Distributions, Non-smooth Manifolds, Transmutations and Boundary Value Problems

Vladimir B. Vasilyev

Abstract One discusses the problem of constructing the theory of pseudo differential equations on manifolds with a non-smooth boundary. Using special factorization principle and transmutation operators we consider some general boundary value problems for elliptic pseudo-differential equations in canonical non-smooth manifolds.

Keywords Non-smooth manifold · Pseudo-differential operator · Elliptic symbol · Boundary value problem

2010 Mathematics Subject Classification Primary: 35S15; Secondary: 58J05

1 Introduction

We study Fredholm properties of elliptic pseudo-differential operators (or equations) in Sobolev–Slobodetskii spaces on manifolds with a boundary but in our case the boundary may be non-smooth.

Basic principles for studying such equations are the following:

- a local principle or freezing coefficients principle;
- factorizability principle for an elliptic symbol at boundary point;
- a pluralism principle for singular boundary points which implies distinct types of local operators.

Local principle and factorizability was first introduced in papers I.B. Simonenko [16] (for multidimensional singular integral operators in Lebesgue L_p-spaces) and M.I. Vishik–G.I. Eskin [2] (for pseudo-differential operators in Sobolev–Slobodetskii H^s-spaces). For manifolds with a smooth boundary one uses an idea

V. B. Vasilyev (✉)
Chair of Differential Equations, Belgorod National Research University, Belgorod, Russia
e-mail: vbv57@inbox.ru; vladimir.b.vasilyev@gmail.com

V. V. Kravchenko, S. M. Sitnik (eds.), *Transmutation Operators and Applications*, Trends in Mathematics, https://doi.org/10.1007/978-3-030-35914-0_14

of "rectification of a boundary", and the problems reduces to a half-space case, for which a factorizability principle holds immediately because under localization at a boundary point and applying the Fourier transform we obtain well known one-dimensional classical Riemann boundary value problem for upper and lower complex half-planes with a multidimensional parameter. This approach does not work if a boundary has at least one singular point like a conical point. One needs here other considerations and approaches.

The wave factorization principle was introduced by the author in 90th [18, 19] to extend the Vishik–Eskin theory to manifolds with a singular boundary. Such approach requires a special factorization for an elliptic symbol, and it leads to multidimensional variant of classical Riemann boundary value problem and multidimensional analogues of the Cauchy type integrals. It was shown [20, 28] these multidimensional analogues transform to the Cauchy type integral with a parameter for limit cases.

The third principle asserts that there are a lot of singularities at a boundary. Every singularity requires a separate studying to obtain solvability conditions for corresponding model equation. Common part of such studying is requiring the wave factorization for an elliptic symbol with respect to corresponding cone. If we have such factorization then we can describe needed solvability conditions (see, for example, [22–27]).

2 Domains and Operators

We consider a certain integro-differential operator A on m-dimensional compact manifold M with a boundary. This operators is defined by the function $A(x,\xi)$, $(x,\xi) \in \mathbb{R}^{2m}$. There are some smooth compact sub-manifolds M_k of dimension $0 \le k \le m-1$ on the boundary ∂M of manifold M which are singularities of a boundary. These singularities are described by a local representative of operator A in a point $x_0 \in M$ on the map $U \ni x_0$ in the following way

$$(A_{x_0}u)(x) = \int\limits_{D_{x_0}} \int\limits_{\mathbb{R}^m} e^{i\xi\cdot(x-y)} A(\varphi(x_0),\xi)u(y)d\xi dy, \quad x \in D_{x_0}, \tag{1}$$

where $\varphi : U \to D_{x_0}$ is a diffeomorphism, and the canonical domain D_{x_0} has a distinct form depending on a placement of the point x_0 on manifold M. We consider the following canonical domains D_{x_0}: $\mathbb{R}^m$, $\mathbb{R}^m_+ = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > 0\}$, $W^k = \mathbb{R}^k \times C^{m-k}$, where C^{m-k} is a convex cone in $\mathbb{R}^{m-k}$ non-including a whole line.

Such an operator A will be considered in Sobolev–Slobodetskii spaces $H^s(M)$, and local variants of such spaces will be spaces $H^s(D_{x_0})$. Local principle asserts that for a Fredholm property of the operator A it is necessary and sufficient an invertibility for all "local operators" $A_{x_0}, x_0 \in M$. So, we need to describe the conditions for unique solvability all model equations of the following type

$$(A_{x_0}u)(x) = v(x), \quad x \in D_{x_0}, \tag{2}$$

in corresponding local Sobolev–Slobodetskii spaces $H^s(D_{x_0})$.

2.1 Paired Equations

Such equations appear together with Eq. (2). Paired equation is called the following equation

$$(AP_+ + BP_-)U(x) = V(x), \quad x \in \mathbb{R}^m,$$

where A, B are model elliptic pseudo-differential operators, P_+ is restriction operator on canonical domain D, P_- is restriction operator on $\mathbb{R}^m \setminus D$. It is easily to show that solving the Eq. (2) is equivalent to solving the paired equation with $A = A_{x_0}$ and $B = I$ (identity). For solving such paired equations they apply the factorization technique and complex variables [2].

2.2 Singularities and Distributions

Author's point of view is the following. Each boundary point of manifold M is served by a special distribution. Such a distribution is the Fourier transform of an indicator of canonical domain. Using these distributions we reduce the Eq. (2) to a certain variant of the Riemann boundary value problem in the function theory of complex variables (one or many) [1, 2, 4, 7, 9, 19–21].

2.3 Complex Variables and Wave Factorization

To obtain the conditions for unique solvability for the Eq. (2) (or equivalently invertibility conditions for the operator (1)) we introduce the following concept. Let us denote [32]

$$\overset{*}{C}{}^{m-k} = \{x \in \mathbb{R}^{m-k} : x \cdot y > 0, y \in C^{m-k}\}$$

Taking into account local principle we will consider only symbols non-depending on spatial variables x and satisfying the condition

$$c_1(1+|\xi|)^\alpha \le |A(\xi)| \le c_2(1+|\xi|)^\alpha. \tag{3}$$

Definition 1 k-Wave factorization of elliptic symbol $A(\xi)$ with respect to the C^{m-k} is called its representation in the form

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ must satisfy the following conditions:

(1) $A_{\neq}(\xi)$, $A_{=}(\xi)$ are defined forall $\xi \in \mathbb{R}^m$ without may be the points $\mathbb{R}^k \times \partial\left(\overset{*}{C}{}^{m-k} \cup (-\overset{*}{C}{}^{m-k})\right)$;

(2) $A_{\neq}(\xi)$, $A_{=}(\xi)$ admit analytic continuation into radial tube domains $T(\overset{*}{C}{}^{m-k})$, $T(-\overset{*}{C}{}^{m-k})$ for almost all $\xi'' \in \mathbb{R}^k$ respectively with estimates

$$|A_{\neq}^{\pm 1}(\xi'', \xi' + i\tau)| \le c_1(1+|\xi|+|\tau|)^{\pm æ_k},$$

$$|A_{=}^{\pm 1}(\xi'', \xi' - i\tau)| \le c_2(1+|\xi|+|\tau|)^{\pm(\alpha - æ_k)}, \ \forall \tau \in \overset{*}{C}{}^{m-k}.$$

The number $æ_k \in \mathbb{R}$ is called index of k-wave factorization.

Existence of such factorization permits to describe solvability picture for model pseudo-differential equation (2) for $m-k=2$ [19, 20], but in a general case we need to know the general form of a distribution supported on a conical surface (we can't find such form in [5]). We try to reduce the problem to a half-space case using transmutation operators.

3 Transmutations, Distributions and the Fourier Transform

Below we consider the case $k=0$ because all conclusions will be the same, only k-dimensional parameter can be appear. Let C be a convex cone in the space $\mathbb{R}^m$, and this cone does not include any whole straight line, it is important because we use the theory of analytic functions of several complex variables [1, 31, 32]. Moreover we suppose that a surface of this cone is given by the equation $x_m = \varphi(x')$, $x' = (x_1, \cdots, x_{m-1})$, where $\varphi : \mathbb{R}^{m-1} \to \mathbb{R}$ is a smooth function in $\mathbb{R}^{m-1} \setminus \{0\}$, and $\varphi(0) = 0$.

Let us introduce the following change of variables [14, 29, 30]

$$\begin{cases} t_1 = x_1 \\ t_2 = x_2 \\ \cdots \\ t_{m-1} = x_{m-1} \\ t_m = x_m - \varphi(x') \end{cases}$$

and we denote this operator by $T_\varphi : \mathbb{R}^m \to \mathbb{R}^m$.

Obviously, this is a smooth transformation excluding an origin. Let f be a local integrable function which generates a distribution defined by the formula

$$(f, \psi) = \int\limits_{\mathbb{R}^m} f(x)\psi(x)dx.$$

We define a functional $T_\varphi f$ by the formula

$$(T_\varphi f, \psi) = (f, T_\varphi^{-1}\psi).$$

According to the Schwartz theorem on one-dimensional distribution from $S'(\mathbb{R})$ supported at the origin 0 [5, 32] we can conclude that if a distribution $f \in S'(\mathbb{R}^m)$ supported in the hyper-plane $x_m = 0$ then it has the following form

$$f(x) = \sum_{k=0}^{n} c_k(x') \otimes \delta^{(k)}(x_m), \quad x = (x', x_m),$$

where $c_k \in S'(\mathbb{R}^{m-1})$, $k = 0, 1, \cdots, n$, are arbitrary distributions.

Therefore we can assert that if a distribution $f \in S'(\mathbb{R}^m)$ is supported on ∂C then $T_\varphi f$ is supported on $\mathbb{R}^{m-1}$.

An arbitrary distribution $f \in S'(\mathbb{R}^m)$ supported on conical surface ∂C can written in the form

$$f(x) = T_\varphi^{-1}\left(\sum_{k=0}^{n} c_k(y') \otimes \delta^{(k)}(y_m)\right), \tag{4}$$

where $c_k \in S'(\mathbb{R}^{m-1})$, $k = 0, 1, \cdots, n$, are arbitrary distributions.

Further, for functions $u(x)$ from $S(\mathbb{R}^m)$ their Fourier transform is defined by the formula

$$(Fu)(\xi) \equiv \tilde{u}(\xi) = \int\limits_{\mathbb{R}^m} e^{ix\cdot\xi} u(x)dx.$$

The Fourier transform for distributions is defined as follows

$$(Ff, \psi) = (f, F\psi),$$

therefore

$$(FT_\varphi f, \psi) = (f, T_\varphi^{-1} F\psi).$$

Let $f \in S'(\mathbb{R}^m)$ be a distribution supported on ∂C. According to the above conclusions it has the special form (4). Using properties of T_φ and F we will find

$$Ff = V_\varphi \left(\sum_{k=0}^{n} \tilde{c}_k(\xi') \xi_m^k \right),$$

where

$$FT_\varphi^{-1} F^{-1} \equiv V_\varphi.$$

For a distribution $f \in S'(\mathbb{R}^m)$ the transform V_φ is given by the formula

$$(V_\varphi \tilde{f}, \psi) \equiv (\tilde{f}, V_{-\varphi}\psi), \quad \forall \psi \in S(\mathbb{R}^m).$$

If $\hat{u}(x', \xi_m)$ denotes the Fourier transform of the function $u(x', x_m)$ with respect to a variable x_m then one can make the following conclusion. Let us denote

$$F_{x' \to \xi'}(e^{-i\xi_m \varphi(x')}) \equiv K_\varphi(\xi', \xi_m),$$

and after this we obtain an integral representation for the operator V_φ:

$$(FT_\varphi^{-1} u)(\xi) = \int_{\mathbb{R}^m} K_\varphi(\xi' - \eta', \xi_m) \tilde{u}(\eta', \xi_m) d\eta'.$$

3.1 Examples

3.1.1 Plane Sector

The case $m = 2$ is a very good, there is only one mentioned cone. We write it as follows

$$C_+^a = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_2 > a|x_1|, a > 0\},$$

and further evaluate:

$$(FT_\varphi^{-1}u)(\xi) = \frac{\tilde{u}(\xi_1 + a\xi_2, \xi_2) + \tilde{u}(\xi_1 - a\xi_2, \xi_2)}{2} +$$

$$v.p.\frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 + a\xi_2 - \eta} - v.p.\frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 - a\xi_2 - \eta} \equiv (V_\varphi\tilde{u})(\xi).$$

We denote by $S_1\tilde{u}$ the operator

$$(S_1\tilde{u})(\xi_1, \xi_2) = v.p.\frac{i}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{\tilde{u}(\eta, \xi_2)d\eta}{\xi_1 - \eta}$$

and analogously S_2 for the second variable.

3.1.2 Standard Cone

As it was shown the kernel K_φ is computable for concrete function $\varphi(x')$. Let $\varphi(x') = a|x'|, a > 0$,. If we will look at the formulas from [31] (see also [16] in which a real analogue of these formulas is given as the Poisson kernel) we will find

$$K_\varphi(\xi', \xi_m) = \frac{a2^{m-1}\pi^{\frac{m-2}{2}}\Gamma(m/2)}{\left(|\xi'|^2 - a^2\xi_m^2\right)^{m/2}}.$$

Therefore for such multidimensional cone the operator V_φ looks as follows

$$(V_\varphi\tilde{u})(\xi) = \int\limits_{\mathbb{R}^{m-1}}\frac{a2^{m-1}\pi^{\frac{m-2}{2}}\Gamma(m/2)\tilde{u}(\eta', \xi_m)d\eta'}{\left(|\xi' - \eta'|^2 - a^2\xi_m^2\right)^{m/2}}.$$

In our opinion we could call it *a conical potential*.

Of course this formula should be treated in a distribution sense. Below we give such definition for the operator V_φ in the space $S'(\mathbb{R}^m)$.

3.1.3 Three-Wedged Pyramid

This cone looks as follows

$$C_+^{\mathbf{a}} = \{x \in \mathbb{R}^3 : x_3 > a_1|x_1| + a_2|x_2|, a_1, a_2 > 0\}$$

For this case the operator V_φ is constructed exactly using two operators S_1, S_2 (see below)

4 Potentials Generated by Transmutations

4.1 General Situation

Now we see that the main problem is to study

Let C be a convex cone non-including a whole straight line. Let us introduce the Bochner kernel [1, 31, 32]

$$B_m(z) = \int\limits_C e^{ix\cdot z}dx, \quad z = \xi + i\tau,$$

and related integral operator

$$(B_m u)(x) = \lim_{\tau\to 0+} \int\limits_{\mathbb{R}^m} B_m(x - y + i\tau)u(y)dy, \quad x \in \mathbb{R}^m.$$

Theorem 1 *If the symbol $A(\xi)$ admits the wave factorization with the index æ, $æ - s = n+\delta, n \in \mathbb{N}, |\delta| < 1/2$, then a general solution of the Eq. (2) in Fourier images is given by the formula*

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi)Q_n(\xi)B_m Q_n^{-1}(\xi)A_{=}^{-1}(\xi)\tilde{lf}(\xi)+$$

$$+A_{\neq}^{-1}(\xi)V_\varphi^{-1}F\left(\sum_{k=1}^n c_k(x')\delta^{(k-1)}(x_m)\right),$$

where $c_k(x') \in H^{s_k}(\mathbb{R}^{m-1})$ are arbitrary functions, $s_k = s - æ + k - 1/2,\ k = 1, 2, \ldots, n$, lf is an arbitrary continuation of f onto $H^{s-\alpha}(\mathbb{R}^m)$, Q_n is an arbitrary polynomial satisfying the condition (3) for $\alpha = n$.

Using these results one needs to add some additional conditions to determine uniquely unknown functions c_k. We will consider certain particular case in the next section.

Some special cases are very interesting, for example if $C = C_+^a = \{x \in \mathbb{R}^m : x = (x', x_m), x_m > a|x'|, a > 0\}$. Using evaluations from [17] we can obtain the following result.

Corollary 1 *If $f \equiv 0, n = 1$, then we have the following form for a general solution in the space $H^s(C_+^a)$*

$$\tilde{u}_+(\xi) = A_{\neq}^{-1}(\xi) \int\limits_{\mathbb{R}^{m-1}} \frac{a2^{m-1}\pi^{\frac{m-2}{2}}\Gamma(m/2)\tilde{c}(\eta')d\eta'}{\left(|\xi' - \eta'|^2 - a^2\xi_m^2\right)^{m/2}},$$

where $c(x') \in H^{s-æ+1/2}(\mathbb{R}^{m-1})$ is an arbitrary function.

5 Boundary Value Problems

According to Theorem 1 we can consider different types of boundary value problems with boundary conditions or with co-boundary operators.

Let us consider a simple boundary value problem for the equation

$$(Au)(x) = 0, \quad x \in C_+^{\mathbf{a}} \tag{5}$$

for the case $æ - s = 1 + \delta, |\delta| < 1/2$, where A is an elliptic pseudo-differential operator with the symbol $A(\xi)$ satisfying the condition (3) and admitting the wave factorization with respect to the cone $C_+^{\mathbf{a}}$.

According to Theorem 1 we have the formula for a general solution, for our case it can be written as

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)(V_{-\mathbf{a}}Fc_0)(\xi), \tag{6}$$

where $c_0(x')$ is an arbitrary function from $H^{s_0}(\mathbb{R}^2)$.

Now we will write an expression for $V_{-\mathbf{a}}Fc_0$ and then we will see what kind of conditions for a solution u is more preferable. Direct calculations led to the following expression

$$\begin{aligned} A_{\neq}(\xi)\tilde{u}(\xi) = \tilde{C}_1(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) + \tilde{C}_2(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) + \\ \tilde{C}_3(\xi_1 + a_1\xi_3, \xi_2 - a_2\xi_3) + \tilde{C}_1(\xi_1 + a_1\xi_3, \xi_2 + a_2\xi_3), \end{aligned} \tag{7}$$

where

$$\tilde{C}_1(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) -$$

$$-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3) + (S_1S_2\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 - a_2\xi_3);$$

$$\tilde{C}_2(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) = \frac{1}{4}\tilde{c}_0(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) - \frac{1}{2}(S_1\tilde{c}_0)(\xi_1 - a_1\xi_3, \xi_2 + a_2\xi_3) +$$

$$+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1-a_1\xi_3,\xi_2+a_2\xi_3)-(S_1S_2\tilde{c}_0)(\xi_1-a_1\xi_3,\xi_2+a_2\xi_3);$$

$$\tilde{C}_3(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3)=\frac{1}{4}\tilde{c}_0(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3)-$$

$$-\frac{1}{2}(S_2\tilde{c}_0)(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3)-(S_1S_2\tilde{c}_0)(\xi_1+a_1\xi_3,\xi_2-a_2\xi_3);$$

$$\tilde{C}_4(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3)=\frac{1}{4}\tilde{c}_0(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3)+\frac{1}{2}(S_1\tilde{c}_0)(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3)+$$

$$+\frac{1}{2}(S_2\tilde{c}_0)(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3)+(S_1S_2\tilde{c}_0)(\xi_1+a_1\xi_3,\xi_2+a_2\xi_3).$$

It seems the problem of finding the unknown function $c_0(\xi_1,\xi_2)$ is very hard, but we suppose that we know the following function $\tilde{u}(\xi_1,\xi_2,0)$. It means that we know the following integral

$$\int\limits_{-\infty}^{+\infty} u(x_1,x_2,x_3)dx_3 \equiv g(x_1,x_2), \tag{8}$$

thus

$$\tilde{u}(\xi_1,\xi_2,0)=\tilde{g}(\xi_1,\xi_2). \tag{9}$$

The formula (7) includes a representation for $V_{-\mathbf{a}}\tilde{c}_0$, where $\tilde{c}_0(\xi')$ is a function of two variables. Thus, if $\tilde{c}_0(\xi_1,\xi_2)$ depends on two variables ξ_1,ξ_2 then $V_{-\mathbf{a}}\tilde{c}_0$ depends on all three variables ξ_1,ξ_2,ξ_3.

Substituting (9) into (7) and collecting similar summands we obtain the following equation for the unknown $\tilde{c}_0(\xi')$

$$A_{\neq}^{-1}(\xi',0)(\tilde{c}_0(\xi'))=\tilde{g}(\xi'),$$

or if we designate $A_{\neq}(\xi',0)\tilde{g}(\xi')\equiv f(\xi')$

$$\tilde{c}_0(\xi')=\tilde{f}(\xi')$$

Now if we have found $\tilde{c}_0(\xi')$ we have the solution of the problem (5) and (8).

Also we can give a priori estimates for the solution.

Theorem 2 *Let $A(\xi)$ admits the wave factorization with respect to the $C^{\mathbf{a}}_+$. Then the boundary value problem (5) and (8) has a unique solution for an arbitrary $g \in H^{s+1/2}(\mathbb{R}^2)$ in the space $H^s(C^{\mathbf{a}}_+)$. This solution can be constructed explicitly by the*

Fourier transform and the one-dimensional singular integral operator. The a priori estimate

$$||u||_s \leq c[g]_{s+1/2}$$

holds for $-1/2 < \delta < 0$.

6 Thin Cones

As we see all local operators includes some parameters (sizes of cones) which can be small or large. These situations correspond to so called thin cones or a half-space case (see, for example, [20] were some calculations were given). Singularities at a boundary can be of distinct dimensions and it is possible such singularities of a low dimension can be obtained from analogous singularities of full dimension. It means we need to find distributions for limit cases when some of parameters of singularities tend to zero. This approach was partially realized in author's papers [22, 23], and the latest paper [27] is devoted to multi-dimensional constructions. The further author's idea is the following. If we know the limit operator for a thin singularity then possible it is zero approximation for a such thin singularity. It is desirable to obtain an asymptotic expansion with a small parameter for the distribution corresponding to a such singularity. We will consider here a two-dimensional case.

To describe a solvability picture for a model elliptic pseudo differential equation with an operator A

$$(Au)(x) = v(x), \tag{10}$$

in two-dimensional cone $C_+^a = \{x \in \mathbb{R}^2 : x_2 > a|x_1|, a > 0\}$ the author earlier considered a special singular integral operator [18, 19]

$$(K_a u)(x) = \frac{a}{2\pi^2} \lim_{\tau \to 0+} \int\limits_{\mathbb{R}^2} \frac{u(y)dy}{(x_1 - y_1)^2 - a^2(x_2 - y_2 + i\tau)^2}.$$

This operator served a conical singularity in the general theory of boundary value problems for elliptic pseudo differential equations on manifolds with a non-smooth boundary. This operator is a convolution operator, and the parameter a is a size of an angle, $x_2 > a|x_1|$, $a = \cot\alpha$.

We will consider two spaces of basic functions for distributions. If $D(\mathbb{R}^2)$ denotes a space of infinitely differentiable functions with a compact support then $D'(\mathbb{R}^2)$ is the corresponding space of distributions over the space $D(\mathbb{R}^2)$, analogously if $S(\mathbb{R}^2)$ is the Schwartz space of infinitely differentiable rapidly decreasing at infinity functions then $S'(\mathbb{R}^2)$ is a corresponding space of distributions over $S(\mathbb{R}^2)$.

When $a \to +\infty$ one obtains [20] the following limit distribution

$$\lim_{a\to\infty} \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2} = \frac{i}{2\pi} \mathcal{P}\frac{1}{\xi_1} \otimes \delta\,(\xi_2)\,,$$

where the notation for distribution $\mathcal{P}$ is taken from V.S. Vladimirov's books [31, 32], and $\otimes$ denotes the direct product of distributions. Here δ denotes one-dimensional Dirac mass-function which acts on $\varphi \in D(\mathbb{R})$ by the following way

$$(\delta, \varphi) = \varphi(0),$$

and the distribution $\mathcal{P}\frac{1}{x}$ is defined by the formula

$$(\mathcal{P}\frac{1}{x}, \varphi) = v.p. \int\limits_{-\infty}^{+\infty} \frac{\varphi(x)dx}{x} \equiv \lim_{\varepsilon\to 0+} \left(\int\limits_{-\infty}^{-\varepsilon} + \int\limits_{\varepsilon}^{+\infty} \right) \frac{\varphi(x)dx}{x}.$$

We would like to obtain an asymptotical expansion for the two-dimensional distribution

$$K_a(\xi_1, \xi_2) \equiv \frac{a}{2\pi^2} \frac{1}{\xi_1^2 - a^2\xi_2^2}$$

with respect to small a^{-1}. It is defined by the corresponding formula $\forall\varphi \in D(\mathbb{R}^2)$

$$(K_a, \varphi) = \frac{a}{2\pi^2} \int\limits_{\mathbb{R}^2} \frac{\varphi(\xi_1, \xi_2)d\xi}{\xi_1^2 - a^2\xi_2^2}.$$

For $K_a \in D'(\mathbb{R}^2)$ we can suggest the following decomposition [28]

$$K_a(\xi_1, \xi_2) = \frac{i}{2\pi} \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!a^n} \mathcal{P}\frac{1}{\xi_1} \otimes \delta^{(n)}(\xi_2).$$

But for $K_a \in S'(\mathbb{R}^2)$ we have more explicit result [28].

Theorem 3 *The following formula*

$$K_a(\xi_1, \xi_2) = \frac{i}{2\pi} \mathcal{P}\frac{1}{\xi_1} \otimes \delta(\xi_2) + \sum_{m,n} c_{m,n}(a)\widetilde{\delta^{(m)}}(\xi_1) \otimes \delta^{(n)}(\xi_2),$$

where $c_{m,n}(a) \to 0$, $a \to +\infty$, *holds in a distribution sense.*

Let us return to the Eq. (10). For $|æ - s| < 1/2$ one has the existence and uniqueness theorem [18]

$$\widetilde{u}(\xi) = A_{\neq}^{-1}(\xi)(K_a\widetilde{lv})(\xi),$$

where lv is an arbitrary continuation of v on the whole $H^s(\mathbb{R}^2)$.

Below we denote $lv \equiv V$.

Theorem 4 *If the symbol $A(\xi)$ admits a wave factorization with respect to the cone C_+^a and $|æ - s| < 1/2$ then Eq. (1) has a unique solution in the space $H^s(C_+^a)$, and for a large a it can be represented in the form*

$$\widetilde{u}(\xi) = \frac{i}{2\pi}A_{\neq}^{-1}(\xi)v.p.\int_{-\infty}^{+\infty}\frac{(A_{=}^{-1}\widetilde{V})(\eta_1,\xi_2)d\eta_1}{\xi_1 - \eta_1} +$$

$$A_{\neq}^{-1}(\xi)\sum_{m,n}c_{m,n}(a)\int_{-\infty}^{+\infty}(\xi_1 - \eta_1)^m(A_{=}^{-1}\widetilde{V})_{\xi_2}^{(n)}(\eta_1,\xi_2)d\eta_1$$

assuming $\widetilde{V} \in S(\mathbb{R}^2)$, $A_{=}^{-1}\widetilde{V}$ means the function $A_{=}^{-1}(\xi)\widetilde{V}(\xi)$.

7 Conclusion

This paper is a brief description of latest author's studies on elliptic pseudo-differential equations and boundary value problems on manifolds with non-smooth boundaries. Other approaches, similar problems, interesting statements can be found in books and monographs [3, 6–8, 10–13, 15].

References

1. S. Bochner, W.T. Martin, *Several Complex Variables* (Princeton University Press, Princeton, 1948)
2. G. Eskin, *Boundary Value Problems for Elliptic Pseudodifferential Equations* (American Mathematical Society, Providence, 1981)
3. H.G. Feichtinger, B. Helffer, M. Lamoureux, N. Lerner, J. Toft, *Pseudo-Differential Operators: Quantization and Signals*, ed. by L. Rodino, M.W. Wong. Lecture Notes Mathematics (Springer, Berlin, 2008)
4. F.D. Gakhov, *Boundary Value Problems* (Dover Publications, New York, 1981)
5. I.M. Gel'fand, G.E. Shilov, *Generalized Functions, Vol. 1. Properties and Operations* (Academic, New York, 1964)
6. Y.I. Karlovich, L.G. Rodino, B. Silbermann (eds.), *Operator Theory, Pseudo-Differential Equations, and Mathematical Physics* (Birkhäuser, Basel, 2013)

7. C. Kottke, R.B. Melrose, Generalized blow-up of corners and fiber products. Trans. Am. Math. Soc. **367**(1), 651–705 (2015)
8. S.G. Milkhin, S. Prößdorf, *Singular Integral Operators* (Akademie-Verlag, Berlin, 1986)
9. N.I. Muskhelishvili, *Singular Integral Equations* (North Holland, Amsterdam, 1976)
10. V.E. Nazaikinskii, A.Yu. Savin, B.-W. Schulze, B. Yu. Sternin, *Elliptic Theory on Singular Manifolds* (Chapman & Hall/CRC, Boca Raton, 2006)
11. V. Nazaikinskii, B.-W. Schulze, B. Sternin, *The Localization Problem in Index Theory of Elliptic Operators. Pseudo-Differential Operators*. Theory and Applications, vol. 10 (Birkhäuser, Basel, 2014)
12. S.A. Nazarov, B.A. Plamenevsky, *Elliptic Problems in Domains with Piecewise Smooth Boundaries* (Walter de Gruyter, Berlin, 1994)
13. L. Rodino, B.-W. Schulze, M.W. Wong (eds.), *Pseudo-Differential Operators: Partial Differential Equations and Time-Frequency Analysis (Fields Institute Communications)* (AMS, Providence, 2007)
14. S. Samko, *Hypersingular Integrals and Their Applications* (CRC Press, London, 2001)
15. B.-W. Schulze, B. Sternin, V. Shatalov, *Differential Equations on Singular Manifolds: Semiclassical Theory and Operator Algebras* (Wiley-VCH, Berlin, 1998)
16. I.B. Simonenko, *Local Method in the Theory of Translation Invariant Operators and Their Envelopes* (CVVR, Rostov on Don, 2007, in Russian)
17. E.M. Stein, G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces* (Princeton University Press, Princeton, 1971)
18. V.B. Vasil'ev, *Wave Factorization of Elliptic Symbols: Theory and Applications. Introduction to the Theory of Boundary Value Problems in Non-smooth Domains* (Kluwer Academic Publishers, Dordrecht, 2000)
19. V.B. Vasilyev, *Multipliers of Fourier Integrals, Pseudodifferential Equations, the Wave Factorization, Boundary Value Problems* (in Russian), 2nd edn. (Editorial URSS, Moscow, 2010)
20. V.B. Vasilyev, Asymptotical analysis of singularities for pseudo differential equations in canonical non-smooth domains, in *Integral Methods in Science and Engineering*. Computational and Analytic Aspects (Birkhäuser, Boston, 2011), pp. 379–390
21. V.B. Vasilyev, Pseudo differential equations on manifolds with non-smooth boundaries, in *Differential and Difference Equations and Applications*. Springer Proceedings in Mathematics & Statistics, vol. 47 (2013), pp. 625–637
22. V.B. Vasilyev, General boundary value problems for pseudo differential equations and related difference equations. Adv. Differ. Equ. **2013**(289), 1–7 (2013)
23. V.B. Vasilyev, On certain elliptic problems for pseudo differential equations in a polyhedral cone. Adv. Dyn. Syst. Appl. **9**(2), 227–237 (2014)
24. V.B. Vasilyev, On the Dirichlet and Neumann problems in multi-dimensional cone. Math. Bohem. **139**(2), 333–340 (2014)
25. V.B. Vasilyev, New constructions in the theory of elliptic boundary value problems, in *Integral Methods in Science and Engineering. Proceedings of IMSE Conference, Karlsruhe, 2014* (Birkhäuser, Basel, 2015), pp. 629–641
26. V.B. Vasilyev, Potentials for elliptic boundary value problems in cones (in Russian). Siberian Electron. Math. Repts. **13**, 1129–1149 (2016)
27. V.B. Vasilyev, Potentials for elliptic boundary value problems in cones (in Russian). Sib. Elektron. Mat. Izv. **13**, 1129–1149 (2016)
28. V.B. Vasilyev, On the asymptotic expansion of certain plane singular integral operators. Bound. Value Problems **116**, 1–13 (2017)
29. V.B. Vasilyev, Pseudo-differential equations, wave factorization, and related problems. Math. Methods Appl. Sci. **41**(18), 9252–9263 (2018)
30. V.B. Vasilyev, On some distributions associated to boundary value problems. Complex Var. Elliptic Equ. **64**(5), 888–898 (2019)
31. V.S. Vladimirov, *Generalized Functions in Mathematical Physics* (Mir, Moscow, 1979)
32. V.S. Vladimirov, *Methods of the Theory of Functions of Many Complex Variables* (Dover Publications, Mineola, 2007)