A DESCRIPTION OF SEISMIC ACOUSTIC WAVE PROPAGATION IN POROUS MEDIA VIA HOMOGENIZATION

ANVARBEK MEIRMANOV†

Abstract. We consider a linear system of differential equations describing the joint motion of an elastic porous body and a fluid occupying the porous space. A rigorous justification is performed for the homogenization procedures under various conditions imposed on the physical parameters as the dimensionless size of the pores tends to zero, while the porous body is geometrically periodic and the process's characteristic time is sufficiently small. Such models describe the propagation of seismic acoustic waves. In the present paper, we derive the homogenized equations, which are different types of nonstandard wave equations depending on the relations between the physical parameters. The proofs are based on Nguetseng's two-scale convergence method of homogenization in periodic structures.

Key words. Stokes equations, Lamé's equations, wave equation, two-scale convergence, homogenization of periodic structures

AMS subject classifications. 35M20, 74F10, 76S05

1. Introduction. In the present paper, we deal with a problem of joint motion of a deformable solid (the *elastic skeleton*) perforated by a system of channels or pores (the *pore space*) and a fluid occupying the pore space. In a domain $\Omega \subset \mathbf{R}^3$, the dimensionless displacement vector $\boldsymbol{w}$ of the continuum medium in the dimensionless variables

$$\boldsymbol{x}' = L\boldsymbol{x}, \quad t' = \tau t, \quad \boldsymbol{w}' = \frac{L^2}{g\tau^2}\boldsymbol{w}, \quad \rho_s' = \rho_0\rho_s, \quad \rho_f' = \rho_0\rho_f, \quad \boldsymbol{F}' = g\boldsymbol{F}$$

satisfies the differential equation

$$\bar{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = \mathrm{div} P + \bar{\rho}\boldsymbol{F}, \tag{1.1}$$

where

$$P = \bar{\chi}P^f + (1-\bar{\chi})P^s, \tag{1.2}$$

$$P^f = \alpha_\mu D\left(x, \frac{\partial \boldsymbol{w}}{\partial t}\right) - \left(p_f - \alpha_\nu \mathrm{div}\frac{\partial \boldsymbol{w}}{\partial t}\right) I, \tag{1.3}$$

$$P^s = \alpha_\lambda D(x, \boldsymbol{w}) + \alpha_\eta (\mathrm{div}\boldsymbol{w}) I, \tag{1.4}$$

$$p_f + \bar{\chi}\alpha_p \mathrm{div}\boldsymbol{w} = 0. \tag{1.5}$$

Hereafter, we use the notation

$$D(x, \boldsymbol{u}) = (1/2)\left(\nabla_x \boldsymbol{u} + (\nabla_x \boldsymbol{u})^T\right), \quad \bar{\rho} = \bar{\chi}\rho_f + (1-\bar{\chi})\rho_s,$$

†Mathematics Department, Belgorod State University, ul.Pobedi 85, 308015 Belgorod, Russia (meirmanov@bsu.edu.ru).

where I is the unit tensor, the given function $\bar{\chi}(\boldsymbol{x})$ is the characteristic function of the pore space, the given function $\boldsymbol{F}(\boldsymbol{x},t)$ is the dimensionless vector of distributed mass forces, P^f is the liquid stress tensor, P^s is the stress tensor in the solid skeleton, and p_f is the liquid pressure.

Equations (1.1)–(1.5) mean that the displacement vector $\boldsymbol{w}$ satisfies the Stokes equations in the pore space Ω_f and the Lamé equations in the solid skeleton Ω_s.

On the "solid skeleton–pore space" common boundary Γ, the displacement vector $\boldsymbol{w}$ and the liquid pressure p_f satisfy the usual continuity condition

$$[\boldsymbol{w}](\boldsymbol{x}_0,t)=0, \quad \boldsymbol{x}_0\in\Gamma,\ t\geq 0, \tag{1.6}$$

and the momentum conservation law in the form

$$[P\cdot\mathbf{n}](\boldsymbol{x}_0,t)=0, \quad \boldsymbol{x}_0\in\Gamma,\ t\geq 0, \tag{1.7}$$

where $\mathbf{n}(\boldsymbol{x}_0)$ is the unit normal to the boundary at the point $\boldsymbol{x}_0\in\Gamma$ and

$$[\varphi](\boldsymbol{x}_0,t)=\varphi_{(s)}(\boldsymbol{x}_0,t)-\varphi_{(f)}(\boldsymbol{x}_0,t),$$

$$\varphi_{(s)}(\boldsymbol{x}_0,t)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_s}}\varphi(\boldsymbol{x},t), \quad \varphi_{(f)}(\boldsymbol{x}_0,t)=\lim_{\substack{\boldsymbol{x}\to\boldsymbol{x}_0\\ \boldsymbol{x}\in\Omega_f}}\varphi(\boldsymbol{x},t).$$

The problem is endowed with the homogeneous initial and boundary conditions

$$\boldsymbol{w}(\boldsymbol{x},0)=0, \quad \frac{\partial\boldsymbol{w}}{\partial t}(\boldsymbol{x},0)=0, \quad \boldsymbol{x}\in\Omega, \tag{1.8}$$

$$\boldsymbol{w}(\boldsymbol{x},t)=0, \quad \boldsymbol{x}\in S=\partial\Omega, \quad t\geq 0. \tag{1.9}$$

The dimensionless constants α_i $(i=\tau,\nu,\ldots)$ are defined by the formulas

$$\alpha_\mu=\frac{2\mu\tau}{L^2\rho_0}, \quad \alpha_\lambda=\frac{2\lambda\tau^2}{L^2\rho_0}, \quad \alpha_\nu=\frac{\nu\tau}{L^2\rho_0},$$

$$\alpha_p=\rho_f c_f^2\frac{\tau^2}{L^2}, \quad \alpha_\eta=\frac{\eta\tau^2}{L^2\rho_0}=\rho_s c_s^2\frac{\tau^2}{L^2},$$

where μ is the fluid viscosity, ν is the bulk fluid viscosity, λ and η are elastic Lamé's constants, c_f is the speed of sound in fluids, c_s is the speed of sound in solids, L is the characteristic size of the domain under study, τ is the characteristic time of the process, ρ_f and ρ_s are the respective mean dimensionless densities of the liquid and solid phases correlated with the mean density of water ρ_0, and g is the value of acceleration due to gravity.

The corresponding mathematical model described by system (1.1)–(1.9) is commonly used (see [2], [9]) and contains a natural small parameter ε, which is the pore characteristic size l divided by the characteristic size L of the entire porous body:

$$\varepsilon=\frac{l}{L}.$$

Our aim is to derive all possible limiting regimes (the homogenized equations) as $\varepsilon\searrow 0$. Such an approximation significantly simplifies the original problem and at the same time preserves all of its main features. But even this approach is too difficult to be realized, and some additional simplifying assumptions are necessary. In terms of geometrical properties of the medium, it is most expedient to simplify the problem by postulating that the porous structure is periodic.

We impose the following constraints.

Assumption 1.

(1) The domain $\Omega = (0,1)^3$ is a periodic repetition of an elementary cell $Y^\varepsilon = \varepsilon Y$, where $Y = (0,1)^3$ and the quantity $1/\varepsilon$ is an integer so that Ω always contains an integer number of elementary cells Y^ε.

(2) Let Y_s be the "solid part" of Y, and let the "liquid part" Y_f of Y be its open complement. We write $\gamma = \partial Y_f \cap \partial Y_s$ and assume that γ is a Lipschitz continuous surface.

(3) The pore space Ω_f^ε is a periodic repetition of the elementary cell εY_f, and the solid skeleton Ω_s^ε is a periodic repetition of the elementary cell εY_s. The Lipschitz continuous boundary $\Gamma^\varepsilon = \partial\Omega_s^\varepsilon \cap \partial\Omega_f^\varepsilon$ is a periodic repetition in Ω of the boundary $\varepsilon\gamma$.

(4) *The "solid skeleton"* Ω_s^ε *and the "pore space"* Ω_f^ε *are connected domains.*

Here the essential assumptions are those last three on the geometry of the elementary cells Y_s and Y_f and the domains Ω_s^ε and Ω_f^ε. As for the first assumption, we take the simplest structure of Ω (namely, the cube) just to simplify the procedure. In principle, for the domain Ω we can choose any bounded domain with a Lipschitz continuous boundary $S = \partial\Omega$.

Under these assumptions, we have

$$\bar{\chi}(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x}) = \chi\left(\frac{\boldsymbol{x}}{\varepsilon}\right),$$

$$\bar{\rho} = \rho^\varepsilon(\boldsymbol{x}) = \chi^\varepsilon(\boldsymbol{x})\rho_f + (1 - \chi^\varepsilon(\boldsymbol{x}))\rho_s,$$

where $\chi(\boldsymbol{y})$ is the characteristic function of Y_f in Y.

We assume that all dimensionless parameters depend on the small parameter ε and the (finite or infinite) limits exist:

$$\lim_{\varepsilon\searrow 0}\alpha_\mu(\varepsilon) = \mu_0, \quad \lim_{\varepsilon\searrow 0}\alpha_\lambda(\varepsilon) = \lambda_0, \quad \lim_{\varepsilon\searrow 0}\alpha_\nu(\varepsilon) = \nu_0,$$

$$\lim_{\varepsilon\searrow 0}\alpha_\eta(\varepsilon) = \eta_0, \quad \lim_{\varepsilon\searrow 0}\alpha_p(\varepsilon) = p_*,$$

$$\lim_{\varepsilon\searrow 0}\frac{\alpha_\mu}{\varepsilon^2} = \mu_1, \quad \lim_{\varepsilon\searrow 0}\frac{\alpha_\lambda}{\varepsilon^2} = \lambda_1.$$

The first research aiming to find the limiting regimes in the case where the skeleton was an absolutely rigid body was carried out by Sanchez-Palencia and Tartar. Sanchez-Palencia [9, sect. 7.2] formally obtained Darcy's law of filtration using the method of two-scale asymptotic expansions, and Tartar [9, Appendix] rigorously justified the homogenization procedure. Using the same method of two-scale expansions, Burridge and Keller [2] formally derived a system of Biot's equations from problem (1.1)–(1.9) in the case where the parameter α_μ was of order ε^2 and the rest of the coefficients were fixed independent of ε. Under the same assumptions as in [2], a rigorous justification of Biot's model was given by Nguetseng [8] and later by Clopeaut et al. [3]. The most general case of problem (1.1)–(1.9) where

$$\mu_0,\, \lambda_0^{-1},\, \nu_0,\, p_*^{-1},\, \eta_0^{-1} < \infty$$

was studied in [6].

All these authors used Nguetseng's two-scale convergence method [7, 5].

In the present paper, we use the same method to investigate all possible limiting regimes in problem (1.1)–(1.9) in the cases where

$$\nu_0,\, p_*,\, \eta_0 < \infty; \quad \mu_0 = \lambda_0 = 0, \quad 0 < p_*,\, \eta_0.$$

These cases correspond to the seismic acoustic wave propagation, where all the processes on distances of tens of thousands of meters ($L \nearrow \infty$) come to an end in several seconds ($\tau \searrow 0$).

We show that the homogenized equations are different types of nonstandard wave equations for a two- or one-velocity continuum (Theorem 2.2).

This is a very interesting fact: initially a one-velocity continuum becomes a two-velocity continuum after the homogenization procedure, which appears to be the result of different smoothness of the solution in the solid and liquid components:

$$\int_\Omega \alpha_\mu(\varepsilon)\chi^\varepsilon |\nabla \boldsymbol{w}^\varepsilon|^2 dx \leq C_0, \quad \int_\Omega \alpha_\lambda(\varepsilon)(1-\chi^\varepsilon)|\nabla \boldsymbol{w}^\varepsilon|^2 dx \leq C_0,$$

where C_0 is a constant independent of the small parameter ε. To preserve the best properties of the solution, we must use the well-known extension lemma [1, 4] and extend the solution from the solid part to the liquid part and conversely. At this stage, the criteria μ_1 and λ_1 become crucial. Namely, let $\boldsymbol{w}_f^\varepsilon$ ($\boldsymbol{w}_s^\varepsilon$) be an extension of the liquid (solid) displacements to the solid (liquid) part, and let $\mu_1 = \lambda_1 = \infty$. Then the limiting (homogenized) system describes the one-velocity continuum. This is because of the fact that each of the sequences $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, and $\{\boldsymbol{w}_s^\varepsilon\}$ two-scale converges to a function independent of the fast variable. This statement easily follows from Nguetseng's theorem.

If $\mu_1 < \infty$ and $\lambda_1 = \infty$ or $\mu_1 = \infty$ and $\lambda_1 < \infty$, then the homogenized systems describe the two-velocity continuum.

Finally, we note that, in practice, to solve a real physical problem in, for example, acoustics, one does not want to carry out the limiting procedure but, instead, wants to find a simple and reliable mathematical model describing the process. But there is only one exact (sufficiently reliable) mathematical model (1.1)–(1.9) with given physical constants (densities, viscosities, etc.), the characteristic size L of the physical domain under study, and the characteristic time τ of the physical process. The small parameter ε and the dimensionless quantities α_μ, α_λ, α_p, ... are functions of them. Changing the values of L and τ within some reasonable limits, one may find some rules for the behavior of the dimensionless quantities as the small parameter tends to zero. All possible limits of these quantities are described by conditions on μ_0, λ_0, μ_1, ... and, as was mentioned above, each homogenized system corresponds to a given combination of them. Thus, for a given physical situation, there exists a combination of dimensionless criteria, which would suggest the choice of the form of the homogenized system for obtaining the exact mathematical model. Therefore, to find all possible homogenized systems is very important from both mathematical and practical standpoints.

2. Main results. There are various equivalent (in the sense of distributions) forms of representation of (1.1) and boundary conditions (1.6)–(1.7). In what follows, it is convenient to write them in the form of the integral identities.

We say that four functions $\left(\boldsymbol{w}^{\varepsilon}, p_f^{\varepsilon}, p_s^{\varepsilon}, q^{\varepsilon}\right)$ are a generalized solution of problem (1.1)–(1.9) if they satisfy the regularity conditions

$$\boldsymbol{w}^{\varepsilon},\ \nabla\boldsymbol{w}^{\varepsilon},\ p_f^{\varepsilon},\ p_s^{\varepsilon},\ q^{\varepsilon} \in L^2(\Omega_T) \tag{2.1}$$

in the domain $\Omega_T = \Omega \times (0,T)$, boundary condition (1.9) in the trace sense, (1.5) and the equations

$$p_s^{\varepsilon} + (1-\chi^{\varepsilon})\alpha_\eta \operatorname{div}\boldsymbol{w}^{\varepsilon} = 0, \tag{2.2}$$

$$q^{\varepsilon} = p_f^{\varepsilon} + \frac{\alpha_\nu}{\alpha_p}\frac{\partial p_f^{\varepsilon}}{\partial t} \tag{2.3}$$

a.e. in Ω_T, and, finally, the integral identity

$$\left.\begin{aligned} \int_{\Omega_T}\Bigg(\rho^{\varepsilon}\boldsymbol{w}^{\varepsilon}\cdot\frac{\partial^2\boldsymbol{\varphi}}{\partial t^2} - \chi^{\varepsilon}\alpha_\mu D(\boldsymbol{x},\boldsymbol{w}^{\varepsilon}) : D\left(x,\frac{\partial\boldsymbol{\varphi}}{\partial t}\right) - \rho^{\varepsilon}\boldsymbol{F}\cdot\boldsymbol{\varphi} \\ + \{(1-\chi^{\varepsilon})\alpha_\lambda D(x,\boldsymbol{w}^{\varepsilon}) - (q^{\varepsilon}+p_s^{\varepsilon})I\} : D(x,\boldsymbol{\varphi})\Bigg)\,dxdt = 0 \end{aligned}\right\} \tag{2.4}$$

for all smooth vector-functions $\boldsymbol{\varphi} = \boldsymbol{\varphi}(\boldsymbol{x},t)$ such that $\boldsymbol{\varphi}|_{\partial\Omega} = \boldsymbol{\varphi}|_{t=T} = \partial\boldsymbol{\varphi}/\partial t|_{t=T} = 0$.

In this definition, we changed the form of representation of the stress tensor P in the integral identity (2.4) by introducing two new unknown functions, q^{ε} and p_s^{ε}, which in a certain sense have the meaning of pressure. In what follows, we call functions q^{ε} and p_s^{ε} the liquid and the solid pressure, respectively, and regard (2.3) as the state equation and equations (1.5) and (2.2) as the continuity equations. This special choice of the continuity and state equations simplifies the use of the homogenization procedure.

In (2.4), by $A : B$ we denote the convolution (or, equivalently, the inner tensor product) of two second-rank tensors along the both indices, i.e., $A : B = \operatorname{tr}(B^*\circ A) = \sum_{i,j=1}^{3} A_{ij}B_{ji}$.

Theorems 2.1–2.2 are the main results of the paper.

THEOREM 2.1. *Let* $\boldsymbol{F}$ *be bounded in* $L^2(\Omega)$. *Then for all* $\varepsilon > 0$ *on an arbitrary time interval* $[0,T]$, *there exists a unique generalized solution of problem* (1.1)–(1.9) *and*

$$\max_{0\le t\le T}\left\|\left|\frac{\partial\boldsymbol{w}^{\varepsilon}}{\partial t}\right| + \sqrt{\alpha_\mu}\chi^{\varepsilon}|\nabla\boldsymbol{w}^{\varepsilon}| + (1-\chi^{\varepsilon})\sqrt{\alpha_\lambda}|\nabla\boldsymbol{w}^{\varepsilon}|\right\|_{2,\Omega}(t) \le C_0, \tag{2.5}$$

$$\|q^{\varepsilon}\|_{2,\Omega_T} + \|p_f^{\varepsilon}\|_{2,\Omega_T} + \|p_s^{\varepsilon}\|_{2,\Omega_T} \le C_0, \tag{2.6}$$

where C_0 *is independent of the small parameter* ε.

THEOREM 2.2. *Assume that the hypotheses in Theorem* 2.1 *hold. Then there exists a subsequence of small parameters* $\{\varepsilon > 0\}$ *and functions* $\boldsymbol{w}_f^{\varepsilon}$, $\boldsymbol{w}_s^{\varepsilon} \in L^\infty(0,T; W_2^1(\Omega))$ *such that*

$$\boldsymbol{w}_f^{\varepsilon} = \boldsymbol{w}^{\varepsilon} \quad \text{in} \quad \Omega_f^{\varepsilon}\times(0,T), \qquad \boldsymbol{w}_s^{\varepsilon} = \boldsymbol{w}^{\varepsilon} \text{ in } \Omega_s^{\varepsilon}\times(0,T),$$

and the sequences $\{p_f^{\varepsilon}\}$, $\{q^{\varepsilon}\}$, $\{p_s^{\varepsilon}\}$, $\{\boldsymbol{w}^{\varepsilon}\}$, $\{\chi^{\varepsilon}\boldsymbol{w}^{\varepsilon}\}$, $\{(1-\chi^{\varepsilon})\boldsymbol{w}^{\varepsilon}\}$, $\{\boldsymbol{w}_f^{\varepsilon}\}$, *and* $\{\boldsymbol{w}_s^{\varepsilon}\}$ *converge as* $\varepsilon \searrow 0$ *weakly in* $L^2(\Omega_T)$ *to the functions* p_f, q, p_s, $\boldsymbol{w}$, $\boldsymbol{w}^f$, $\boldsymbol{w}^s$, $\boldsymbol{w}_f$, *and* $\boldsymbol{w}_s$, *respectively.*

(I) *If* $\mu_1 = \lambda_1 = \infty$*, then* $\boldsymbol{w}_f = \boldsymbol{w}_s = \boldsymbol{w}$ *and, in* Ω_T*, the functions* $\boldsymbol{w}$*,* p_f*,* q*, and* p_s *satisfy the system of acoustic equations*

$$\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = -\frac{1}{m}\nabla q + \hat{\rho}\boldsymbol{F}, \tag{2.7}$$

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \mathrm{div}\boldsymbol{w} = 0, \tag{2.8}$$

$$q = p_f + \frac{\nu_0}{p_*}\frac{\partial p_f}{\partial t}, \quad \frac{1}{m}q = \frac{1}{1-m}p_s, \tag{2.9}$$

the homogeneous initial conditions

$$\boldsymbol{w}(\boldsymbol{x},0) = \frac{\partial \boldsymbol{w}}{\partial t}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega, \tag{2.10}$$

and the homogeneous boundary condition

$$\boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S,\ t > 0, \tag{2.11}$$

where $\hat{\rho} = m\rho_f + (1-m)\rho_s$ *is the average density of the mixture and* $m = \int_Y \chi dy$ *is the porosity.*

(II) *If* $\mu_1 = \infty$ *and* $\lambda_1 < \infty$*, then, in* Ω_T*, the functions* $\boldsymbol{w}^f = m\boldsymbol{w}_f$*,* $\boldsymbol{w}^s$*,* p_f*,* q*, and* p_s *satisfy the system of acoustic equations consisting of the state equations* (2.9) *and the momentum balance equation*

$$\rho_f m\frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = -\frac{1}{m}\nabla q + \hat{\rho}\boldsymbol{F} \tag{2.12}$$

for the liquid component, the continuity equation

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + m\mathrm{div}\boldsymbol{w}_f + \mathrm{div}\boldsymbol{w}^s = 0, \tag{2.13}$$

and the relation

$$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\frac{\partial \boldsymbol{w}_f}{\partial t} + \int_0^t B_1^s(t-\tau)\cdot\boldsymbol{z}^s(\boldsymbol{x},\tau)d\tau, \tag{2.14}$$

where

$$\boldsymbol{z}^s(\boldsymbol{x},t) = -\frac{1}{m}\nabla q(\boldsymbol{x},t) + \rho_s\boldsymbol{F}(\boldsymbol{x},t) - \rho_s\frac{\partial^2 \boldsymbol{w}_f}{\partial t^2}(\boldsymbol{x},t),$$

in the case of $\lambda_1 > 0$ *or the momentum balance equation in the form*

$$\rho_s\frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s B_2^s\cdot\frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + ((1-m)I - B_2^s)\cdot\left(-\frac{1}{m}\nabla q + \rho_s\boldsymbol{F}\right) \tag{2.15}$$

in the case of $\lambda_1 = 0$ *for the solid component. Problem* (2.9), (2.12)–(2.15) *is supplemented with the homogeneous initial conditions* (2.10) *for displacements in the liquid and the solid components and the homogeneous boundary condition* (2.11) *for the displacements* $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$*.*

In (2.14)–(2.15), *the matrices* $B_1^s(t)$ *and* B_2^s *are defined below by formulas* (5.37) *and* (5.39), *where the matrix* $((1-m)I - B_2^s)$ *is symmetric and strictly positively definite.*

(III) *If* $\mu_1 < \infty$ *and* $\lambda_1 = \infty$, *then, in* Ω_T, *the functions* $\boldsymbol{w}^f$, $\boldsymbol{w}^s = (1-m)\boldsymbol{w}_s$, p_f, q, *and* p_s *satisfy the system of acoustic equations consisting of the state equations* (2.9) *and the momentum balance equation*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} + \rho_s(1-m)\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} = -\frac{1}{m}\nabla q + \hat{\rho}\boldsymbol{F} \tag{2.16}$$

for the solid component, the continuity equation

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\boldsymbol{w}^f + (1-m)\operatorname{div}\boldsymbol{w}_s = 0, \tag{2.17}$$

and the relation

$$\frac{\partial \boldsymbol{w}^f}{\partial t} = m\frac{\partial \boldsymbol{w}_s}{\partial t} + \int_0^t B_1^f(t-\tau)\cdot \boldsymbol{z}^f(\boldsymbol{x},\tau)d\tau, \tag{2.18}$$

where

$$\boldsymbol{z}^f(\boldsymbol{x},t) = -\frac{1}{m}\nabla q(\boldsymbol{x},t) + \rho_f\boldsymbol{F}(\boldsymbol{x},t) - \rho_f\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2}(\boldsymbol{x},t),$$

in the case of $\mu_1 > 0$ *or the momentum balance equation in the form*

$$\rho_f\frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} = \rho_f B_2^f\cdot\frac{\partial^2 \boldsymbol{w}_s}{\partial t^2} + (mI - B_2^f)\cdot\left(-\frac{1}{m}\nabla q + \rho_f\boldsymbol{F}\right) \tag{2.19}$$

in the case of $\mu_1 = 0$ *for the liquid component. Problem* (2.9), (2.16)–(2.19) *is supplemented with homogeneous initial conditions* (2.10) *for displacements in the liquid and solid components and the homogeneous boundary condition* (2.11) *for the displacements* $\boldsymbol{w} = \boldsymbol{w}^f + (1-m)\boldsymbol{w}_s$.

In (2.18)–(2.19), *the matrices* $B_1^f(t)$ *and* B_2^f *are given below by formulas* (5.44)–(5.45), *where the matrix* $(mI - B_2^f)$ *is symmetric and strictly positively definite.*

(IV) *If* $\mu_1 < \infty$ *and* $\lambda_1 < \infty$, *then, in* Ω_T, *the functions* $\boldsymbol{w}$, p_f, q, *and* p_s *satisfy the system of acoustic equations consisting of the continuity and state equations* (2.8) *and* (2.9) *and the relation*

$$\frac{\partial \boldsymbol{w}}{\partial t} = \int_0^t B(t-\tau)\cdot\nabla q(\boldsymbol{x},\tau)d\tau + \boldsymbol{f}(\boldsymbol{x},t), \tag{2.20}$$

where $B(t)$ *and* $\boldsymbol{f}(\boldsymbol{x},t)$ *are given below by* (5.58) *and* (5.59).

Problem (2.8), (2.9), (2.20) *is supplemented with homogeneous initial and boundary conditions* (2.10) *and* (2.11).

As was mentioned above, even in the most simple case (I) with $\nu_0 = 0$, Theorem 2.2 gives the standard wave equation for the solid pressure p_s but with a completely new speed of sound in the mixture, which includes the porosity, densities, and speeds of sound in the solid and liquid components.

In the next simple case (IV) with $\nu_0 = 0$, Theorem 2.2 gives a new wave equation for the solid pressure in the form

$$\frac{\partial p_s}{\partial t} = \int_0^t \operatorname{div}\big(\widetilde{B}(t-\tau)\cdot\nabla p_s(\boldsymbol{x},\tau)\big)d\tau. \tag{2.21}$$

Here $\widetilde{B}(0) = c^2 I$, where the time derivative of the matrix $\widetilde{B}(t)$ is generally unbounded at $t = 0$. This equation has no simple solutions like traveling waves and requires a special analysis even for the smooth matrix $\widetilde{B}(t)$.

The rest of the homogenized models described by Theorem 2.2 are much more complicated than the model (2.21). This is natural, because one cannot expect that a simple model gives an "accurate" approximation of the very complicated original model (1.1)–(1.9).

3. Preliminaries.

3.1. Two-scale convergence. The justification of Theorem 2.2 is based on a systematic use of the two-scale convergence method, which was proposed by Nguetseng [7] and has been recently used in a wide range of homogenization problems (see, for example, the survey [5]).

DEFINITION 3.1. *A sequence $\{w^\varepsilon\} \subset L^2(\Omega_T)$ is said to be two-scale convergent to the limit $W \in L^2(\Omega_T \times Y)$ if and only if the limiting relation*

$$\lim_{\varepsilon \searrow 0} \int_{\Omega_T} w^\varepsilon(\boldsymbol{x}, t)\, \sigma\left(\boldsymbol{x}, t, \frac{\boldsymbol{x}}{\varepsilon}\right) d\boldsymbol{x} dt = \int_{\Omega_T} \int_Y W(\boldsymbol{x}, t, \boldsymbol{y})\, \sigma(\boldsymbol{x}, t, \boldsymbol{y}) d\boldsymbol{y} d\boldsymbol{x} dt \tag{3.1}$$

holds for any function $\sigma = \sigma(\boldsymbol{x}, t, \boldsymbol{y}) \in C^\infty(\Omega_T \times Y)$ one-periodic in $\boldsymbol{y}$ and finite in Ω_T.

The existence and the main properties of weakly convergent sequences are established by the following fundamental theorem [7, 5].

THEOREM 3.2 (Nguetseng's theorem). 1. *Any sequence bounded in $L^2(\Omega_T)$ contains a subsequence two-scale convergent to some limit $W \in L^2(\Omega_T \times Y)$.*

2. *Let the sequences $\{w^\varepsilon\}$ and $\{\varepsilon \nabla_x w^\varepsilon\}$ be bounded in $L^2(\Omega_T)$. Then there exists a function $W = W(\boldsymbol{x}, t, \boldsymbol{y})$ one-periodic in $\boldsymbol{y}$ and a subsequence $\{w^\varepsilon\}$ such that W, $\nabla_y W \in L^2(\Omega_T \times Y)$, and the subsequences $\{w^\varepsilon\}$ and $\{\varepsilon \nabla_x w^\varepsilon\}$ two-scale converge to W and $\nabla_y W$, respectively.*

COROLLARY 3.3. *Let $\sigma \in L^2(Y)$ and $\sigma^\varepsilon(\boldsymbol{x}) = \sigma(\boldsymbol{x}/\varepsilon)$. Assume that a sequence $\{w^\varepsilon\} \subset L^2(\Omega_T)$ two-scale converges to $W \in L^2(\Omega_T \times Y)$. Then the sequence $\{\sigma^\varepsilon w^\varepsilon\}$ two-scale converges to the function σW.*

3.2. An extension lemma. A typical difficulty in homogenization problems like problem (1.1)–(1.7) arises in passing to the limit as $\varepsilon \searrow 0$ because of the fact that the bounds on the displacement gradient $\nabla \boldsymbol{w}^\varepsilon$ may be different in the liquid and solid components. The classical approach to overcoming this difficulty consists in constructing an extension of the displacement field defined merely on Ω_s or Ω_f to the whole Ω. The following lemma is valid due to the well-known results from [1, 4, 8]. We formulate it in the form convenient for us.

LEMMA 3.4. *Suppose that Assumption 1 on the geometry of the periodic structure is satisfied and $\boldsymbol{w}^\varepsilon \in \overset{\circ}{W}{}^1_2(\Omega)$. Then there exist functions $\boldsymbol{w}_f^\varepsilon$, $\boldsymbol{w}_s^\varepsilon \in W_2^1(\Omega)$ such that their respective restrictions on the subdomains Ω_f^ε and Ω_s^ε coincide with w^ε, i.e.,*

$$\chi^\varepsilon(\boldsymbol{x})(\boldsymbol{w}_f^\varepsilon(\boldsymbol{x}) - \boldsymbol{w}^\varepsilon(\boldsymbol{x})) = 0, \quad (1 - \chi^\varepsilon(\boldsymbol{x}))(\boldsymbol{w}_s^\varepsilon(\boldsymbol{x}) - \boldsymbol{w}^\varepsilon(\boldsymbol{x})) = 0, \quad \boldsymbol{x} \in \Omega, \tag{3.2}$$

and, in addition, the estimate

$$\|\boldsymbol{w}_i^\varepsilon\|_{2,\Omega} \leq C \|\boldsymbol{w}^\varepsilon\|_{2,\Omega_i^\varepsilon}, \quad \|D(x, \boldsymbol{w}_i^\varepsilon)\|_{2,\Omega} \leq C \|D(x, \boldsymbol{w}^\varepsilon)\|_{2,\Omega_i^\varepsilon}, \ i = f, s, \tag{3.3}$$

holds true, where the constant C depends only on the geometry of Y and is independent of ε.

3.3. Some notation. Further we denote the following:
(1)

$$\langle\Phi\rangle_Y = \int_Y \Phi dy, \quad \langle\Phi\rangle_{Y_f} = \int_Y \chi\Phi dy, \quad \langle\Phi\rangle_{Y_s} = \int_Y (1-\chi)\Phi dy.$$

(2) If $\mathbf{a}$ and $\mathbf{b}$ are two vectors, then the matrix $\mathbf{a}\otimes\mathbf{b}$ is defined by the formula

$$(\mathbf{a}\otimes\mathbf{b})\cdot\mathbf{c} = \mathbf{a}(\mathbf{b}\cdot\mathbf{c})$$

for any vector $\mathbf{c}$.

4. Proof of Theorem 2.1. Estimates (2.5)–(2.6) follow from the energy equality in the form

$$\frac{d}{dt}\left\{\int_\Omega \rho^\varepsilon\left(\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\right)^2 + \alpha_\lambda\int_\Omega (1-\chi^\varepsilon)D(x,\boldsymbol{w}^\varepsilon) : D(x,\boldsymbol{w}^\varepsilon)dx\right.$$

$$\left. + \alpha_p\int_\Omega \chi^\varepsilon(\mathrm{div}\boldsymbol{w}^\varepsilon)^2 dx + \alpha_\eta\int_\Omega (1-\chi^\varepsilon)(\mathrm{div}\boldsymbol{w}^\varepsilon)^2 dx\right\} + \alpha_\nu\int_\Omega \chi^\varepsilon\left(\mathrm{div}\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\right)^2 dx$$

$$+ \alpha_\mu\int_\Omega \chi^\varepsilon D\left(x,\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\right) : D\left(x,\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}\right) dx = \int_\Omega \rho^\varepsilon\frac{\partial \boldsymbol{F}}{\partial t}\cdot\frac{\partial \boldsymbol{w}^\varepsilon}{\partial t}dx \tag{4.1}$$

if we use Hölder, Gronwall, and Korn inequalities and extension Lemma 3.4. In turn, the energy equality (4.1) follows from (1.1) if we express the stress tensor P and the liquid pressure p_f using state equations (1.2)–(1.4) and continuity equation (1.5), multiply the result by $\partial\boldsymbol{w}^\varepsilon/\partial t$, and integrate by parts. Note that all terms on the "solid skeleton–pore space" interface Γ^ε disappear due to boundary conditions (1.6)–(1.7).

The same estimates (2.5)–(2.6) guarantee the existence and uniqueness of the generalized solution for problem (1.1)–(1.9).

5. Proof of Theorem 2.2.

5.1. Weak and two-scale limits of sequences of displacements and pressures. First, we use Lemma 3.4 and conclude that there are functions $\boldsymbol{w}_f^\varepsilon$, $\boldsymbol{w}_s^\varepsilon \in L^\infty(0,T;W_2^1(\Omega))$ such that

$$\boldsymbol{w}_f^\varepsilon = \boldsymbol{w}^\varepsilon \text{ in } \Omega_f^\varepsilon\times(0,T), \quad \boldsymbol{w}_s^\varepsilon = \boldsymbol{w}^\varepsilon \text{ in } \Omega_s^\varepsilon\times(0,T).$$

By Theorem 2.1, the sequences $\{p_f^\varepsilon\}$, $\{q^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, $\{\sqrt{\alpha_\mu}\nabla\boldsymbol{w}_f^\varepsilon\}$, $\{\boldsymbol{w}_s^\varepsilon\}$, and $\{\sqrt{\alpha_\lambda}\nabla\boldsymbol{w}_s^\varepsilon\}$ are bounded in $L^2(\Omega_T)$. Hence there exists a subsequence of small parameters $\{\varepsilon>0\}$ and functions p_f, q, p_s, $\boldsymbol{w}$, $\boldsymbol{w}_f$, and $\boldsymbol{w}_s$ such that

$$p_f^\varepsilon \rightharpoonup p_f, \quad q^\varepsilon \rightharpoonup q, \quad p_s^\varepsilon \rightharpoonup p_s, \quad \boldsymbol{w}^\varepsilon \rightharpoonup \boldsymbol{w}, \quad \boldsymbol{w}_f^\varepsilon \rightharpoonup \boldsymbol{w}_f, \quad \boldsymbol{w}_s^\varepsilon \rightharpoonup \boldsymbol{w}_s \tag{5.1}$$

weakly in $L^2(\Omega_T)$ as $\varepsilon\searrow 0$.

Note also that

$$(1-\chi^\varepsilon)\alpha_\lambda D(x,\boldsymbol{w}_s^\varepsilon)\to 0, \quad \chi^\varepsilon\alpha_\mu D(x,\boldsymbol{w}_f^\varepsilon)\to 0 \tag{5.2}$$

strongly in $L^2(\Omega_T)$ as $\varepsilon\searrow 0$.

Relabeling if necessary, we assume that the sequences themselves converge.

By Nguetseng's theorem, there exist functions $P_f(\boldsymbol{x},t,\boldsymbol{y})$, $P_s(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}_s(\boldsymbol{x},t,\boldsymbol{y})$ that are one-periodic in $\boldsymbol{y}$ and satisfy the condition that the sequences $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, and $\{\boldsymbol{w}_s^\varepsilon\}$ two-scale converge to $P_f(\boldsymbol{x},t,\boldsymbol{y})$, $P_s(\boldsymbol{x},t,\boldsymbol{y})$, $Q(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})$, $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})$, and $\boldsymbol{W}_s(\boldsymbol{x},t,\boldsymbol{y})$, respectively.

LEMMA 5.1. *If* $\mu_1=\infty$ $(\lambda_1=\infty)$, *then* $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{w}_f(\boldsymbol{x},t)$, $\chi(\boldsymbol{y})\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})=\chi(\boldsymbol{y})\boldsymbol{w}_f(\boldsymbol{x},t)$, *and* $\boldsymbol{w}^f=\langle\boldsymbol{W}\rangle_{Y_f}=m\boldsymbol{w}_f$ $\big(\boldsymbol{W}_s(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{w}_s(\boldsymbol{x},t)$, $(1-\chi(\boldsymbol{y}))\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})=(1-\chi(\boldsymbol{y}))\boldsymbol{w}_s(\boldsymbol{x},t)$ *and* $\boldsymbol{w}^s=\langle\boldsymbol{W}\rangle_{Y_s}=(1-m)\boldsymbol{w}_s\big)$.

Proof. Suppose that $\mu_1=\infty$, and let $\Psi(\boldsymbol{x},t,\boldsymbol{y})$ be an arbitrary smooth scalar function periodic in $\boldsymbol{y}$. The sequence $\{\sigma_{ij}^\varepsilon\}$, where

$$\sigma_{ij}^\varepsilon=\int_\Omega\sqrt{\alpha_\lambda}\frac{\partial w_{f,i}^\varepsilon}{\partial x_j}(\boldsymbol{x},t)\Psi(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)dx,\quad \boldsymbol{w}_f^\varepsilon=(w_{f,1}^\varepsilon,w_{f,2}^\varepsilon,w_{f,3}^\varepsilon),$$

is uniformly bounded in ε. Therefore,

$$\int_\Omega\varepsilon\frac{\partial w_{f,i}^\varepsilon}{\partial x_j}(\boldsymbol{x},t)\Psi(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)dx=\frac{\varepsilon}{\sqrt{\alpha_\lambda}}\sigma_{ij}^\varepsilon\to 0$$

as $\varepsilon\searrow 0$, which is equivalent to

$$\int_\Omega\int_Y W_{f,i}(\boldsymbol{x},t,\boldsymbol{y})\frac{\partial\Psi}{\partial y_j}(\boldsymbol{x},t,\boldsymbol{y})dxdy=0,\quad \boldsymbol{W}_f=(W_{f,1},W_{f,2},W_{f,3}),$$

or $\boldsymbol{W}_f(\boldsymbol{x},t,\boldsymbol{y})=\boldsymbol{w}_f(\boldsymbol{x},t)$. Therefore taking the two-scale limit as $\varepsilon\searrow 0$ in the relation $\chi^\varepsilon(\boldsymbol{w}^\varepsilon-\boldsymbol{w}_f^\varepsilon)=0$, we arrive at

$$\chi(\boldsymbol{y})\boldsymbol{W}(\boldsymbol{x},t,\boldsymbol{y})=\chi(\boldsymbol{y})\boldsymbol{w}_f. \qquad \square$$

5.2. Micro- and macroscopic equations. We start the proof of the theorem from the macro- and microscopic equations related to the continuity equations.

LEMMA 5.2. *For almost all* $\boldsymbol{x}\in\Omega$ *and* $\boldsymbol{y}\in Y$, *the weak and two-scale limits of the sequences* $\{p_f^\varepsilon\}$, $\{p_s^\varepsilon\}$, $\{q^\varepsilon\}$, $\{\boldsymbol{w}^\varepsilon\}$, $\{\boldsymbol{w}_f^\varepsilon\}$, *and* $\{\boldsymbol{w}_s^\varepsilon\}$ *satisfy the relations*

$$Q=q\chi/m,\ P_f=p_f\chi/m,\ P_s=p_s(1-\chi)/(1-m),\ Q=P+\nu_0p_*^{-1}\partial P/\partial t; \tag{5.3}$$

$$q/m=p_s/(1-m),\quad q=p_f+\nu_0p_*^{-1}\partial p_f/\partial t; \tag{5.4}$$

$$p_f/p_*+p_s/\eta_0+\operatorname{div}\boldsymbol{w}=0; \tag{5.5}$$

$$\boldsymbol{w}(\boldsymbol{x},t)\cdot\boldsymbol{n}(\boldsymbol{x})=0,\quad \boldsymbol{x}\in S,\ t>0; \tag{5.6}$$

$$\operatorname{div}_y\boldsymbol{W}=0; \tag{5.7}$$

$$\boldsymbol{W}=\chi\boldsymbol{W}_f+(1-\chi)\boldsymbol{W}_s. \tag{5.8}$$

Proof. In order to prove (5.3), into (2.4) we substitute the test function $\boldsymbol{\psi}^\varepsilon=\varepsilon\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{x}/\varepsilon)$, where $\boldsymbol{\psi}(\boldsymbol{x},t,\boldsymbol{y})$ is an arbitrary one-periodic function of $\boldsymbol{y}$ that is finite on Y_f (or finite on Y_s or finite on Y). Passing to the limit as $\varepsilon\searrow 0$, we obtain

$$\nabla_yQ=0,\ \boldsymbol{y}\in Y_f;\ \nabla_yP_s=0,\ \boldsymbol{y}\in Y_s;\ \nabla_y(Q+P_s)=0,\ \boldsymbol{y}\in Y. \tag{5.9}$$

Next, fulfilling the two-scale passage to the limit in the state equation (2.3) and in the relations

$$(1-\chi^\varepsilon)q^\varepsilon=0,\quad \chi^\varepsilon p_s^\varepsilon=0$$

we arrive at the last equation in (5.3) and the relations

$$(1-\chi)Q = 0, \quad \chi P_s = 0,$$

which together with the first two equations in (5.9) prove the first three equations in (5.3).

The second equation in (5.4) is the result of integration of the last equation in (5.3) over the domain Y_f.

The first relation in (5.4) follows from (5.3) and the last equation in (5.9): the sequence $\{(q^\varepsilon + p_s^\varepsilon)\}$ two-scale converges to $(Q + P_s) = (q + p_s)$.

Equations (5.5)–(5.7) appear as a result of the two-scale passage to the limit in (1.5) and (2.2) with the proper test functions being involved. Thus, for example, (5.5) and (5.6) arise if we consider the linear combination of (1.5) and (2.2)

$$\frac{1}{\alpha_p} p_f^\varepsilon + \frac{1}{\alpha_\eta} p_s^\varepsilon + \operatorname{div} \boldsymbol{w}^\varepsilon = 0, \tag{5.10}$$

multiply it by an arbitrary function independent of the "fast" variable $\boldsymbol{x}/\varepsilon$, and then pass to the limit as $\varepsilon \searrow 0$. To prove (5.7), it suffices to consider the two-scale limiting relations in (5.10) as $\varepsilon \searrow 0$ with the test functions $\varepsilon\psi\left(\boldsymbol{x}/\varepsilon\right) h(\boldsymbol{x}, t)$, where ψ and h are arbitrary smooth functions.

To prove (5.8), it suffices to consider the two-scale limiting relations in

$$\boldsymbol{w}^\varepsilon = \chi^\varepsilon \boldsymbol{w}_f^\varepsilon + (1-\chi^\varepsilon)\boldsymbol{w}_s^\varepsilon. \qquad \square$$

LEMMA 5.3. *For almost all $(\boldsymbol{x}, t) \in \Omega_T$, the relation*

$$\rho_f \frac{\partial^2 \boldsymbol{w}^f}{\partial t^2} + \rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = -\frac{1}{m}\nabla q + \hat{\rho}\boldsymbol{F} \tag{5.11}$$

holds true.

Proof. Substituting a test function of the form $\boldsymbol{\psi} = \boldsymbol{\psi}(\boldsymbol{x}, t)$ into integral identity (2.4) and passing to the limit as $\varepsilon \searrow 0$, we arrive at (5.11). $\square$

LEMMA 5.4. *Let $\mu_1 = \infty$ and $\lambda_1 < \infty$. Then, in Y_s, the functions $\boldsymbol{W}^s = (1-\chi)\boldsymbol{W}$, $\boldsymbol{w}_f$, and q satisfy the microscopic relations*

$$\rho_s \frac{\partial^2 \boldsymbol{W}^s}{\partial t^2} = \lambda_1 \triangle_y \boldsymbol{W}^s - \nabla_y R^s - \frac{1}{m}\nabla q + \rho_s \boldsymbol{F}, \quad \boldsymbol{y} \in Y_s, \tag{5.12}$$

$$\boldsymbol{W}^s = \boldsymbol{w}_f, \quad \boldsymbol{y} \in \gamma, \tag{5.13}$$

in the case $\lambda_1 > 0$ or the microscopic relations

$$\rho_s \frac{\partial^2 \boldsymbol{W}^s}{\partial t^2} = -\nabla_y R^s - \frac{1}{m}\nabla q + \rho_s \boldsymbol{F}, \quad \boldsymbol{y} \in Y_s, \tag{5.14}$$

$$(\boldsymbol{W}^s - \boldsymbol{w}_f) \cdot \boldsymbol{n} = 0, \quad \boldsymbol{y} \in \gamma, \tag{5.15}$$

in the case $\lambda_1 = 0$.

The problem is endowed with the homogeneous initial data

$$\boldsymbol{W}^s(\boldsymbol{y}, 0) = \frac{\partial \boldsymbol{W}^s}{\partial t}(\boldsymbol{y}, 0) = 0, \quad \boldsymbol{y} \in Y_s. \tag{5.16}$$

In (5.15) $\boldsymbol{n}$ *is the unit normal to* γ.

Proof. The differential equations (5.12) and (5.14) follow as $\varepsilon \searrow 0$ from integral equality (2.4) with the test function $\boldsymbol{\psi} = \boldsymbol{\varphi}(x\varepsilon^{-1}) \cdot h(\boldsymbol{x}, t)$, where $\boldsymbol{\varphi}$ is solenoidal and finite in Y_s.

The boundary condition (5.13) is a consequence of the two-scale convergence of the sequence $\{\sqrt{\alpha_\lambda}\nabla_x \boldsymbol{w}^\varepsilon\}$ to the function $\sqrt{\lambda_1}\nabla_y \boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y})$. By this convergence, the function $\nabla_y \boldsymbol{W}(\boldsymbol{x}, t, \boldsymbol{y})$ is L^2-integrable in Y. The boundary condition (5.15) follows from (5.7)–(5.8) and the relation $\boldsymbol{W}_f = \boldsymbol{w}_f$. □

In the same way, one can prove the following lemma.

LEMMA 5.5. *Let* $\mu_1 < \infty$ *and* $\lambda_1 = \infty$. *Then, in* Y_f, *the functions* $\boldsymbol{W}^f = \chi \boldsymbol{W}$, $\boldsymbol{w}_s$, *and* q *satisfy the microscopic relations*

$$\rho_f \frac{\partial^2 \boldsymbol{W}^f}{\partial t^2} = \mu_1 \triangle_y \frac{\partial \boldsymbol{W}^f}{\partial t} - \nabla_y R^f - \frac{1}{m}\nabla q + \rho_f \boldsymbol{F}, \quad \boldsymbol{y} \in Y_f, \tag{5.17}$$

$$\boldsymbol{W}^f = \boldsymbol{w}_s, \quad \boldsymbol{y} \in \gamma, \tag{5.18}$$

in the case $\mu_1 > 0$ *or the microscopic relations*

$$\rho_f \frac{\partial^2 \boldsymbol{W}^f}{\partial t^2} = -\nabla_y R^f - \frac{1}{m}\nabla q + \rho_f \boldsymbol{F}, \quad \boldsymbol{y} \in Y_f, \tag{5.19}$$

$$(\boldsymbol{W}^f - \boldsymbol{w}_s) \cdot \boldsymbol{n} = 0, \quad \boldsymbol{y} \in \gamma, \tag{5.20}$$

in the case $\mu_1 = 0$. *The problem is endowed with the homogeneous initial data*

$$\boldsymbol{W}^f(\boldsymbol{y}, 0) = \frac{\partial \boldsymbol{W}^f}{\partial t}(\boldsymbol{y}, 0) = 0, \quad \boldsymbol{y} \in Y_f. \tag{5.21}$$

LEMMA 5.6. *Let* $\mu_1 < \infty$, $\lambda_1 < \infty$, *and* $\tilde{\rho} = \rho_f \chi + \rho_s(1 - \chi)$. *Then, in* Y, *the functions* $\boldsymbol{W}$ *and* q *satisfy the microscopic equation*

$$\left.\begin{aligned} &\tilde{\rho}\partial^2 \boldsymbol{W}/\partial t^2 + 1/m \nabla q - \tilde{\rho}\boldsymbol{F} \\ &= \operatorname{div}_y\{\mu_1 \chi D(y, \partial \boldsymbol{W}/\partial t) + \lambda_1(1 - \chi)D(y, \boldsymbol{W}) - RI\} \end{aligned}\right\} \tag{5.22}$$

and the homogeneous initial data

$$\boldsymbol{W}(\boldsymbol{y}, 0) = \frac{\partial \boldsymbol{W}}{\partial t}(\boldsymbol{y}, 0) = 0, \quad \boldsymbol{y} \in Y. \tag{5.23}$$

In the proof of the last lemma, we additionally use Nguetseng's theorem, which states that the sequence $\{\varepsilon D(x, \boldsymbol{w}^\varepsilon)\}$ two-scale converges to the function $D(y, \boldsymbol{W})$.

5.3. Homogenized equations. Lemmas 5.2 and 5.3 imply the following lemma.

LEMMA 5.7. *Let* $\mu_1 = \lambda_1 = \infty$. *Then* $\boldsymbol{w}_f = \boldsymbol{w}_s = \boldsymbol{w}$ *and, in* Ω_T, *the functions* $\boldsymbol{w}$, p_f, q, *and* p_s *satisfy the system of acoustic equations*

$$\hat{\rho}\frac{\partial^2 \boldsymbol{w}}{\partial t^2} = -\frac{1}{m}\nabla q + \hat{\rho}\boldsymbol{F}, \tag{5.24}$$

$$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + \operatorname{div}\boldsymbol{w} = 0, \tag{5.25}$$

(5.26) $$q = p_f + \frac{\nu_0}{p_*}\frac{\partial p_f}{\partial t}, \quad \frac{1}{m}q = \frac{1}{1-m}p_s,$$

the homogeneous initial conditions

(5.27) $$\boldsymbol{w}(\boldsymbol{x},0) = \frac{\partial \boldsymbol{w}}{\partial t}(\boldsymbol{x},0) = 0, \quad \boldsymbol{x} \in \Omega,$$

and the homogeneous boundary condition

(5.28) $$\boldsymbol{w}(\boldsymbol{x},t) \cdot \boldsymbol{n}(\boldsymbol{x}) = 0, \quad \boldsymbol{x} \in S,\ t > 0.$$

LEMMA 5.8. *Let $\mu_1 = \infty$ and $\lambda_1 < \infty$. Then, in Ω_T, the functions $\boldsymbol{w}_f$, $\boldsymbol{w}^s$, p_f, q, and p_s satisfy the system of acoustic equations consisting of the state equations* (5.26)*, the momentum balance equation for the liquid component*

(5.29) $$\rho_f m \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + \rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = -\frac{1}{m}\nabla q + \hat{\rho}\boldsymbol{F},$$

the continuity equation

(5.30) $$\frac{1}{p_*}p_f + \frac{1}{\eta_0}p_s + m\mathrm{div}\boldsymbol{w}_f + \mathrm{div}\boldsymbol{w}^s = 0,$$

and the relation

(5.31) $$\frac{\partial \boldsymbol{w}^s}{\partial t} = (1-m)\frac{\partial \boldsymbol{w}_f}{\partial t} + \int_0^t B_1^s(t-\tau) \cdot \boldsymbol{z}^s(\boldsymbol{x},\tau)d\tau,$$

where

$$\boldsymbol{z}^s(\boldsymbol{x},t) = -\frac{1}{m}\nabla q(\boldsymbol{x},t) + \rho_s \boldsymbol{F}(\boldsymbol{x},t) - \rho_s \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2}(\boldsymbol{x},t),$$

in the case of $\lambda_1 > 0$ or the momentum balance equation for the solid component in the form

(5.32) $$\rho_s \frac{\partial^2 \boldsymbol{w}^s}{\partial t^2} = \rho_s B_2^s \cdot \frac{\partial^2 \boldsymbol{w}_f}{\partial t^2} + ((1-m)I - B_2^s) \cdot \left(-\frac{1}{m}\nabla q + \rho_s \boldsymbol{F}\right)$$

in the case of $\lambda_1 = 0$. Problem (5.26), (5.29)–(5.32) *is supplemented by homogeneous initial conditions* (5.27) *for displacements in the liquid and solid components and the homogeneous boundary condition* (5.28) *for the displacements $\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s$.*

In (5.31)–(5.32)*, the matrices $B_1^s(t)$ and B_2^s are defined by formulas* (5.37) *and* (5.39)*, where the matrix $((1-m)I - B_2^s)$ is symmetric and strictly positive definite.*

Proof. Equation (5.29) follows directly from (5.11). The continuity equation (5.30) follows from (5.5) if we take into account that

$$\boldsymbol{w} = m\boldsymbol{w}_f + \boldsymbol{w}^s.$$

To find the last two equations (5.31) and (5.32), we just have to solve the system of microscopic equations (5.7), (5.12)–(5.16) and use the formula

$$\boldsymbol{w}^s = \langle \boldsymbol{W} \rangle_{Y_s}.$$

There are two different cases.

(a) If $\lambda_1 > 0$, then the solution of the system of microscopic equations (5.7), (5.12), and (5.13) supplemented with the homogeneous initial data (5.16) is given by the formulas

$$\boldsymbol{W}^s = \int_0^t \left(\boldsymbol{v}(\boldsymbol{x},\tau) + \sum_{i=1}^{3} \boldsymbol{W}^{s,i}(\boldsymbol{y}, t-\tau) z_i^s(\boldsymbol{x},\tau) \right) d\tau,$$

$$R^s = \int_0^t \sum_{i=1}^{3} R^{s,i}(\boldsymbol{y}, t-\tau) z_i^s(\boldsymbol{x},\tau) d\tau, \quad \boldsymbol{z}^s = \left(z_1^s, z_2^s, z_3^s\right),$$

and the functions $\boldsymbol{W}^{s,i}(\boldsymbol{y},t)$ and $R^{s,i}(\boldsymbol{y},t)$ are defined by virtue of the periodic initial boundary value problem

$$\rho_s \frac{\partial^2 \boldsymbol{W}^{s,i}}{\partial t^2} - \lambda_1 \triangle \boldsymbol{W}^{s,i} + \nabla R^{s,i} = 0, \quad \boldsymbol{y} \in Y_s,\ t > 0, \tag{5.33}$$

$$\operatorname{div}_y \boldsymbol{W}^{s,i} = 0, \quad \boldsymbol{y} \in Y_s,\ t > 0, \tag{5.34}$$

$$\boldsymbol{W}^{s,i} = 0, \quad \boldsymbol{y} \in \gamma,\ t > 0, \tag{5.35}$$

$$\boldsymbol{W}^{s,i}(y,0) = 0, \quad \rho_s \frac{\partial \boldsymbol{W}^{s,i}}{\partial t}(\boldsymbol{y},0) = \boldsymbol{e}_i, \quad \boldsymbol{y} \in Y_s. \tag{5.36}$$

In (5.36), $\boldsymbol{e}_i$ is the standard Cartesian basis vector.

Therefore,

$$B_1^s(t) = \left\langle \sum_{i=1}^{3} \frac{\partial \boldsymbol{W}^{s,i}}{\partial t} \right\rangle_{Y_s} (t) \otimes \boldsymbol{e}_i. \tag{5.37}$$

Note that (5.33) is understood in the sense of distributions and the function $B_1^s(t)$ has no time derivative at $t = 0$.

(b) If $\lambda_1 = 0$, then in solving the system (5.7), (5.14), (5.15), and (5.16), we first find the pressure $R^s(\boldsymbol{x},t,\boldsymbol{y})$ by solving the Neumann problem for the Laplace equation in Y_s in the form

$$R^s(\boldsymbol{x},t,\boldsymbol{y}) = \sum_{i=1}^{3} R_{s,i}(\boldsymbol{y}) z_i^s(\boldsymbol{x},t),$$

where $R_{s,i}(\boldsymbol{y})$ is the solution of the problem

$$\triangle_y R_{s,i} = 0, \quad \boldsymbol{y} \in Y_s; \quad \nabla_y R_{s,i} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{e}_i, \quad \boldsymbol{y} \in \gamma; \quad \langle R_{s,i} \rangle_{Y_s} = 0. \tag{5.38}$$

Formula (5.32) is the result of integration of (5.14) over the domain Y_s and

$$B_2^s = \sum_{i=1}^{3} \langle \nabla R_{s,i} \rangle_{Y_s} \otimes \boldsymbol{e}_i, \tag{5.39}$$

where the matrix $B=((1-m)I-B_2^s)$ is symmetric and strictly positive definite. In fact, let $\tilde{R}=\sum_{i=1}^{3}R_{s,i}\xi_i$ for any unit vector $\boldsymbol{\xi}=(\xi_1,\xi_2,\xi_3)$. Then

$$(B\cdot\boldsymbol{\xi})\cdot\boldsymbol{\xi}=\langle(\boldsymbol{\xi}-\nabla\tilde{R})^2\rangle_{Y_f}$$

and $(B\cdot\boldsymbol{\xi})\cdot\boldsymbol{\xi}=0$ if and only if $\tilde{R}$ is a linear function in $\boldsymbol{y}$. On the other hand, it follows from the assumption about the geometry of the domain Y_s that all linear periodic functions on Y_s are constant. Finally, the normalization condition $\langle R_{s,i}\rangle_{Y_s}=0$ yields that $\tilde{R}=0$. However, this is impossible, because the functions $R_{s,i}$ are linearly independent. □

LEMMA 5.9. *Let $\mu_1<\infty$ and $\lambda_1=\infty$. Then, in Ω_T, the functions $\boldsymbol{w}^f$, $\boldsymbol{w}_s$, p_f, q, and p_s satisfy the system of acoustic equations consisting of the state equations* (5.26), *the momentum balance equation for the solid component*

$$\rho_f\frac{\partial^2\boldsymbol{w}^f}{\partial t^2}+\rho_s(1-m)\frac{\partial^2\boldsymbol{w}_s}{\partial t^2}=-\frac{1}{m}\nabla q+\hat{\rho}\boldsymbol{F}, \tag{5.40}$$

the continuity equation

$$\frac{1}{p_*}p_f+\frac{1}{\eta_0}p_s+\mathrm{div}\boldsymbol{w}^f+(1-m)\mathrm{div}\boldsymbol{w}_s=0, \tag{5.41}$$

and the relation

$$\frac{\partial\boldsymbol{w}^f}{\partial t}=m\frac{\partial\boldsymbol{w}_s}{\partial t}+\int_0^t B_1^f(t-\tau)\cdot\boldsymbol{z}^f(\boldsymbol{x},\tau)d\tau, \tag{5.42}$$

where

$$\boldsymbol{z}^f(\boldsymbol{x},t)=-\frac{1}{m}\nabla q(\boldsymbol{x},t)+\rho_f\boldsymbol{F}(\boldsymbol{x},t)-\rho_f\frac{\partial^2\boldsymbol{w}_s}{\partial t^2}(\boldsymbol{x},t),$$

in the case of $\mu_1>0$ or the momentum balance equation for the liquid component in the form

$$\rho_f\frac{\partial^2\boldsymbol{w}^f}{\partial t^2}=\rho_f B_2^f\cdot\frac{\partial^2\boldsymbol{w}_s}{\partial t^2}+(mI-B_2^f)\cdot\left(-\frac{1}{m}\nabla q+\rho_f\boldsymbol{F}\right) \tag{5.43}$$

in the case of $\mu_1=0$. Problem (5.26), (5.40)–(5.43) *is supplemented with the homogeneous initial conditions* (5.27) *for displacements in the liquid and solid components and the homogeneous boundary condition* (5.28) *for the displacements $\boldsymbol{w}=\boldsymbol{w}^f+(1-m)\boldsymbol{w}_s$.*

In (5.42)–(5.43), *the matrices $B_1^f(t)$ and B_2^f are given below by formulas* (5.44)–(5.45), *where the matrix $(mI-B_2^f)$ is symmetric and strictly positive definite.*

Proof. The proof of this lemma repeats that of the previous lemma. Here we have to solve the system of microscopic equations (5.7), (5.17)–(5.21) and use the formula

$$\boldsymbol{w}^f=\langle\boldsymbol{W}\rangle_{Y_f}.$$

Thus,

$$B_1^f(t)=\left\langle\sum_{i=1}^{3}\frac{\partial\boldsymbol{W}^{f,i}}{\partial t}\right\rangle_{Y_f}(t)\otimes\boldsymbol{e}_i, \tag{5.44}$$

$$B_2^f = \sum_{i=1}^{3} \langle \nabla R_{f,i} \rangle_{Y_f} \otimes \boldsymbol{e}_i, \tag{5.45}$$

where the functions $\boldsymbol{W}^{f,i}(\boldsymbol{y},t)$ solve the periodic initial boundary value problem

$$\rho_f \frac{\partial^2 \boldsymbol{W}^{f,i}}{\partial t^2} - \mu_1 \triangle \frac{\partial \boldsymbol{W}^{f,i}}{\partial t} + \nabla R^{f,i} = 0, \quad \boldsymbol{y} \in Y_f,\ t > 0, \tag{5.46}$$

$$\mathrm{div}_y \boldsymbol{W}^{f,i} = 0, \quad \boldsymbol{y} \in Y_f,\ t > 0, \tag{5.47}$$

$$\boldsymbol{W}^{f,i} = 0, \quad \boldsymbol{y} \in \gamma,\ t > 0, \tag{5.48}$$

$$\boldsymbol{W}^{f,i}(y,0) = 0, \quad \rho_f \frac{\partial \boldsymbol{W}^{f,i}}{\partial t}(y,0) = \boldsymbol{e}_i, \quad \boldsymbol{y} \in Y_f, \tag{5.49}$$

and the functions $R_{f,i}(\boldsymbol{y})$ solve the periodic boundary value problem

$$\triangle_y R_{f,i} = 0, \quad \boldsymbol{y} \in Y_f; \quad \nabla_y R_{f,i} \cdot \boldsymbol{n} = \boldsymbol{n} \cdot \boldsymbol{e}_i, \quad \boldsymbol{y} \in \gamma; \quad \langle R_{f,i} \rangle_{Y_f} = 0. \tag{5.50}$$

Note that, as before, the matrix $(mI - B_2^f)$ is symmetric and strictly positive definite. □

The proof of Theorem 2.2 is completed by the following lemma.

LEMMA 5.10. *Let* $\mu_1 < \infty$ *and* $\lambda_1 < \infty$. *Then, in* Ω_T, *the functions* $\boldsymbol{w}$, p_f, q, *and* p_s *satisfy the system of acoustic equations consisting of the continuity and state equations* (5.25) *and* (5.26) *and the relation*

$$\frac{\partial \boldsymbol{w}}{\partial t} = \int_0^t B(t-\tau) \cdot \nabla q(\boldsymbol{x},\tau) d\tau + \boldsymbol{f}(\boldsymbol{x},t), \tag{5.51}$$

where $B(t)$ *and* $\boldsymbol{f}(\boldsymbol{x},t)$ *are given below by* (5.58) *and* (5.59).

Problem (5.25), (5.26), (5.51) *is supplemented with the homogeneous initial and boundary conditions* (5.27) *and* (5.28).

Proof. To derive the momentum conservation law (5.51), we must solve the system of microscopic equations (5.7), (5.22) with the initial conditions (5.23) and use the formula

$$\boldsymbol{w} = \langle \boldsymbol{W} \rangle_Y.$$

Let

$$\boldsymbol{W} = \int_0^t \sum_{i=1}^{3} \left\{ \boldsymbol{W}^{q,i}(\boldsymbol{y},t-\tau) \frac{\partial q}{\partial x_i}(\boldsymbol{x},\tau) + \boldsymbol{W}^{F,i}(\boldsymbol{y},t-\tau) F_i(\boldsymbol{x},\tau) \right\} d\tau,$$

$$R = \int_0^t \sum_{i=1}^{3} \left\{ R^{q,i}(\boldsymbol{y},t-\tau) \frac{\partial q}{\partial x_i}(\boldsymbol{x},\tau) + R^{F,i}(\boldsymbol{y},t-\tau) F_i(\boldsymbol{x},\tau) \right\} d\tau,$$

where $\boldsymbol{F} = \sum_{i=1}^{3} F_i \boldsymbol{e}_i$.

Then the pair $\{\boldsymbol{W}, R\}$ is a solution of system (5.7), (5.22) and (5.23) if and only if the functions $\{\boldsymbol{W}^{q,i}(\boldsymbol{y},t), R^{q,i}(\boldsymbol{y},t)\}$ and $\{\boldsymbol{W}^{F,i}(\boldsymbol{y},t), R^{F,i}(\boldsymbol{y},t)\}$ are periodic in $\boldsymbol{y}$ solutions of the equations

$$\operatorname{div}_y \left\{ \mu_1 \chi D\left(y, \frac{\partial \boldsymbol{W}^{q,i}}{\partial t}\right) + \lambda_1(1-\chi) D(y, \boldsymbol{W}^{q,i}) - R^{q,i} I \right\} = \tilde{\rho} \frac{\partial^2 \boldsymbol{W}^{q,i}}{\partial t^2}, \tag{5.52}$$

$$\operatorname{div}_y \boldsymbol{W}^{q,i} = 0, \tag{5.53}$$

$$\operatorname{div}_y \left\{ \mu_1 \chi D\left(y, \frac{\partial \boldsymbol{W}^{F,i}}{\partial t}\right) + \lambda_1(1-\chi) D(y, \boldsymbol{W}^{F,i}) - R^{F,i} I \right\} = \tilde{\rho} \frac{\partial^2 \boldsymbol{W}^{F,i}}{\partial t^2}, \tag{5.54}$$

$$\operatorname{div}_y \boldsymbol{W}^{F,i} = 0 \tag{5.55}$$

in the domain Y for $t > 0$ and satisfy the initial conditions

$$\boldsymbol{W}^{q,i}(\boldsymbol{y}, 0) = 0, \quad \tilde{\rho} \frac{\partial \boldsymbol{W}^{q,i}}{\partial t}(\boldsymbol{y}, 0) = -\frac{1}{m} \boldsymbol{e}_i, \quad \boldsymbol{x} \in Y, \tag{5.56}$$

$$\boldsymbol{W}^{F,i}(\boldsymbol{y}, 0) = 0, \quad \frac{\partial \boldsymbol{W}^{F,i}}{\partial t}(\boldsymbol{y}, 0) = \boldsymbol{e}_i, \quad \boldsymbol{x} \in Y. \tag{5.57}$$

Here $\boldsymbol{e}_i$ is the standard Cartesian basis vector.

Therefore,

$$B(t) = \sum_{i=1}^{3} \left\langle \frac{\partial \boldsymbol{W}^{q,i}}{\partial t}(\boldsymbol{y}, t) \right\rangle_Y \otimes \boldsymbol{e}_i, \tag{5.58}$$

$$\mathbf{f}(\boldsymbol{x}, t) = \int_0^t \sum_{i=1}^{3} \left\langle \frac{\partial \boldsymbol{W}^{F,i}}{\partial t}(\boldsymbol{y}, t-\tau) \right\rangle_Y F_i(\boldsymbol{x}, \tau) d\tau. \tag{5.59}$$

The solvability and uniqueness of problems (5.52), (5.53), (5.56) and (5.54), (5.55), (5.57) follow directly from the energy identity

$$\frac{1}{2} \int_Y \left(\tilde{\rho} \left(\frac{\partial \boldsymbol{W}^{j,i}}{\partial t}(\boldsymbol{y}, t) \right)^2 + \lambda_1 D\big(y, \boldsymbol{W}^{j,i}(\boldsymbol{y}, t)\big) : D\big(y, \boldsymbol{W}^{j,i}(\boldsymbol{y}, t)\big) \right) dy$$

$$+ \int_0^t \int_Y \mu_1 D\left(y, \frac{\partial \boldsymbol{W}^{j,i}}{\partial \tau}(\boldsymbol{y}, \tau)\right) : D\left(y, \frac{\partial \boldsymbol{W}^{j,i}}{\partial \tau}(\boldsymbol{y}, \tau)\right) dy d\tau = \frac{1}{2} \beta^j$$

for $i = 1, 2, 3$ and $j = q, F$.

Here

$$\beta^q = \left\langle \frac{1}{\tilde{\rho}} \right\rangle_Y, \quad \beta^F = \langle \tilde{\rho} \rangle_Y.$$

As before, equations (5.51) are understood in the sense of distributions and the function $B(t)$ has no time derivative at $t = 0$. $\square$

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