Function theoretical approach to anisotropic plane elasticity

A. Soldatov

[1] Belgorod State University

The representation of general solutions of Lame system of plane elasticity is given with the help of so-called Douglis analytic functions. Using integral representation of these functions the basic boundary value problems for Lame system are reduced to equivalent singular integral equations on the boundary.

The plane elastic medium is characterized by the displacement vector $u = (u_1, u_2)$ and by stress and deformation tensors

$$\sigma = \begin{pmatrix} \sigma_1 & \sigma_3 \\ \sigma_3 & \sigma_2 \end{pmatrix}, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 & \varepsilon_3 \\ \varepsilon_3 & \varepsilon_2 \end{pmatrix},$$

where

$$\varepsilon_1 = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_2 = \frac{\partial u_2}{\partial x_2}, \quad 2\varepsilon_3 = \frac{\partial u_1}{\partial x_2} + \frac{\partial u_1}{\partial x_2}.$$

They are connected by Hooke law i.e. by linear relation

$$\tilde{\sigma} = \alpha\tilde{\varepsilon}, \quad \alpha = \begin{pmatrix} \alpha_1 & \alpha_4 & \alpha_6 \\ \alpha_4 & \alpha_2 & \alpha_5 \\ \alpha_6 & \alpha_5 & \alpha_3 \end{pmatrix} > 0,$$

where $\tilde{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, $\tilde{\varepsilon} = (\varepsilon_1, \varepsilon_2, 2\varepsilon_3)$. If the external forces are absent then from the equilibrium equations and the Hooke law we reduce the Lame system for the displacement vector $u = (u_1, u_2)$. This is the second order elliptic system

$$a_{11}\frac{\partial^2 u}{\partial x_1^2} + (a_{12} + a_{21})\frac{\partial^2 u}{\partial x_1 \partial x_2} + a_{22}\frac{\partial^2 u}{\partial x_2^2} = 0 \qquad (1)$$

with the coefficients

$$a_{11} = \begin{pmatrix} \alpha_1 & \alpha_6 \\ \alpha_6 & \alpha_3 \end{pmatrix}, \quad a_{12} = \begin{pmatrix} \alpha_6 & \alpha_4 \\ \alpha_3 & \alpha_5 \end{pmatrix}, \quad a_{21} = \begin{pmatrix} \alpha_6 & \alpha_3 \\ \alpha_4 & \alpha_5 \end{pmatrix}, \quad a_{22} = \begin{pmatrix} \alpha_3 & \alpha_5 \\ \alpha_5 & \alpha_2 \end{pmatrix}.$$

For the roots ν_1, ν_2 in upper half-plane of the forth order characteristical polynomial $\chi(z) = \det[a_{11} + (a_{12} + a_{21})z + a_{22}z^2]$ we have two cases (i) $\nu_1 \neq \nu_2$ and (ii) $\nu_1 = \nu_2 = \nu$. Accordingly to these cases let us consider the first order elliptic system

$$\frac{\partial \phi}{\partial x_2} - J\frac{\partial \phi}{\partial x_1} = 0, \quad \text{where} \quad (i) \; J = \begin{pmatrix} \nu_1 & 0 \\ 0 & \nu_2 \end{pmatrix}, \quad (ii) \; J = \begin{pmatrix} \nu & 1 \\ 0 & \nu \end{pmatrix}. \qquad (2)$$

The main elements of function theory hold for the solutions ϕ of this system. For Hankel matrixes J it is studied by Douglis [2] in terms of supercomplex numbers. A general solution of (2) can be described throw analytic vector $\psi = (\psi_1, \psi_2)$ by the formula

$$\begin{array}{ll} (i) & \phi_1(x) = \psi_1(x_1 + \nu_1 x_2), \; \phi_2(x) = \psi_2(x_1 + \nu_2 x_2); \\ (ii) & \phi_1(x) = \psi_1(x_1 + \nu x_2) + x_2\psi_2'(x_1 + \nu x_2), \; \phi_2(x) = \psi_2(x_1 + \nu x_2). \end{array} \qquad (3)$$

The system (2) is closely connected with Lame system [1].

Theorem 0.1 *A general solution $u = (u_1, u_2)$ of (1) and columns $\sigma_{(1)} = (\sigma_1, \sigma_3)$, $\sigma_{(2)} = (\sigma_3, \sigma_2)$ of the corresponding stress tensor σ can be described throw solution ϕ of (2) by the following formulas:*

$$u = \mathrm{Re}\, B\phi, \quad \sigma_{(1)} = \mathrm{Re}\, CJ\frac{\partial \phi}{\partial x_1}, \quad \sigma_{(2)} = -\mathrm{Re}\, C\frac{\partial \phi}{\partial x_1}, \qquad (4)$$

where the matrixes B, C are invertible and defined from relations $a_{11}B + (a_{12} + a_{21})BJ + a_{22}BJ^2 = 0$, $C = -(a_{21}B + a_{22}BJ)$.

The substitution (3) into (4) gives the known representation [3, 4] of a general solution of Lame system throw analytic functions. The another closed approaches were developed in [5] – [7].

Theorem 1 permits a direct study of Dirichlet problem for Lame system. Let us introduce the matrix-valued function $x_1 + x_2 J$ on the plane, where x_1 denotes the scalar matrix. This matrix is invertible for $x \neq 0$ and $[2\pi i(x_1 + x_2 J)]^{-1}$ is the fundamental solution of the Douglis system (2).

Let D be finite domain bounded by Lyapunov contour Γ and let $n(y) = [n_1(y), n_2(y)]$ denote the external unit normal at the point $y \in \Gamma$. It follows from theorem 1 that the function

$$(I\varphi)(x) = \frac{1}{\pi}\int_\Gamma Q[y - x, n(y)]\varphi(y)ds_y, \quad Q(\xi, n) = \mathrm{Im}\,[B(\xi_1 + \xi_2 J)^{-1}(-n_2 + n_1 J)B^{-1}], \tag{5}$$

satisfies (1) in the domain D for each real vector – valued function $\varphi \in C(\Gamma)$. The matrix- valued kernel $Q(\xi, n)$ of this integral operator is odd and homogeneous of degree -1. In the explicite form

$$Q(\xi, n) = Q_0(\xi, n)H(\xi), \quad Q_0(\xi, n) = \frac{\xi_1 n_1 + \xi_2 n_2}{|\xi|^2},$$

where accordingly two cases (i) and (ii)

$$H(\xi) = \begin{pmatrix} (\mathrm{Im}\,\nu_1)|\zeta_1|^2 & 0 \\ 0 & (\mathrm{Im}\,\nu_2)|\zeta_2|^2 \end{pmatrix} + \mathrm{Im}\left[\frac{\zeta_1\zeta_2(\nu_1 - \nu_2)}{\det B}\begin{pmatrix} B_{12}B_{21} & -B_{11}B_{12} \\ B_{22}B_{21} & -B_{12}B_{21} \end{pmatrix}\right], \quad \zeta_j = \frac{|\xi|}{\xi_1 + \nu_j\xi_2},$$

$$H(\xi) = (\mathrm{Im}\,\nu)|\zeta|^2 + \mathrm{Im}\left[\frac{\zeta^2}{\det B}\begin{pmatrix} -B_{11}B_{21} & B_{11}^2 \\ -B_{21}^2 & B_{11}B_{21} \end{pmatrix}\right], \quad \zeta = \frac{|\xi|}{\xi_1 + \nu\xi_2}.$$

Let $(K\varphi)(x)$ be defined by (5) for $x \in \Gamma$. In this case there exists a number $0 < \alpha < 1$ such that $|y - x|^\alpha Q[y - x, n(y)] \in C(\Gamma \times \Gamma)$. So the operator K is compact in the space $C(\Gamma)$.

Theorem 0.2 *(a) The operator I is bounded $C(\Gamma) \to C(\overline{D})$ and $(I\varphi)|_\Gamma = \varphi + K\varphi$.*

(b) The Fredholm equation $\varphi + K\varphi = f$ is one-to -one solvable in the class $C(\Gamma)$.

(c) The Dirichlet problem $u|_\Gamma = f$ for Lame system is one-to -one solvable in the class $C(\overline{D})$ and its solution $u = I(1 + K)^{-1}f$.

The matrixes B, C in (4) can be explicitly described. Let us write

$$a_{11} + (a_{12} + a_{21})z + a_{22}z^2 = \begin{pmatrix} p_1(z) & p_3(z) \\ p_3(z) & p_2(z) \end{pmatrix}, \quad (a_{21} + a_{22}z)\begin{pmatrix} -p_2(z) & p_3(z) \\ p_3(z) & -p_1(z) \end{pmatrix} = \begin{pmatrix} q_1(z) & q_3(z) \\ q_2(z) & q_4(z) \end{pmatrix}.$$

In particular the characteristical polynomial $\chi(z) = p_1(z)p_2(z) - p_3^2(z)$. In the case (i) let us put

$$B_1 = \begin{pmatrix} p_2(\nu_1) & p_2(\nu_2) \\ -p_3(\nu_1) & -p_3(\nu_2) \end{pmatrix}, B_2 = \begin{pmatrix} -p_3(\nu_1) & -p_3(\nu_2) \\ p_1(\nu_1) & p_1(\nu_2) \end{pmatrix}, C_1 = \begin{pmatrix} q_1(\nu_1) & q_1(\nu_2) \\ q_2(\nu_1) & q_2(\nu_2) \end{pmatrix}, C_2 = \begin{pmatrix} q_3(\nu_1) & q_3(\nu_2) \\ q_4(\nu_1) & q_4(\nu_2) \end{pmatrix}.$$

If the polynomial $p_3 \neq 0$, then $|\lambda_1| + |\lambda_2| > 0$, $\lambda_j = \det(B_j C_j)$, and we write $B = B_j$, $C = C_j$ for $\lambda_j \neq 0$. If $p_3(z) = \alpha_6 + (\alpha_3 + \alpha_4)z + \alpha_5 z^2 \equiv 0$, then we put $B = 1$, $C = -a_{21} - a_{22}J$. In the case (ii) we can put

$$B = \begin{pmatrix} p_2(\nu) & p_2'(\nu) \\ -p_3(\nu) & -p_3'(\nu) \end{pmatrix}, \quad C = \begin{pmatrix} q_1(\nu) & q_1'(\nu) \\ q_2(\nu) & q_2'(\nu) \end{pmatrix}.$$

In the orthotropic case $\alpha_5 = \alpha_6 = 0$ the characteristical polynomial χ is biquadratic and the matrixes B, C can be explicitly expressed throw the modulus α_j. In particular for the isotropic medium, when $\alpha_5 = \alpha_6 = 0, \alpha_1 = \alpha_2 = \lambda + 2\mu, \alpha_3 = \mu$, $\alpha_4 = \lambda$, we have the case (ii) with $\nu = i$ and

$$B = \begin{pmatrix} 1 & 0 \\ i & -æ \end{pmatrix}, \quad C = \mu\begin{pmatrix} 2i & æ - 1 \\ 2 & i(æ + 1) \end{pmatrix}, \quad æ = \frac{\lambda + 3\mu}{\lambda + \mu}.$$

Acknowledgements The work was supported by RFBR (project No. 070100299)

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