers of the Bessel operator, the absolute error, and illustrations are given. The list of references contains sources with known results on similar fractional operators, as well as applications of them.

Keywords: *Bessel operator, fractional power, Hankel integral transform, composition method, transmutation operators.*

Introduction

We will consider fractional powers $(B_\gamma)^\alpha$, $\alpha \in \mathbb{R}$ of differential Bessel operator in the form

$$B_\gamma = D^2 + \frac{\gamma}{x} D, \qquad \gamma \geq 0, \qquad D := \frac{d}{dx}. \tag{1}$$

This fractional power will be constructed using Integral Transforms Composition Method (ITCM) [1–3].

In fact there are three main approaches to define and study fractional powers of the Bessel operator.

1. The most natural direct approach defining fractional powers *by explicit differential-integral operators, like Riemann — Liouville or Gerasimov — Caputo fractional operators.* Among many advantages of this approach there are obvious connections with the Riemann — Liouville operators as special cases and direct path to applications to integro–differential equations.
2. The definition via the Hankel transform in terms of the power multiplicators. This approach is appropriate to study the potential theory and special types of potentials using generalized translations instead of the classical shift operators. Among disadvantages of this approach there are difficulties with inverse operators due to divergent integrals, it demands usage of the distribution theory and different kinds of the regularization procedures.
3. Definition via Balakrishnan (and even earlier) approach via general theory of operator fractional powers [4]. In fact it is based on refined facts of the functional analysis and seems having not practical applications to integral and differential equations.

For more detailed discussion of the fractional powers of (1) on a segment and an semi-axes we refer to [5; 6]. Negative fractional degrees of operator (1) obtained in this way are *one-dimensional Riesz B-potentials* studied in [7–11]. Note that in [5; 6] it was shown that approaches 1 and 2 from the list above are quiet different. It means that action of the Hankel transform on fractional powers of (1) as in [5; 6] is not reduced again to the Hankel transform, but involves more complicated integral operators with special function kernels.

The potential theory comes from the mathematical physics. The most well-known areas of its application are electrostatic and gravitational theory, probability theory, scattering theory, biological systems and other.

First application of classical Riesz potentials was given by M. Riesz himself and it was a solution of Maxwell's equations for the electromagnetic field (see [12, p. 146] and [13]). Maxwell's equations are fundamental equations of the classical electrodynamics and optics. The equations completely describe all electromagnetic phenomena in an arbitrary environment and provide a mathematical model for electric, optical and radio technologies, such as power generation, electric motors, wireless communication, lenses, radar etc. So Riesz potentials can be used for studying of realistic single particle energy levels.

An interesting fact was noticed in [14]. Namely, in this paper it was shown that Riesz potential can be interpreted as a transmutation operator. More precisely, the operator square root of the Laplacian was obtained from the harmonic extension problem to the upper half space as the operator that maps the Dirichlet boundary condition to the Neumann condition. The same result but for hyperbolic Riesz potentials was obtained in [15].

1. Construction of fractional power of Bessel operator by ITCM

In this section using integral transform composition method we construct the operator $\mathbf{I}^\alpha_\gamma$ which is the negative fractional power of Bessel operator: $(B_\gamma)^\alpha = \mathbf{I}^\alpha_\gamma$, $\alpha > 0$. To do this, we use a suitable integral transformation, it is the Hankel transform.

Let $\gamma > 0$. The one-dimentional *Hankel transform* of a function f, such that $f(x)x^\gamma \in L_1(0,\infty)$ is expressed as

$$H_\gamma[f](\xi) = H_\gamma[f(x)](\xi) = f(\xi) = \int\limits_0^\infty f(x)\, j_{\frac{\gamma-1}{2}}(x\xi) x^\gamma dx, \qquad (2)$$

where the symbol j_ν is used for the normalized Bessel function of the first kind (see [16, p. 10; 17]):

$$j_\nu(x)=\frac{2^\nu\Gamma(\nu+1)}{x^\nu}\;J_\nu(x),$$

where J_ν is the Bessel function of the first kind.

Let $f(x)x^\gamma\in L_1(0,\infty)$ and f has bounded variation in a neighborhood of a point x of continuity of f. Then the inversion formula

$$H_\gamma^{-1}[\widehat{f}(\xi)](x)=f(x)=\frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(x\xi)\widehat{f}(\xi)\xi^\gamma\,d\xi$$

holds.

We are looking for operator $\mathbf{I}_\gamma^\alpha$ in the factorized form

$$\mathbf{I}_\gamma^\alpha=H_\gamma^{-1}(t^{-2\alpha}H_\gamma),$$

where H_γ is a Hankel transform (2). It is exactly the form which is provided by ITCM. Next we will consider even function $f=f(x)$ from the Schwartz space, $x\in\mathbb{R}$.

Let $x>0$. We have

$$\left(\mathbf{I}_\gamma^\alpha f\right)(x)=H_\gamma^{-1}\left[t^{-2\alpha}H_\gamma[f](t)\right](x)=$$

$$=\frac{2^{1-\gamma}}{\Gamma^2\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(xt)\,t^{\gamma-2\alpha}\,dt\int\limits_0^\infty j_{\frac{\gamma-1}{2}}(ty)f(y)y^\gamma dy=$$

$$=\int\limits_0^\infty (xt)^{\frac{1-\gamma}{2}}J_{\frac{\gamma-1}{2}}(xt)t^{\gamma-2\alpha}\,dt\int\limits_0^\infty (ty)^{\frac{1-\gamma}{2}}J_{\frac{\gamma-1}{2}}(ty)f(y)y^\gamma dy=$$

$$=x^{\frac{1-\gamma}{2}}\int\limits_0^\infty y^{\frac{\gamma+1}{2}}f(y)dy\int\limits_0^\infty t^{1-2\alpha}J_{\frac{\gamma-1}{2}}(xt)J_{\frac{\gamma-1}{2}}(ty)dt=$$

$$=x^{\frac{1-\gamma}{2}}\int\limits_0^x y^{\frac{\gamma+1}{2}}f(y)dy\int\limits_0^\infty t^{1-2\alpha}J_{\frac{\gamma-1}{2}}(xt)J_{\frac{\gamma-1}{2}}(ty)dt+$$

$$+x^{\frac{1-\gamma}{2}}\int\limits_x^\infty y^{\frac{\gamma+1}{2}}f(y)dy\int\limits_0^\infty t^{1-2\alpha}J_{\frac{\gamma-1}{2}}(xt)J_{\frac{\gamma-1}{2}}(ty)dt.$$

Using formula 2.12.31.1 from [18, p. 209] of the form

$$\int\limits_0^\infty t^{\beta-1}\,J_\rho(xt)J_\nu(yt)\,dt=$$

$$=\begin{cases}2^{\beta-1}x^{-\nu-\beta}y^\nu\frac{\Gamma\left(\frac{\nu+\rho+\beta}{2}\right)}{\Gamma(\nu+1)\Gamma\left(\frac{\rho-\nu-\beta}{2}+1\right)}{}_2F_1\left(\frac{\nu+\rho+\beta}{2},\frac{\nu-\rho+\beta}{2};\nu+1;\frac{y^2}{x^2}\right), & 0<y<x;\\ 2^{\beta-1}x^\rho y^{-\rho-\beta}\frac{\Gamma\left(\frac{\nu+\rho+\beta}{2}\right)}{\Gamma(\rho+1)\Gamma\left(\frac{\nu-\rho-\beta}{2}+1\right)}{}_2F_1\left(\frac{\nu+\rho+\beta}{2},\frac{\beta+\rho-\nu}{2};\rho+1;\frac{x^2}{y^2}\right), & 0<x<y,\end{cases}$$

$$x,y,\operatorname{Re}(\beta+\rho+\nu)>0;\ \operatorname{Re}\beta<2,$$

where ${}_2F_1$ is hypergeometric Gauss function which inside the circle $|z|<1$ determined as the sum of the hypergeometric series (see [19, p. 373, formula 15.3.1])

$$ {}_2F_1(a,b;c;z)=F(a,b,c;z)=\sum_{k=0}^{\infty}\frac{(a)_k(b)_k}{(c)_k}\frac{z^k}{k!}, \tag{3}$$

and for $|z|\geq 1$ it is obtained by analytic continuation of this series. In (3) parameters a,b,c and variable z can be complex, and $c\neq 0,-1,-2,\dots$ Multiplier $(a)_k$ is the Pohgammer symbol is defined by $(a)_k=a(a+1)...(a+k-1)$, $k=1,2,...,$ $(a)_0\equiv 1$.

Putting $\beta=2-2\alpha$, $\rho=\frac{\gamma-1}{2}$, $\nu=\frac{\gamma-1}{2}$ we obtain

$$\int\limits_0^{\infty} t^{1-2\alpha}J_{\frac{\gamma-1}{2}}(xt)J_{\frac{\gamma-1}{2}}(ty)dt=$$

$$=\begin{cases}\frac{2^{1-2\alpha}y^{\frac{\gamma-1}{2}}}{x^{2-2\alpha+\frac{\gamma-1}{2}}}\frac{\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma(\alpha)}{}_2F_1\left(\frac{\gamma+1}{2}-\alpha,1-\alpha;\frac{\gamma+1}{2};\frac{y^2}{x^2}\right), & 0<y<x;\\ \frac{2^{1-2\alpha}x^{\frac{\gamma-1}{2}}}{y^{2-2\alpha+\frac{\gamma-1}{2}}}\frac{\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma(\alpha)}{}_2F_1\left(\frac{\gamma+1}{2}-\alpha,1-\alpha;\frac{\gamma+1}{2};\frac{x^2}{y^2}\right), & 0<x<y,\end{cases}$$

$$\mathrm{Re}\,(\gamma+1-2\alpha)>0;$$

and

$$\left(\mathbf{I}_{\gamma}^{\alpha}f\right)(x)=\frac{2^{1-2\alpha}\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\alpha\right)}\left(x^{2\alpha-\gamma-1}\int\limits_0^x f(y){}_2F_1\left(\frac{\gamma+1}{2}-\alpha,1-\alpha;\frac{\gamma+1}{2};\frac{y^2}{x^2}\right)y^{\gamma}dy+\right.$$

$$\left.+\int\limits_x^{\infty} f(y){}_2F_1\left(\frac{\gamma+1}{2}-\alpha,1-\alpha;\frac{\gamma+1}{2};\frac{x^2}{y^2}\right)y^{2\alpha-1}dy\right).$$

In [20] the next formula is proved

$${}_2F_1\left(a,a-b+\frac{1}{2},b+\frac{1}{2};z^2\right)=(1+z)^{-2a}\,{}_2F_1\left(a,b,2b;\frac{4z}{(1+z)^2}\right),$$

using which we get for $a=\frac{\gamma+1}{2}-\alpha$, $b=\frac{\gamma}{2}$,

$$x^{2\alpha-\gamma-1}y^{\gamma}{}_2F_1\left(\frac{\gamma+1}{2}-\alpha,1-\alpha;\frac{\gamma+1}{2};\frac{y^2}{x^2}\right)=$$

$$=y^{\gamma}(x+y)^{2\alpha-\gamma-1}{}_2F_1\left(\frac{\gamma+1}{2}-\alpha,\frac{\gamma}{2};\gamma;\frac{4xy}{(x+y)}\right),$$

$$y^{2\alpha-1}{}_2F_1\left(\frac{\gamma+1}{2}-\alpha,1-\alpha;\frac{\gamma+1}{2};\frac{x^2}{y^2}\right)=$$

$$=y^{\gamma}(x+y)^{2\alpha-\gamma-1}{}_2F_1\left(\frac{\gamma+1}{2}-\alpha,\frac{\gamma}{2};\gamma;\frac{4xy}{(x+y)}\right).$$

So we get a definition of a negative fractional power of the Bessel operator for $0<\alpha<\frac{\gamma+1}{2}$

$$(B_{\gamma})^{-\alpha}f(x)=\left(\mathbf{I}_{\gamma}^{\alpha}f\right)(x)=$$

$$= \frac{2^{1-2\alpha}\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\alpha\right)} \int\limits_0^\infty f(y)(x+y)^{2\alpha-\gamma-1} {}_2F_1\left(\frac{\gamma+1}{2}-\alpha, \frac{\gamma}{2}; \gamma; \frac{4xy}{(x+y)}\right) y^\gamma dy. \quad (4)$$

Since the hypergeometric series (3) converges only in the unit circle of the complex plane, it is necessary to construct an analytic continuation of the hypergeometric function beyond the boundary of this circle, to the entire complex plane. One of the ways to continue analytically is to use the Euler integral representation of the form

$${}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(b-c)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt, \quad (5)$$

$$0 < \operatorname{Re} b < \operatorname{Re} c, \qquad |\arg(1-z)| < \pi,$$

in which the right side is defined under the specified conditions, ensuring the convergence of the integral.

2. Generalized translation and fractional power of Bessel operator

We will use the subspace of the space of rapidly decreasing functions:

$$S_{ev} = \left\{ f \in C^\infty_{ev} : \sup_{x\in[0,\infty)} \left|x^\alpha D^\beta f(x)\right| < \infty \quad \forall \alpha,\beta \in \mathbb{Z}_+ \right\},$$

where C^∞_{ev} is a class of infinetly differentiable functions on $[0,\infty)$, such that $\left.\frac{d^{2k+1}f}{dx_i^{2k+1}}\right|_{x=0} = 0$ for all non-negative integer k.

We consider $\gamma \geq 0$. Let $L_p^\gamma(0,\infty) = L_p^\gamma$, $1{\le}p{<}\infty$, be the space of all measurable in $(0,\infty)$ functions, admitting even continuation on $\mathbb{R}$, such that

$$\int\limits_0^\infty |f(x)|^p x^\gamma dx < \infty.$$

For a real number $p \geq 1$, the L_p^γ–norm of f is defined by

$$||f||_{p,\gamma} = \left(\int\limits_0^\infty |f(x)|^p x^\gamma dx \right)^{1/p}.$$

It is known (see [16]) that L_p^γ is a Banach space. For every $1 \leq p < \infty$ the Schwartz class S_{ev} is dense in L_p^γ.

In this section we consider the transmutation operator called the *generalized translation* of the form (see [17]):

$${}^\gamma T_x^y f(x) = C(\gamma) \int\limits_0^\pi f(\sqrt{x^2+y^2-2xy\cos\varphi}) \sin^{\gamma-1}\varphi d\varphi, \qquad \gamma > 0, \quad (6)$$

$$C(\gamma) = \left(\int\limits_0^\pi \sin^{\gamma-1}\varphi d\varphi \right)^{-1} = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{\gamma}{2}\right)}, \qquad \gamma > 0.$$

For $\gamma=0$ generalized translation ${}^\gamma T_x^y$ is

$$^0T_x^y=T_x^yf(x)=\frac{f(x+y)+f(x-y)}{2}.$$

It is known [17] that

$$^\gamma T_x^y j_{\frac{\gamma-1}{2}}(x\xi)=j_{\frac{\gamma-1}{2}}(x\xi)\,j_{\frac{\gamma-1}{2}}(y\xi). \tag{7}$$

Let consider some properties of the generalized translation.

Property 1. *For the generalized translation operator* ${}^\gamma T_x^y$ *the representation*

$$^\gamma T_x^y f(x)=2^{\gamma-1}C(\gamma)\int\limits_0^1 f\left((x+y)\sqrt{1-\frac{4xy}{(x+y)^2}z}\right)z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz \tag{8}$$

is valid.

Proof. We transform the generalized translation operator as follows. First in (6) putting $\varphi=2\alpha$ we obtain

$$^\gamma T_x^y f(x)=2C(\gamma)\int\limits_0^{\pi/2} f(\sqrt{x^2+y^2+2xy\cos 2\alpha})\sin^{\gamma-1}(2\alpha)d\alpha=$$

$$=2^\gamma C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2+2xy(\cos^2\alpha-\sin^2\alpha)}\right)\sin^{\gamma-1}\alpha\cos^{\gamma-1}\alpha d\alpha=$$

$$=2^\gamma C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2+2xy(1-2\sin^2\alpha)}\right)\sin^{\gamma-1}\alpha(1-\sin^2\alpha)^{\frac{\gamma-1}{2}}d\alpha.$$

Now let $\sin\alpha=t$ then $\alpha=0$ for $t=0$, $\alpha=\pi/2$ for $t=1$, $d\alpha{=}\frac{dt}{(1-t^2)^{1/2}}$ and

$$^\gamma T_x^y f(x)=2^\gamma C(\gamma)\int\limits_0^1 f(\sqrt{x^2+y^2+2xy(1-2t^2)})t^{\gamma-1}(1-t^2)^{\frac{\gamma}{2}-1}dt=\{t^2=z\}=$$

$$=2^{\gamma-1}C(\gamma)\int\limits_0^1 f(\sqrt{x^2+y^2+2xy(1-2z)})z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz=$$

$$=2^{\gamma-1}C(\gamma)\int\limits_0^1 f(\sqrt{(x+y)^2-4xyz})z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz=$$

$$=2^{\gamma-1}C(\gamma)\int\limits_0^1 f\left((x+y)\sqrt{1-\frac{4xy}{(x+y)^2}z}\right)z^{\frac{\gamma}{2}-1}(1-z)^{\frac{\gamma}{2}-1}dz.$$

The proof is complete. □

Property 2. *The generalized translation operator* ${}^\gamma T_x^y$ *has the form*

$$({}^\gamma T_x^y f)(x){=}\frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}}\int\limits_{|x-y|}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz. \tag{9}$$

Proof. Changing the variable φ to the 2α in (6) we obtain

$$ {}^\gamma T_x^y f(x) = 2C(\gamma)\int\limits_0^{\pi/2} f(\sqrt{x^2+y^2-2xy\cos 2\alpha})\sin^{\gamma-1}(2\alpha)d\alpha = $$

$$ = 2^\gamma C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2-2xy(\cos^2\alpha-\sin^2\alpha)}\right)\sin^{\gamma-1}\alpha\cos^{\gamma-1}\alpha d\alpha = $$

$$ = 2^\gamma C(\gamma)\int\limits_0^{\pi/2} f\left(\sqrt{x^2+y^2-2xy(1-2\sin^2\alpha)}\right)\sin^{\gamma-1}\alpha(1-\sin^2\alpha)^{\frac{\gamma-1}{2}}d\alpha. $$

Now putting $\sin\alpha = t$ we get for $\alpha=0$, $t=0$, for $\alpha=\pi/2$, $t=1$, $d\alpha=\frac{dt}{(1-t^2)^{1/2}}$ and

$$ {}^\gamma T_x^y f(x) = 2^\gamma C(\gamma)\int\limits_0^1 f(\sqrt{x^2+y^2-2xy(1-2t^2)})t^{\gamma-1}(1-t^2)^{\frac{\gamma}{2}-1}dt. $$

Introducing the variable z by the equality $\sqrt{x^2+y^2-2xy(1-2t^2)}=z$ we obtain

$$ t=\left(\frac{z^2-(x-y)^2}{4xy}\right)^{1/2}, \qquad dt=\frac{zdz}{(4xy)^{1/2}(z^2-(x-y)^2)^{1/2}}, $$

$z=|x-y|$ when $t=0$, $z=x+y$ when $t=1$ and

$$ {}^\gamma T_x^y f(x) = \frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}}\int\limits_{|x-y|}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz. $$

The proof is complete. □

Property 3. *If f is a Schwartz function, g is a continuous function, then*

$$ \int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy = \int\limits_0^\infty f(y)\,{}^\gamma T_x^y g(x)y^\gamma dy. \tag{10} $$

Proof. Applying to $\int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy$ the representation (9) we obtain

$$ \int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy = $$

$$ = (4x)^{1-\gamma}2^\gamma C(\gamma)\int\limits_0^\infty yg(y)dy\int\limits_{|x-y|}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz = $$

$$ = (4x)^{1-\gamma}2^\gamma C(\gamma)\Bigg[\int\limits_0^x yg(y)dy\int\limits_{x-y}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz+ $$

$$+\int\limits_x^\infty yg(y)dy\int\limits_{y-x}^{x+y} zf(z)[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz\Bigg].$$

Converting an expression $(z^2-(x-y)^2)((x+y)^2-z^2)$ and changing the order of integration we get

$$\int\limits_0^\infty {}^\gamma T_x^y f(x)g(y)y^\gamma dy =$$

$$=(4x)^{1-\gamma}2^\gamma C(\gamma)\Bigg[\int\limits_0^x zf(z)dz\int\limits_{x-z}^{x+z} yg(y)[((z+x)^2-y^2)(y^2-(z-x)^2)]^{\frac{\gamma}{2}-1}dy+$$

$$+\int\limits_x^\infty zf(z)dz\int\limits_{z-x}^{x+z} yg(y)[((z+x)^2-y^2)(y^2-(z-x)^2)]^{\frac{\gamma}{2}-1}dy\Bigg]=\int\limits_0^\infty f(z)\,{}^\gamma T_x^z g(y)z^\gamma dz.$$

The commutation is proved. □

Property 4. *For $x>0$ the formula representing a generalized translation ${}^\gamma T_x^y$ of power function x^α is*

$${}^\gamma T_x^y x^\alpha=(x+y)^\alpha\,{}_2F_1\left(-\frac{\alpha}{2},\frac{\gamma}{2};\gamma;\frac{4xy}{(x+y)^2}\right), \tag{11}$$

where ${}_2F_1$ is the Gaussian hypergeometric function.

Proof. Let first $x\neq y$. Using formula (8) let find ${}^\gamma T_x^y$ of x^α. We have

$${}^\gamma T_x^y x^\alpha=\frac{2^{\gamma-1}\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)}(x+y)^\alpha\int\limits_0^1\left(1-\frac{4xy}{(x+y)^2}z\right)^{\frac{\alpha}{2}}(1-z)^{\frac{\gamma}{2}-1}z^{\frac{\gamma}{2}-1}dz.$$

The last integral is the Gaussian hypergeometric function (5) for $z=\frac{4xy}{(x+y)^2}$, $a=-\frac{\alpha}{2}$, $b=\frac{\gamma}{2}$, $c=2b=\gamma$, $(c>b>0)$, thus

$${}^\gamma T_x^y x^\alpha=\frac{2^{\gamma-1}\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma\left(\frac{\gamma}{2}\right)}{\sqrt{\pi}\Gamma(\gamma)}(x+y)^\alpha\,{}_2F_1\left(-\frac{\alpha}{2},\frac{\gamma}{2},\gamma;\frac{4xy}{(x+y)^2}\right).$$

Using the doubling formula for gamma function

$$\Gamma(2z)=\frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right)$$

we obtain (11). □

Property 5. *The generalized translation ${}^\gamma T_x^y$ of e^{-x^2}, $x>0$ is*

$${}^\gamma T_x^y e^{-x^2}=\Gamma\left(\tfrac{\gamma+1}{2}\right)(xy)^{\frac{1-\gamma}{2}}e^{-x^2-y^2}I_{\frac{\gamma-1}{2}}(2xy). \tag{12}$$

Proof. Using the formula (9) we obtain

$${}^\gamma T_x^y e^{-x^2}=\frac{2^\gamma C(\gamma)}{(4xy)^{\gamma-1}}\int\limits_{|x-y|}^{x+y} ze^{-z^2}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz.$$

Find the integral

$$I=\int\limits_{|x-y|}^{x+y} ze^{-z^2}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz=\{z^2=t\}=$$

$$=\frac{1}{2}\int\limits_{(x-y)^2}^{(x+y)^2} e^{-t}[(t-(x-y)^2)((x+y)^2-t)]^{\frac{\gamma}{2}-1}dt=\{t-(x-y)^2=w\}=$$

$$=\frac{1}{2}e^{-(x-y)^2}\int\limits_0^{4xy} e^{-w}[w(4xy-w)]^{\frac{\gamma}{2}-1}dw.$$

Applying the formula 2.3.6.2 from [21] of the form

$$\int\limits_0^a x^{\alpha-1}(a-x)^{\alpha-1}e^{-px}dx=\sqrt{\pi}\Gamma(\alpha)\left(\frac{a}{p}\right)^{\alpha-1/2}e^{-ap/2}I_{\alpha-1/2}(ap/2),\quad \mathrm{Re}\,\alpha>0,$$

we get

$$I=2^{\gamma-2}\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)e^{-x^2-y^2}(xy)^{\frac{\gamma-1}{2}}I_{\frac{\gamma-1}{2}}(2xy).$$

Then

$${}^\gamma T_x^y e^{-x^2}=\frac{2^\gamma}{(4xy)^{\gamma-1}}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{\gamma}{2}\right)}\int\limits_{|x-y|}^{x+y} ze^{-z^2}[(z^2-(x-y)^2)((x+y)^2-z^2)]^{\frac{\gamma}{2}-1}dz=$$

$$=\frac{2^\gamma}{(4xy)^{\gamma-1}}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\sqrt{\pi}\;\Gamma\left(\frac{\gamma}{2}\right)}2^{\gamma-2}\sqrt{\pi}\Gamma\left(\frac{\gamma}{2}\right)e^{-x^2-y^2}(xy)^{\frac{\gamma-1}{2}}I_{\frac{\gamma-1}{2}}(2xy).$$

After simplification we get (12). □

Using (11) we can rewrite (4) as

$$(B_\gamma)^{-\alpha}f(x)=\left(\mathbf{I}_\gamma^\alpha f\right)(x)=\frac{2^{1-2\alpha}\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma(\alpha)}\int\limits_0^\infty({}^\gamma T_x^y x^{2\alpha-\gamma-1})f(y)y^\gamma dy.$$

From (10) it follows that

$$(B_\gamma)^{-\alpha}f(x)=\left(\mathbf{I}_\gamma^\alpha f\right)(x)=\frac{2^{1-2\alpha}\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma(\alpha)}\int\limits_0^\infty({}^\gamma T_x^y f(x))x^{2\alpha-1}dy. \tag{13}$$

It is easy to see that for Schwartz functions this integral converges for all $\alpha>0$. Expression (13) we will call *one-dimensional Riesz B-potential* or *Bessel–Riesz fractional integral.*

Let verify that for $\alpha=1$ and for $f=f(x)$ from the Schwartz space we get $\left(\mathbf{I}_\gamma^1 B_\gamma f\right)(x)=f(x)$:

$$(B_\gamma)^{-\alpha}B_\gamma f(x)=\left(\mathbf{I}_\gamma^1 B_\gamma f\right)(x)=\frac{\Gamma\left(\frac{\gamma-1}{2}\right)}{2\Gamma\left(\frac{\gamma+1}{2}\right)}\int\limits_0^\infty\left({}^\gamma T_x^y(B_\gamma)_x f(x)\right)y\,dy=$$

$$= \frac{1}{\gamma-1}\int\limits_0^\infty ((B_\gamma)_y\, {}^\gamma T_x^y f(x))\, y dy = \frac{1}{\gamma-1}\int\limits_0^\infty \left(\frac{d}{dy}y^\gamma \frac{d}{dy}\, {}^\gamma T_x^y f(x)\right) y^{1-\gamma} dy =$$

$$= \frac{1}{\gamma-1}\left(y^\gamma \frac{d}{dy}\, {}^\gamma T_x^y f(x)\right) y^{1-\gamma}\Bigg|_{y=0}^\infty - \frac{1}{\gamma-1}\int\limits_0^\infty \left(y^\gamma \frac{d}{dy}\, {}^\gamma T_x^y f(x)\right)\frac{d}{dy} y^{1-\gamma} dy =$$

$$= \int\limits_0^\infty \frac{d}{dy}\, {}^\gamma T_x^y f(x) dy = {}^\gamma T_x^y f(x)\Bigg|_{y=0}^\infty = f(x).$$

For $\gamma = 0$, $\alpha \neq 1, 3, 5, \ldots$ and for even function $f(x)$ we get

$$\left(\mathbf{I}_0^\alpha f\right)(x) = \frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}\int\limits_0^\infty \left[f(x+y) - f(x-y)\right] y^{2\alpha-1} dy =$$

$$= \frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}\left(\int\limits_0^\infty f(x+y) y^{2\alpha-1} dy - \int\limits_0^\infty f(x-y) y^{2\alpha-1} dy\right) =$$

$$= \frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}\left(\int\limits_{-\infty}^0 f(x-y)|y|^{2\alpha-1} dy - \int\limits_0^\infty f(x-y) y^{2\alpha-1} dy\right) =$$

$$= \frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}\int\limits_{-\infty}^\infty f(x-y)|y|^{2\alpha-1} dy = \frac{\Gamma\left(\frac{1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{1}{2}\right)\Gamma(\alpha)}\int\limits_{-\infty}^\infty f(y)|y-x|^{2\alpha-1} dy.$$

Since

$$\Gamma\left(\frac{1}{2}-\alpha\right) = \frac{\pi}{\sin\left(\frac{\pi+\alpha\pi}{2}\right)\Gamma\left(\frac{1}{2}+\alpha\right)}, \quad \Gamma(\alpha)\Gamma\left(\frac{1}{2}+\alpha\right) = \sqrt{\pi}2^{1-2\alpha}\Gamma(2\alpha)$$

we get the Riesz potential (see formula 12.1 in [22, p. 214])

$$\left(\mathbf{I}_0^\alpha f\right)(x) = \frac{1}{2\Gamma(2\alpha)\cos(\alpha\pi/2)}\int\limits_{-\infty}^\infty f(y)|y-x|^{2\alpha-1} dy.$$

Now let define the Bessel — Riesz fractional derivative. To do this, we will use the finite-difference method. Due to its versatility and simplicity this method is currently finding wider application to the problems of mathematical physics in particular with fractional derivatives. It is applicable both for theoretical studies of various problems, and for their approximate numerical results. We consider generalized finite differences defined by the formula

$$(\boxdot_t^l f)(x) = \sum_{k=0}^l (-1)^k C_l^k\, {}^\gamma T_x^{kt} f(x).$$

Let $l = 2[\alpha]+1$, $0 < \alpha$ is not entire. Namely, *the Bessel — Riesz fractional derivative* is

$$\left(\mathbf{B}_\gamma^\alpha f\right)(x) = \lim_{\substack{(L_p^\gamma)\\ \varepsilon\longrightarrow +0}} \frac{1}{d_{l,\gamma}(\alpha)}\int\limits_\varepsilon^\infty \frac{(\boxdot_t^l f)(x)}{t^{1+2\alpha}}\, dt = \frac{1}{d_{l,\gamma}(\alpha)}\int\limits_0^\infty \frac{(\boxdot_t^l f)(x)}{t^{1+2\alpha}}\, dt.$$

Here

$$d_{l,\gamma}(\alpha) = \frac{\pi\,\Gamma\left(\frac{\gamma+1}{2}\right)}{2^{2\alpha+1}\,\Gamma\left(\frac{1+\gamma}{2}+\alpha\right)\Gamma(\frac{1}{2}+\alpha)}\,\frac{\sum\limits_{k=0}^{l}(-1)^{k+1}C_l^k k^{2\alpha}}{\sin\alpha\pi}.$$

For example, for $0<\alpha<1$

$$\left(\mathbf{B}_\gamma^\alpha f\right)(x) = \frac{1}{d_{1,\gamma}(\alpha)}\int\limits_0^\infty \frac{f(x) - {}^\gamma T_x^t f(x)}{t^{1+2\alpha}}\,dt.$$

3. Numerical scheme and examples

Numerical methods for fractional integrals and derivatives are usually based on the Grunwald — Letnikov formulas, which are a generalization of formulas with finite differences and Riemannian sums, or on the use of the representation of solutions by infinite series. In order to find the Bessel — Riesz fractional integral the Gauss — Laguerre quadrature (see [23; 24]) can be used. Namely, the numerical integration formula of Gaussian type

$$\int\limits_0^\infty e^{-x}f(x)dx = \sum_{k=0}^{n} a_k f(x_k) + \frac{(n!)^2}{(2n)!}f^{(2n)}(\xi), \qquad 0<\xi<\infty.$$

For numerical implementation, it is convenient to use the recurrence formulas for Laguerre polynomials [25]. The first two polynomials are defined as

$$L_0(x)=1, \qquad L_1(x) = 1-x$$

and then using the following recurrence relation for any $k\geq 1$:

$$L_{k+1}(x) = \frac{(2k+1-x)L_k(x) - kL_{k-1}(x)}{k+1}.$$

We have

$$\int\limits_0^\infty f(y)\left({}^\gamma T_x^y x^{2\alpha-\gamma-1}\right)y^\gamma\,dy = \sum_{i=1}^{n} w_i g(y_i) + E_n(\xi),$$

where y_i is the i-th root of Laguerre polynomial $L_n(x)$ and the w_i is given by

$$w_i = \frac{y_i}{(n+1)^2\left[L_{n+1}(x_i)\right]^2}, \quad g(y) = e^y f(y)\left({}^\gamma T_x^y x^{2\alpha-\gamma-1}\right)y^\gamma,$$

$$E_n(\xi) = \frac{(n!)^2}{(2n)!}g^{(2n)}(\xi), \qquad 0<\xi<\infty.$$

If $f(x)$ decrease exponentially as $x\to\infty$ then Gauss — Laguerre quadrature is best:

$$E_n(\xi) < C\cdot\frac{2n+1}{4^n}.$$

For integrands that are $O(1/x^p)$, $p>2\alpha-1$, as $x\to\infty$, the convergence rate becomes

$$E_n(\xi) < \frac{C}{n^{p-2\alpha+1}}.$$

Example 1. Let find $\mathbf{I}_\gamma^\alpha e^{-x^2}$. Using (12) and formula 2.15.6.4 from [18] we obtain

$$\mathbf{I}_\gamma^\alpha e^{-x^2} = \frac{2^{1-2\alpha}\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma\left(\frac{\gamma+1}{2}\right)\Gamma(\alpha)}\int\limits_0^\infty \left({}^\gamma T_x^y e^{-x^2}\right) y^{2\alpha-1}dy =$$

$$= \frac{2^{1-2\alpha}\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{\Gamma(\alpha)} x^{\frac{1-\gamma}{2}} e^{-x^2} \int\limits_0^\infty e^{-y^2} I_{\frac{\gamma-1}{2}}(2xy)\, y^{2\alpha-1+\frac{1-\gamma}{2}}dy =$$

$$= \frac{\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{\gamma+1}{2}\right)} e^{-x^2}\, {}_1F_1\left(\alpha, \frac{\gamma+1}{2}; x^2\right).$$

For $n=10$, $\alpha=0.7$, $\gamma=0.5$ we get next results.

Table 1

x	Numerical calculation of $\mathbf{I}_\gamma^\alpha e^{-x^2}$, $\alpha=0.7$, $\gamma=0.5$	Exact value of $\mathbf{I}_\gamma^\alpha e^{-x^2}$, $\alpha=0.7$, $\gamma=0.5$	Absolute error
0.01	6.020591617	6.020591621	3.84693E-09
0.2	6.004767496	6.0047675	3.69646E-09
0.39	5.962266113	5.962266116	3.30447E-09
0.58	5.898170155	5.898170158	2.74828E-09
0.77	5.81943073	5.819430732	2.1265E-09
0.96	5.733357751	5.733357752	1.53078E-09
1.15	5.646313464	5.646313465	1.02519E-09
1.34	5.562932683	5.562932684	6.38767E-10
1.53	5.485940593	5.485940594	3.7027E-10
1.72	5.416426922	5.416426922	1.99684E-10
1.91	5.354341757	5.354341757	1.00189E-10
2.1	5.299001572	5.299001572	4.67759E-11
2.29	5.249480524	5.249480524	2.03046E-11
2.48	5.204851927	5.204851927	8.22364E-12
2.67	5.16430296	5.16430296	3.09441E-12
2.86	5.127166869	5.127166869	1.09246E-12
3.05	5.0929132	5.0929132	3.58824E-13
3.24	5.061123119	5.061123119	1.14575E-13
3.43	5.031463853	5.031463853	4.44089E-14
3.62	5.003667528	5.003667528	1.42109E-14
3.81	4.977515233	4.977515233	5.32907E-15
4	4.952825466	4.952825466	2.66454E-15
4.19	4.929445799	4.929445799	7.99361E-15
4.38	4.9072468	4.9072468	1.33227E-14
4.57	4.886117511	4.886117511	3.81917E-14

As for numerical scheme for the Bessel — Riesz fractional derivative we should notice that just as ordinary shift operators can be expanded in powers of the differentiation operator, the operators ${}^\gamma T_x^y$ can be expanded in powers of the Bessel operator by the formula [17]

$${}^\gamma T_x^y f(x) = \sum_{k=0}^m \varphi_k(y) B_\gamma^k f(x) + R_m(x,y),$$

where

$$\varphi_k(x) = \frac{1}{k!}\frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}+k\right)}\left(\frac{x}{2}\right)^{2k}, \quad R_m(x,y) = \varphi_{m+1}(y) B_\gamma^{m+1} f(\xi)|_{\xi=x+\theta y}, \ -1<\theta<1.$$

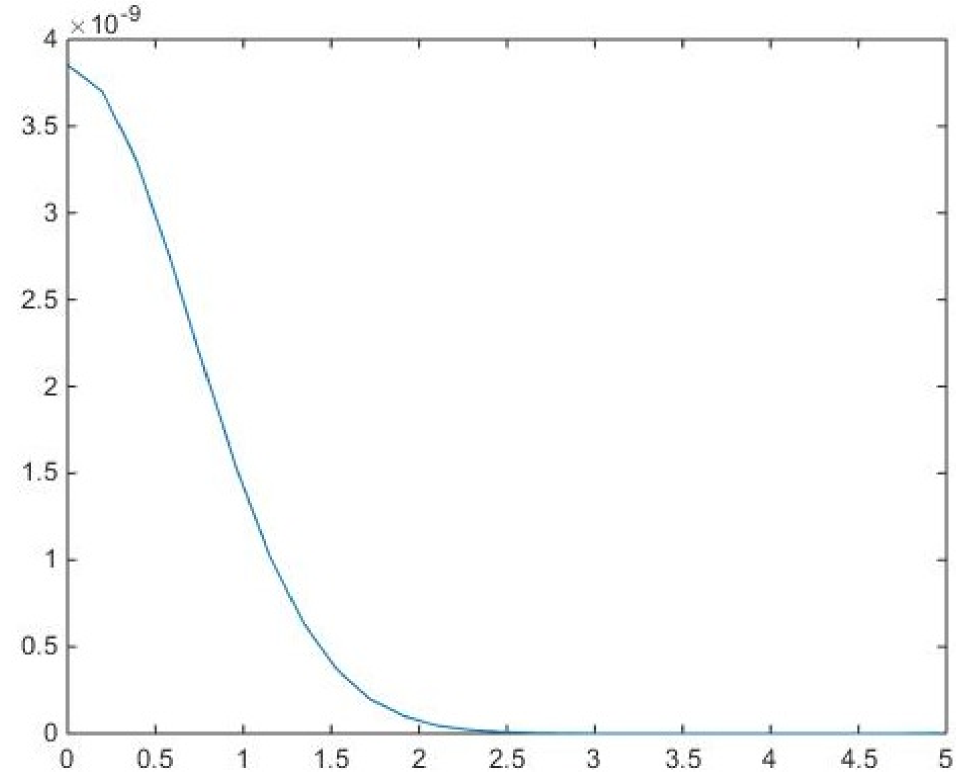

Fig. 1. $\mathbf{I}_\gamma^\alpha e^{-x^2} = \frac{\Gamma\left(\frac{\gamma+1}{2}-\alpha\right)}{2^{2\alpha}\Gamma\left(\frac{\gamma+1}{2}\right)} e^{-x^2} \, {}_1F_1\left(\alpha, \frac{\gamma+1}{2}; x^2\right)$

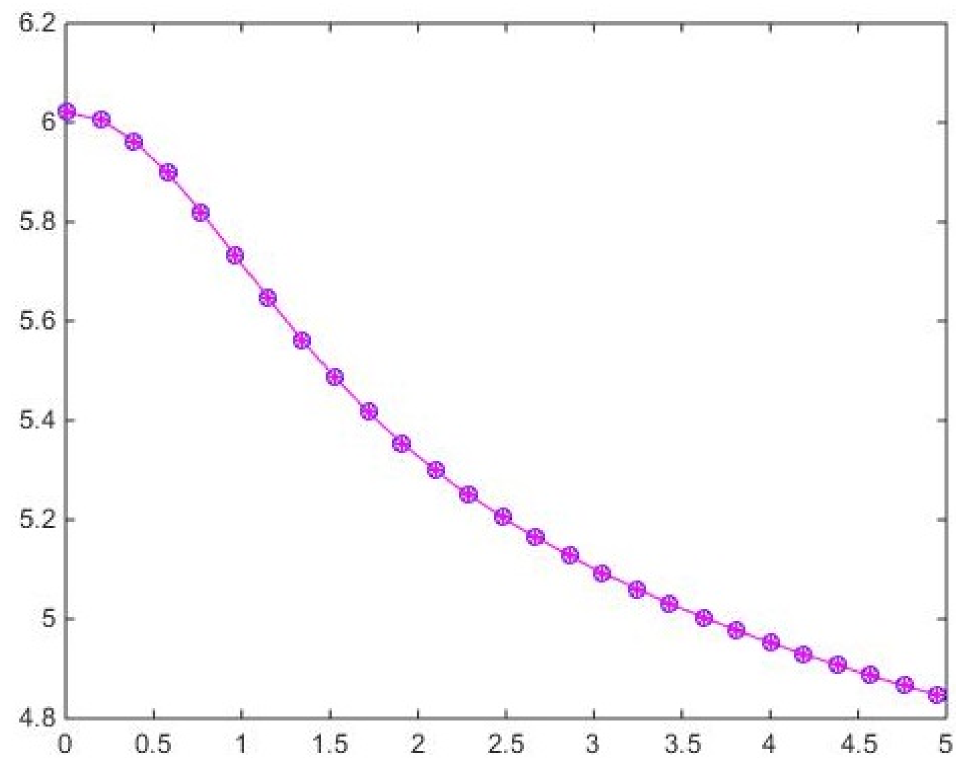

Fig. 2. Absolute error of numerical calculation of $\mathbf{I}_\gamma^\alpha e^{-x^2}$

We have

$$\varphi_0(x) = 1, \qquad \varphi_1(x) = \frac{\Gamma\left(\frac{\gamma+1}{2}\right)}{\Gamma\left(\frac{\gamma+1}{2}+1\right)} \left(\frac{x}{2}\right)^2 = \frac{x^2}{2(\gamma+1)}.$$

Then

$$^\gamma T_x^t f(x) - f(x) = \frac{t^2}{2(\gamma+1)} B_\gamma f(\xi)|_{\xi=x+\theta t}$$

and for $0 < \alpha < 1$

$$\left(\mathbf{B}_\gamma^\alpha f\right)(x) = -\frac{1}{2(\gamma+1)d_{1,\gamma}(\alpha)}\int\limits_0^\infty t^{1-2\alpha}(B_\gamma f(\xi)|_{\xi=x+\theta t})\ dt$$

converges absolutely for $f \in S_{ev}$. Since S_{ev} is dense in L_p^γ we can also take function $f \in L_p^\gamma$, $1 \leq p < \infty$.

Example 2. Let find $\mathbf{B}_\gamma^\alpha j_{\frac{\gamma-1}{2}}(x)$ for $\gamma = 2$, $0 < \alpha < 1$.

Table 2

x	Numerical calculation of $\mathbf{B}_\gamma^\alpha j_{\frac{1}{2}}(x)$, $\alpha = 0.2$, $\gamma = 2$	Exact value of $\mathbf{B}_\gamma^\alpha j_{\frac{1}{2}}(x)$, $\alpha = 0.2$, $\gamma = 2$	Absolute error
0.01	1.397316	1.41372	0.016428
0.3	1.397316	1.392633	0.016183
0.59	1.397316	1.333139	0.015491
0.88	1.397316	1.238213	0.014388
1.17	1.397316	1.112569	0.012928
1.46	1.397316	0.96238	0.011183
1.75	1.397316	0.794917	0.009237
2.04	1.397316	0.618117	0.007183
2.33	1.397316	0.440132	0.005114
2.62	1.397316	0.26886	0.003124
2.91	1.397316	0.11151	0.001296
3.2	1.397316	-0.02579	0.0003
3.49	1.397316	-0.1383	0.001607
3.78	1.397316	-0.22288	0.00259
4.07	1.397316	-0.27812	0.003232
4.36	1.397316	-0.30433	0.003536
4.65	1.397316	-0.30344	0.003526
4.94	1.397316	-0.2788	0.00324
5.23	1.397316	-0.2349	0.00273
5.52	1.397316	-0.17703	0.002057
5.81	1.397316	-0.11089	0.001289
6.1	1.397316	-0.04222	0.000491
6.39	1.397316	0.023587	0.000274
6.68	1.397316	0.081794	0.00095
6.97	1.397316	0.128612	0.001495
7.26	1.397316	0.161377	0.001875
7.55	1.397316	0.178666	0.002076
7.84	1.397316	0.180307	0.002095
8.13	1.397316	0.16731	0.001944
8.42	1.397316	0.141717	0.001647
8.71	1.397316	0.106388	0.001236
9	1.397316	0.064737	0.000752
9.29	1.397316	0.020448	0.000238
9.58	1.397316	-0.02281	0.000265

Using formula (7) and the fact that $j_{1/2}(x) = \frac{\sin x}{x}$ we get:

$$\left(\mathbf{B}_\gamma^\alpha f\right)(x) = \frac{1}{d_{1,2}(\alpha)}\frac{\sin x}{x}\int\limits_0^\infty \frac{t-\sin t}{t^{2\alpha+2}}\ dt =$$

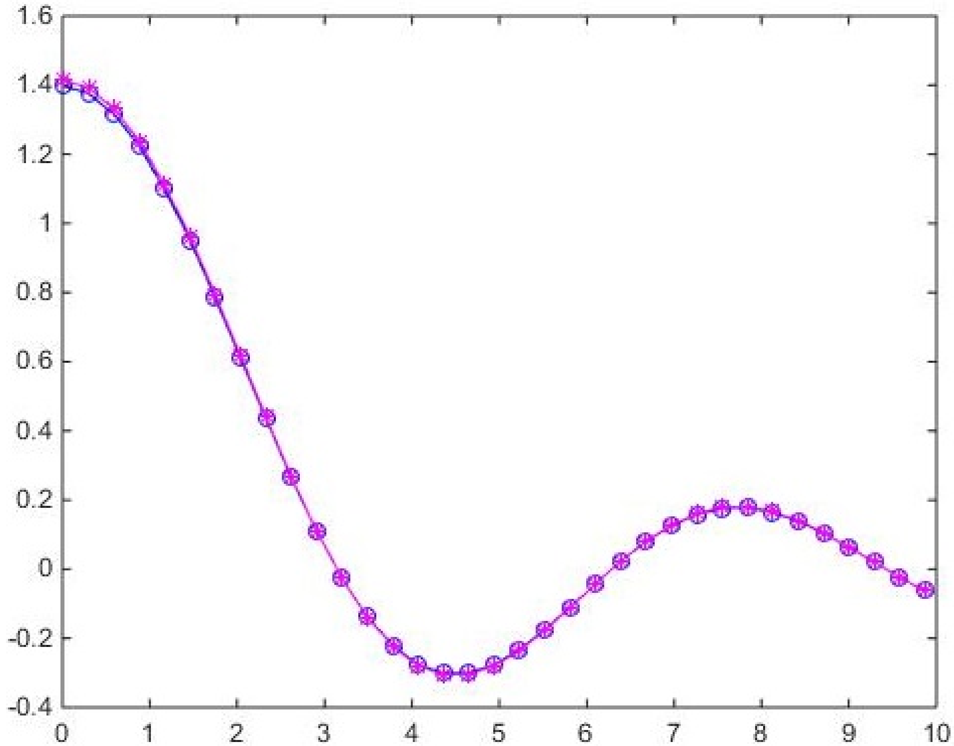

Fig. 3. $\mathbf{B}_\gamma^\alpha j_{\frac{1}{2}}(x) = \frac{2^{2\alpha}\Gamma\left(\frac{3}{2}+\alpha\right)\Gamma\left(\frac{1}{2}+\alpha\right)}{\Gamma\left(\frac{3}{2}\right)\Gamma(2\alpha+2)}\frac{\sin x}{x}$

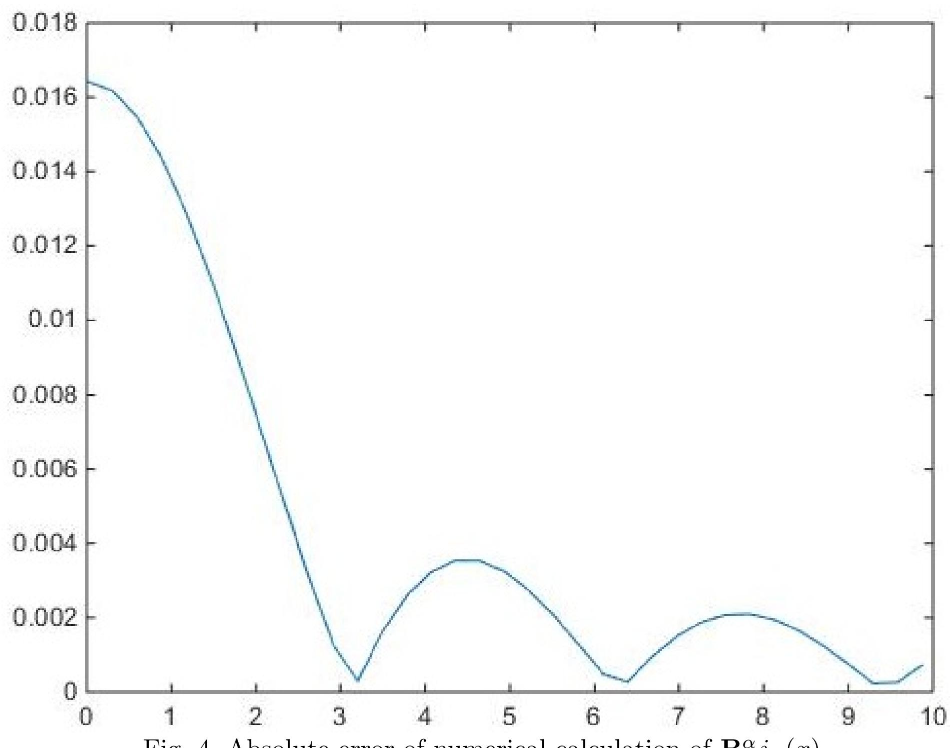

Fig. 4. Absolute error of numerical calculation of $\mathbf{B}_\gamma^\alpha j_{\frac{1}{2}}(x)$

$$= \Gamma(-1-2\alpha)\cos(\alpha\pi)\frac{1}{d_{1,2}(\alpha)}\frac{\sin x}{x} = \frac{\pi\cos(\alpha\pi)}{\Gamma(2\alpha+2)\sin(2\pi\alpha)}\frac{1}{d_{1,2}(\alpha)}\frac{\sin x}{x},$$

since

$$d_{1,2}(\alpha) = \frac{\pi\,\Gamma\left(\frac{3}{2}\right)}{2^{2\alpha+1}\,\Gamma\left(\frac{3}{2}+\alpha\right)\Gamma(\frac{1}{2}+\alpha)}\frac{1}{\sin\alpha\pi}$$

we get

$$\left(\mathbf{B}_{\gamma}^{\alpha} f\right)(x) = \frac{2^{2\alpha}\Gamma\left(\frac{3}{2}+\alpha\right)\Gamma(\frac{1}{2}+\alpha)}{\Gamma\left(\frac{3}{2}\right)\Gamma(2\alpha+2)}\frac{\sin x}{x}.$$

For $n = 10000$, $\alpha = 0.2$, $\gamma = 2$ we get the results in Table 2.

Conclusion

On the Hankel transform basis using the integral transform compositional method proposed by V.V. Katrakhov and S.M. Sitnik, fractional powers of the Bessel operator are constructed. This article also discussed numerical methods for fractional powers of the Bessel operator. Differential and integral operators presented in the article include one-dimensional Riesz B-potentials and the Bessel — Riesz fractional derivative.

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ДРОБНЫЕ СТЕПЕНИ ОПЕРАТОРА БЕССЕЛЯ И ИХ ПРИБЛИЖЁННОЕ ВЫЧИСЛЕНИЕ

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Рассматриваются дробные степени оператора Бесселя и их численная реализация. Обширная литература посвящена изучению дробных степеней оператора Лапласа и их приложениям. Такие степени используются при конструировании функциональных пространств, при естественном обобщении уравнения Шрёдингера в квантовой теории, при построении моделей распространения акустических волн в сложных средах (например, биологических тканях) и пространственно-временных моделей аномальной (очень медленной или очень быстрой) диффузии, в спектральной теории и др. Если предположить радиальность функции, на которую действует оператор Лапласа, то мы придём к задаче о построении дробной степени оператора Бесселя. Мы предлагаем использовать композиционный метод построения указанных операторов, что приводит к конструкциям, близким по своим свойствам к риссовым производным. В качестве базового интегрального преобразования рассматривается преобразование Ханкеля. На его основе композиционным методом, предложенным В. В. Катраховым и С. М. Ситником, строятся отрицательные степени оператора Бесселя. Полученный оператор содержит в ядре гипергеометрическую функцию Гаусса. Для дальнейшего изучения в статье приводится оператор обобщённого сдвига, доказываются его свойства. Известными способами достигается регуляризация интеграла при построении положительной дробной степени оператора Бесселя. Затем предлагается схема численного вычисления дробных степеней оператора Бесселя, основанная на полученной Б. М. Левитаном формуле Тейлора — Дельсарта. Приводятся примеры, содержащие точное и приближённое значения положительной и отрицательной степеней оператора Бесселя, абсолютную погрешность и иллюстрации. В списке литературы приводятся источники, в которых содержатся известные результаты о дробных степенях операторов рассматриваемого в статье типа, а также содержащие их приложения.

Ключевые слова: *оператор Бесселя, дробная степень, преобразование Ханкеля, обобщённый сдвиг, композиционный метод, операторы преобразования.*

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