One-Dimensional and Multi-Dimensional Integral Transforms of Buschman–Erdélyi Type with Legendre Functions in Kernels

Sergei M. Sitnik and Oksana V. Skoromnik

Abstract This paper consists of two parts. In the first part we give a brief survey of results on Buschman–Erdélyi operators, which are transmutations for the Bessel singular operator. Main properties and applications of Buschman–Erdélyi operators are outlined. In the second part of the paper we consider multi-dimensional integral transforms of Buschman–Erdélyi type with Legendre functions in kernels. Complete proofs are given in this part, main tools are based on Mellin transform properties and usage of Fox H-functions.

Keywords Buschman–Erdélyi operators · Multidimensional Buschman–Erdélyi operators · Transmutations · Mellin transform · Fox H-function

MSC 35A22

1 Buschman–Erdélyi Operators

For a given pair of operators (A, B) an operator T is called transmutation (or intertwining) operator if on elements of some functional spaces the following property is valid

$$T\,A = B\,T. \tag{1}$$

And how the transmutations usually works? Suppose we study properties for a rather complicated operator A. But suppose also that we know the corresponding properties for a model more simple operator B and transmutation (1) readily exists.

S. M. Sitnik
Belgorod State National Research University (BelGU), Belgorod, Russia
e-mail: sitnik@bsu.edu.ru

O. V. Skoromnik (✉)
Polotsk State University "PSU", Novopolotsk, Belarus
e-mail: post@psu.by

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Then we usually may copy results for the model operator B to corresponding ones for the more complicated operator A. This is shortly the main idea of transmutations.

Let us consider for example an equation $Au = f$, then applying to it a transmutation with property (1) we consider a new equation $Bv = g$, with $v = Tu$, $g = Tf$. So if we can solve the simpler equation $Bv = g$, then the initial one is also solved and has solution $u = T^{-1}v$. Of course, it is supposed that the inverse operator exists and its explicit form is known. This is a simple application of the transmutation technique for finding and proving formulas for solutions of ordinary and partial differential equations.

The monographs [2, 6–8, 17, 23, 57, 59] are completely devoted to the transmutation theory and its applications, note also author's survey [50]. Moreover, essential parts of monographs [9, 12, 24, 30–32, 34–39, 45, 60], include material on transmutations, the complete list of books which investigate some transmutational problems is now near of 100 items.

The term "Buschman–Erdélyi transmutations" was introduced by the author and is now accepted. Integral equations with these operators were studied in mid-1950th. The author was first to prove the transmutational nature of these operators. The classical Sonine and Poisson operators are special cases of the Buschman–Erdélyi transmutations and Sonine–Dimovski and Poisson–Dimovski transmutations are their generalizations for the hyper-Bessel equations and functions.

The Buschman–Erdélyi transmutations have many modifications. The author introduced convenient classification of them. Due to this classification we introduce Buschman–Erdélyi transmutations of the first kind, their kernels are expressed in terms of Legendre functions of the first kind. In the limiting case we define Buschman–Erdélyi transmutations of zero order smoothness being important in applications. The kernels of Buschman–Erdélyi transmutations of the second kind are expressed in terms of Legendre functions of the second kind. Some combination of operators of the first kind and the second kind leads to operators of the third kind. For the special choice of parameters they are unitary operators in the standard Lebesgue space. The author proposed the terms "Sonine–Katrakhov" and "Poisson–Katrakhov" transmutations in honor of V. Katrakhov who introduced and studied these operators.

The study of integral equations and invertibility for the Buschman–Erdélyi operators was started in 1960-th by P. Buschman and A. Erdélyi, [4, 5, 14, 15]. These operators also were investigated by Higgins, Ta Li, Love, Habibullah, K. N. Srivastava, Ding Hoang An, Smirnov, Virchenko, Fedotova, Kilbas, Skoromnik and others. During this period, for this class of operators were considered only problems of solving integral equations, factorization and invertibility, cf. [44].

The most detailed study of the Buschman–Erdélyi transmutations was taken by the author in 1980–1990th [20, 46, 47] and continued in [19–22, 46–49, 51–56] and some other papers. Interesting and important results were proved by N. Virchenko and A. Kilbas and their disciples [26, 27, 61].

Let us first consider the most well-known transmutations for the Bessel operator and the second derivative:

$$T\,(B_\nu)\, f = \left(D^2\right) T f,\;\; B_\nu = D^2 + \frac{2\nu+1}{x} D,\;\; D^2 = \frac{d^2}{dx^2},\quad \nu \in \mathbb{C}. \tag{2}$$

Definition 1 The Poisson transmutation is defined by

$$P_\nu f = \frac{1}{\Gamma(\nu+1)2^\nu x^{2\nu}} \int_0^x \left(x^2 - t^2\right)^{\nu-\frac{1}{2}} f(t)\, dt,\;\; \Re\nu > -\frac{1}{2}. \tag{3}$$

Respectively, the Sonine transmutation is defined by

$$S_\nu f = \frac{2^{\nu+\frac{1}{2}}}{\Gamma(\frac{1}{2}-\nu)} \frac{d}{dx} \int_0^x \left(x^2 - t^2\right)^{-\nu-\frac{1}{2}} t^{2\nu+1} f(t)\, dt,\;\; \Re\nu < \frac{1}{2}. \tag{4}$$

The operators (3)–(4) intertwine by the formulas

$$S_\nu B_\nu = D^2 S_\nu,\quad P_\nu D^2 = B_\nu P_\nu. \tag{5}$$

The definition may be extended to $\nu \in \mathbb{C}$. We will use more historically exact term as the Sonine–Poisson–Delsarte transmutations [50].

An important generalization for the Sonine–Poisson–Delsarte are the transmutations for the hyper-Bessel operators and functions. Such functions were first considered by Kummer and Delerue. The detailed study on these operators and hyper-Bessel functions was done by Dimovski and further, by Kiryakova. The corresponding transmutations have been called by Kiryakova [31] as the Sonine–Dimovski and Poisson–Dimovski transmutations. In hyper-Bessel operators theory the leading role is for the Obrechkoff integral transform [10, 11, 13, 31]. It is a transform with Meijer's G-function kernel which generalizes the Laplace, Meijer and many other integral transforms introduced by different authors. Various results on the hyper-Bessel functions, connected equations and transmutations were many times reopened. The same is true for the Obrechkoff integral transform. It my opinion, the Obrechkoff transform together with the Laplace, Fourier, Mellin, Stankovic transforms are essential basic elements from which many other transforms are constructed with corresponding applications.

Let us define and study some main properties of the Buschman–Erdélyi transmutations of the first kind. This class of transmutations for some choice of parameters generalize the Sonine–Poisson–Delsart transmutations, Riemann–Liouville and Erdélyi–Kober fractional integrals, Mehler–Fock transform.

Definition 2 Define the Buschman–Erdélyi operators of the first kind by

$$B_{0+}^{\nu,\mu} f = \int_0^x \left(x^2 - t^2\right)^{-\frac{\mu}{2}} P_\nu^\mu \left(\frac{x}{t}\right) f(t) dt, \quad (6)$$

$$E_{0+}^{\nu,\mu} f = \int_0^x \left(x^2 - t^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu \left(\frac{t}{x}\right) f(t) dt, \quad (7)$$

$$B_{-}^{\nu,\mu} f = \int_x^\infty \left(t^2 - x^2\right)^{-\frac{\mu}{2}} P_\nu^\mu \left(\frac{t}{x}\right) f(t) dt, \quad (8)$$

$$E_{-}^{\nu,\mu} f = \int_x^\infty \left(t^2 - x^2\right)^{-\frac{\mu}{2}} \mathbb{P}_\nu^\mu \left(\frac{x}{t}\right) f(t) dt. \quad (9)$$

Here $P_\nu^\mu(z)$ is the Legendre function of the first kind, $\mathbb{P}_\nu^\mu(z)$ is this function on the cut $-1 \le t \le 1$ ([1]), $f(x)$ is a locally summable function with some growth conditions at $x \to 0$, $x \to \infty$. The parameters are $\mu, \nu \in \mathbb{C}$, $\Re\mu < 1$, $\Re\nu \ge -1/2$.

Now consider some main properties for this class of transmutations, following essentially [46, 47], and also [48, 50]. All functions further are defined on positive semiaxis. So we use notations L_2 for the functional space $L_2(0,\infty)$ and $L_{2,k}$ for power weighted space $L_{2,k}(0,\infty)$ equipped with norm

$$\int_0^\infty |f(x)|^2 x^{2k+1}\, dx, \quad (10)$$

$\mathbb{N}$ denotes the set of naturals, $\mathbb{N}_0$-positive integer, $\mathbb{Z}$-integer and $\mathbb{R}$-real numbers.

First, add to Definition 2 a case of parameter $\mu = 1$. It defines a very important class of operators.

Definition 3 Define for $\mu = 1$ the Buschman–Erdélyi operators of zero order smoothness by

$$B_{0+}^{\nu,1} f = {}_1S_{0+}^{\nu} f = \frac{d}{dx} \int_0^x P_\nu \left(\frac{x}{t}\right) f(t)\, dt, \quad (11)$$

$$E_{0+}^{\nu,1} f = {}_1P_{-}^{\nu} f = \int_0^x P_\nu \left(\frac{t}{x}\right) \frac{df(t)}{dt}\, dt, \quad (12)$$

$$B_{-}^{\nu,1} f = {}_1S_{-}^{\nu} f = \int_x^\infty P_\nu \left(\frac{t}{x}\right) (-\frac{df(t)}{dt})\, dt, \quad (13)$$

$$E_{-}^{\nu,1} f = {}_1P_{0+}^{\nu} f = (-\frac{d}{dx}) \int_x^\infty P_\nu \left(\frac{x}{t}\right) f(t)\, dt, \quad (14)$$

where $P_\nu(z) = P_\nu^0(z)$ is the Legendre function.

Theorem 1 *The next formulas hold true for factorizations of Buschman–Erdélyi transmutations for suitable functions via Riemann–Liouville fractional integrals and*

Buschman–Erdélyi operators of zero order smoothness:

$$B_{0+}^{\nu,\mu} f = I_{0+}^{1-\mu} \, {}_1S_{0+}^{\nu} f, \; B_{-}^{\nu,\mu} f = {}_1P_{-}^{\nu} \, I_{-}^{1-\mu} f, \tag{15}$$

$$E_{0+}^{\nu,\mu} f = {}_1P_{0+}^{\nu} \, I_{0+}^{1-\mu} f, \; E_{-}^{\nu,\mu} f = I_{-}^{1-\mu} \, {}_1S_{-}^{\nu} f. \tag{16}$$

These formulas allow to separate parameters ν and μ. We will prove soon that operators (11)–(14) are isomorphisms of $L_2(0,\infty)$ except for some special parameters. So, operators (6)–(9) roughly speaking are of the same smoothness in L_2 as integrodifferentiations $I^{1-\mu}$ and they coincide with them for $\nu = 0$. It is also possible to define Buschman–Erdélyi operators for all $\mu \in \mathbb{C}$.

Definition 4 Define the number $\rho = 1 - Re\,\mu$ as smoothness order for Buschman–Erdélyi operators (6)–(9).

So for $\rho > 0$ (otherwise for $Re\,\mu > 1$) the Buschman–Erdélyi operators are smoothing and for $\rho < 0$ (otherwise for $Re\,\mu < 1$) they decrease smoothness in L_2 spaces. Operators (11)–(14) for which $\rho = 0$ due to Definition 4 are of zero smoothness order in accordance with their definition.

For some special parameters $\nu, \ \mu$ the Buschman–Erdélyi operators of the first kind are reduced to other known operators. So for $\mu = -\nu$ or $\mu = \nu + 2$ they reduce to Erdélyi–Kober operators, for $\nu = 0$ they reduce to fractional integrodifferentiation $I_{0+}^{1-\mu}$ or $I_{-}^{1-\mu}$, for $\nu = -\frac{1}{2}$, $\mu = 0$ or $\mu = 1$ kernels reduce to elliptic integrals, for $\mu = 0$, $x = 1$, $\nu = it - \frac{1}{2}$ the operator $B_{-}^{\nu,0}$ differs only by a constant from Mehler–Fock transform.

As a pair for the Bessel operator consider a connected one

$$L_\nu = D^2 - \frac{\nu(\nu+1)}{x^2} = \left(\frac{d}{dx} - \frac{\nu}{x}\right)\left(\frac{d}{dx} + \frac{\nu}{x}\right), \tag{17}$$

which for $\nu \in \mathbb{N}$ is an angular momentum operator from quantum physics. Their transmutational relations are established in the next theorem.

Theorem 2 *For a given pair of transmutations* X_ν, Y_ν

$$X_\nu L_\nu = D^2 X_\nu, \, Y_\nu D^2 = L_\nu Y_\nu \tag{18}$$

define the new pair of transmutations by formulas

$$S_\nu = X_{\nu-1/2} x^{\nu+1/2}, \, P_\nu = x^{-(\nu+1/2)} Y_{\nu-1/2}. \tag{19}$$

Then for the new pair S_ν, P_ν *the next formulas are valid:*

$$S_\nu B_\nu = D^2 S_\nu, \, P_\nu D^2 = B_\nu P_\nu. \tag{20}$$

Theorem 3 *Let Re* $\mu \le 1$. *Then an operator* $B_{0+}^{\nu,\mu}$ *on proper functions is a Sonine type transmutation and* (18) *is valid.*

The same result holds true for other Buschman–Erdélyi operators, $E_{-}^{\nu,\mu}$ is Sonine type and $E_{0+}^{\nu,\mu}$, $B_{-}^{\nu,\mu}$ are Poisson type transmutations.

From these transmutation connections, we conclude that the Buschman–Erdélyi operators link the corresponding eigenfunctions for the two operators. They lead to formulas for the Bessel functions via exponents and trigonometric functions, and vice versa which generalize the classical Sonine and Poisson formulas.

Now consider factorizations of the Buschman–Erdélyi operators. First let us list the main forms of fractional integrodifferentiations: Riemann–Liouville, Erdélyi–Kober, fractional integral by function $g(x)$, cf. [44],

$$I_{0+,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \tag{21}$$

$$I_{-,x}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t-x)^{\alpha-1} f(t)dt,$$

$$I_{0+,2,\eta}^{\alpha} f = \frac{2x^{-2(\alpha+\eta)}}{\Gamma(\alpha)} \int_0^x \left(x^2-t^2\right)^{\alpha-1} t^{2\eta+1} f(t)dt, \tag{22}$$

$$I_{-,2,\eta}^{\alpha} f = \frac{2x^{2\eta}}{\Gamma(\alpha)} \int_x^{\infty} \left(t^2-x^2\right)^{\alpha-1} t^{1-2(\alpha+\eta)} f(t)dt,$$

$$I_{0+,g}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_0^x (g(x)-g(t))^{\alpha-1} g'(t) f(t)dt, \tag{23}$$

$$I_{-,g}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (g(t)-g(x))^{\alpha-1} g'(t) f(t)dt.$$

In all cases $\Re\alpha > 0$ and the operators may be further defined for all α, see [44]. In the case of $g(x) = x$ (23) reduces to the Riemann–Liouville integral, in the case of $g(x) = x^2$ (23) reduces to the Erdélyi–Kober operator, and in the case of $g(x) = \ln x$—to the Hadamard fractional integrals.

Theorem 4 *The following factorization formulas are valid for the Buschman–Erdélyi operators of the first kind via the Riemann–Liouville and Erdélyi–Kober fractional integrals:*

$$B_{0+}^{\nu,\mu} = I_{0+}^{\nu+1-\mu} I_{0+;\,2,\,\nu+\frac{1}{2}}^{-(\nu+1)} \left(\frac{2}{x}\right)^{\nu+1}, \tag{24}$$

$$E_{0+}^{\nu,\mu} = \left(\frac{x}{2}\right)^{\nu+1} I_{0+;\,2,\,-\frac{1}{2}}^{\nu+1} I_{0+}^{-(\nu+\mu)}, \tag{25}$$

$$B_{-}^{\nu,\,\mu} = \left(\frac{2}{x}\right)^{\nu+1} I_{-;\,2,\,\nu+1}^{-(\nu+1)} I_{-}^{\nu-\mu+2}, \tag{26}$$

$$E_{-}^{\nu,\,\mu} = I_{-}^{-(\nu+\mu)} I_{-;\,2,\,0}^{\nu+1} \left(\frac{x}{2}\right)^{\nu+1}. \tag{27}$$

The Sonine–Poisson–Delsarte transmutations also are special cases for this class of operators.

Now let us study the properties of the Buschman–Erdélyi operators of zero order smoothness, defined by (11)–(14). A similar operator was introduced by Katrakhov by multiplying the Sonine operator with a fractional integral, his aim was to work with transmutation obeying good estimates in $L_2(0,\infty)$.

We use the Mellin transform defined by [40]

$$g(s) = Mf(s) = \int_0^\infty x^{s-1} f(x)\,dx. \tag{28}$$

The Mellin convolution is defined by

$$(f_1 * f_2)(x) = \int_0^\infty f_1\left(\frac{x}{y}\right) f_2(y)\,\frac{dy}{y}, \tag{29}$$

so the convolution operator with kernel K acts under the Mellin transform as a multiplication on multiplicator

$$M[Af](s) = M\,[\int_0^\infty K\left(\frac{x}{y}\right) f(y)\,\frac{dy}{y}](s) = M[K * f](s) = m_A(s) Mf(s), \tag{30}$$

$$m_A(s) = M[K](s).$$

We observe that the Mellin transform is a generalized Fourier transform on semiaxis with Haar measure $\frac{dy}{y}$, [18]. It plays important role for the theory of special functions, for example the gamma function is a Mellin transform of the exponential. With the Mellin transform the important breakthrough in evaluating integrals was done in 1970th when mainly by O. Marichev, the famous Slater's theorem was adapted for calculations. The Slater's theorem taking the Mellin transform as input gives the function itself as output via hypergeometric functions, see [40]. This theorem occurred to be the milestone of powerful computer method for calculating integrals for many problems in differential and integral equations. The package *Mathematica* of Wolfram Research is based on this theorem in calculating integrals.

Theorem 5 *The Buschman–Erdélyi operator of zero order smoothness* ${}_1S^{\nu}_{0+}$ *defined by (11) acts under the Mellin transform as convolution (30) with multiplicator*

$$m(s) = \frac{\Gamma(-s/2 + \frac{\nu}{2} + 1)\Gamma(-s/2 - \frac{\nu}{2} + 1/2)}{\Gamma(1/2 - \frac{s}{2})\Gamma(1 - \frac{s}{2})} \tag{31}$$

for $\Re s < \min(2 + \Re\nu, 1 - \Re\nu)$. *Its norm is a periodic in* ν *and equals*

$$\|B^{\nu,1}_{0+}\|_{L_2} = \frac{1}{\min(1, \sqrt{1 - \sin\pi\nu})}. \tag{32}$$

This operator is bounded in $L_2(0, \infty)$ *if* $\nu \neq 2k + 1/2, k \in \mathbb{Z}$ *and unbounded if* $\nu = 2k + 1/2, k \in \mathbb{Z}$.

Corollary 1 *The norms of operators* (11)–(14) *are periodic in* ν *with period 2* $\|X^{\nu}\| = \|X^{\nu+2}\|$, X^{ν} *is any of operators* (11)–(14).

Corollary 2 *The norms of the operators* ${}_1S^{\nu}_{0+}$, ${}_1P^{\nu}_{-}$ *are not bounded in general, every norm is greater or equals to 1. The norms are equal to 1 if* $\sin\pi\nu \leq 0$. *The operators* ${}_1S^{\nu}_{0+}$, ${}_1P^{\nu}_{-}$ *are unbounded in* L_2 *if and only if* $\sin\pi\nu = 1$ *(or* $\nu = (2k) + 1/2,\ k \in \mathbb{Z}$*).*

Corollary 3 *The norms of the operators* ${}_1P^{\nu}_{0+}$, ${}_1S^{\nu}_{-}$ *are all bounded in* ν, *every norm is not greater then* $\sqrt{2}$. *The norms are equal to 1 if* $\sin\pi\nu \geq 0$. *The operators* ${}_1P^{\nu}_{0+}$, ${}_1S^{\nu}_{-}$ *are bounded in* L_2 *for all* ν. *The maximum of norm equals* $\sqrt{2}$ *is achieved if and only if* $\sin\pi\nu = -1$ *(или* $\nu = -1/2 + (2k),\ k \in \mathbb{Z}$*).*

The most important property of the Buschman–Erdélyi operators of zero order smoothness is the unitarity for integer ν. It is just the case if we interpret for these parameters the operator L_{ν} as angular momentum operator in quantum mechanics.

Theorem 6 *The operators* (11)–(14) *are unitary in* L_2 *if and only if the parameter* ν *is an integer. In this case the pairs of operators* $({}_1S^{\nu}_{0+}, {}_1P^{\nu}_{-})$ *and* $({}_1S^{\nu}_{-}, {}_1P^{\nu}_{0+})$ *are mutually inverse.*

To formulate an interesting special case, let us suppose that operators (11)–(14) act on functions permitting outer or inner differentiation in integrals, it is enough to suppose that $xf(x) \to 0$ for $x \to 0$. Then for $\nu = 1$

$${}_1P^{1}_{0+}f = (I - H_1)f,\ {}_1S^{1}_{-}f = (I - H_2)f, \tag{33}$$

and $H_1,\ H_2$ are the famous Hardy operators,

$$H_1 f = \frac{1}{x}\int_0^x f(y)dy,\ H_2 f = \int_x^\infty \frac{f(y)}{y}dy, \tag{34}$$

I is the identic operator.

Corollary 4 *The operators (33) are unitary in L_2 and mutually inverse. They are transmutations for the pair of differential operators d^2/dx^2 and $d^2/dx^2 - 2/x^2$.*

The unitarity of the shifted Hardy operators (33) in L_2 is a known fact [33]. Below in application section, we introduce a new class of generalizations for the classical Hardy operators.

Now we list some properties of the operators acting as convolutions by the formula (30) and with some multiplicator under the Mellin transform and being transmutations for the second derivative and angular momentum operator in quantum mechanics.

Theorem 7 *Let an operator S_ν act by formulas (30) and (18). Then:*

(a) its multiplicator satisfies a functional equation

$$m(s) = m(s-2)\frac{(s-1)(s-2)}{(s-1)(s-2)-\nu(\nu+1)}; \tag{35}$$

(b) if any function $p(s)$ is periodic with period 2 ($p(s) = p(s-2)$), then a function $p(s)m(s)$ is a multiplicator for a new transmutation operator S_2^ν also acting by the rule (18).

This theorem confirms the importance of studying transmutations in terms of the Mellin transform and multiplicator functions.

Define the Stieltjes transform by (cf. [44])

$$(Sf)(x) = \int_0^\infty \frac{f(t)}{x+t}dt.$$

This operator also acts by the formula (30) with multiplicator $p(s) = \pi/\sin(\pi s)$, it is bounded in L_2. Obviously $p(s) = p(s-2)$. So from Theorem 7 it follows a convolution of the Stieltjes transform with bounded transmutations (11)–(14), also transmutations of the same class bounded in L_2.

In this way many new classes of transmutations were introduced with special functions as kernels.

Now we construct transmutations which are unitary for all ν. They are defined by formulas

$$S_U^\nu f = -\sin\frac{\pi\nu}{2}\,{}_2S^\nu f + \cos\frac{\pi\nu}{2}\,{}_1S_-^\nu f, \tag{36}$$

$$P_U^\nu f = -\sin\frac{\pi\nu}{2}\,{}_2P^\nu f + \cos\frac{\pi\nu}{2}\,{}_1P_-^\nu f. \tag{37}$$

For all values $\nu \in \mathbb{R}$ they are linear combinations of Buschman–Erdélyi transmutations of the first and second kinds of zero order smoothness. Also they are in the defined below class of Buschman–Erdélyi transmutations of the third kind. The following integral representations are valid:

$$S_U^\nu f = \cos\frac{\pi\nu}{2}\left(-\frac{d}{dx}\right)\int_x^\infty P_\nu\left(\frac{x}{y}\right) f(y)\,dy \tag{38}$$

$$+\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(\int_0^x (x^2-y^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{x}{y}\right) f(y)\,dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{x}{y}\right) f(y)\,dy\right),$$

$$P_U^\nu f = \cos\frac{\pi\nu}{2}\int_0^x P_\nu\left(\frac{y}{x}\right)\left(\frac{d}{dy}\right) f(y)\,dy \tag{39}$$

$$-\frac{2}{\pi}\sin\frac{\pi\nu}{2}\left(-\int_0^x (x^2-y^2)^{-\frac{1}{2}} \mathbb{Q}_\nu^1\left(\frac{y}{x}\right) f(y)\,dy - \int_x^\infty (y^2-x^2)^{-\frac{1}{2}} Q_\nu^1\left(\frac{y}{x}\right) f(y)\,dy\right).$$

Theorem 8 *The operators* (36)–(37)*,* (38)–(39) *for all $\nu \in \mathbb{R}$ are unitary, mutually inverse and conjugate in L_2. They are transmutations acting by* (17)*. S_U^ν is a Sonine type transmutation and P_U^ν is a Poisson type one.*

Transmutations like (38)–(39) but with kernels in more complicated form with hypergeometric functions were first introduced by Katrakhov in 1980. Due to this, the author proposed terms for this class of operators as Sonine–Katrakhov and Poisson–Katrakhov. In author's papers these operators were reduced to more simple form of Buschman–Erdélyi ones. It made possible to include this class of operators in general composition (or factorization) method [20, 21, 49].

2 Multi-Dimensional Integral Transforms of Buschman–Erdélyi Type with Legendre Functions in Kernels

In this part we consider generalisations of Buschman–Erdélyi operators for multi-dimensional case.

First introduce integral transforms:

$$\left(\mathrm{H}^{1}_{\sigma,\kappa} f\right)(\mathbf{x}) = \mathbf{x}^{\sigma} \int\limits_{\mathbf{0}}^{\mathbf{x}} \mathrm{H}^{\mathbf{m},\,\mathbf{n}}_{\mathbf{p},\,\mathbf{q}}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,p} \\ (\mathbf{b}_j, \beta_j)_{1,q} \end{matrix}\right] \mathbf{t}^{\kappa} f(\mathbf{t}) \frac{\mathbf{dt}}{\mathbf{t}} (\mathbf{x} > 0); \tag{40}$$

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{x}) = \int\limits_{\mathbf{0}}^{\mathbf{x}} \left(\mathbf{x}^2 - \mathbf{t}^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{x}}{\mathbf{t}}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0); \tag{41}$$

$$\left(\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{x}) = \int\limits_{\mathbf{0}}^{\mathbf{x}} \left(\mathbf{x}^2 - \mathbf{t}^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{t}}{\mathbf{x}}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0); \tag{42}$$

here (see [[43], Section 28.4]) $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathrm{R}^n$; $\mathbf{t} = (t_1, t_2, \ldots, t_n) \in \mathrm{R}^n$, R^n Euclidean n-space; $\mathbf{x} \cdot \mathbf{t} = \sum\limits_{n=1}^{n} x_n t_n$ denotes their scalar product; in particular, $\mathbf{x} \cdot \mathbf{1} = \sum\limits_{n=1}^{n} x_n$ for $\mathbf{1}$= (1,...,1). The expression $\mathbf{x} > \mathbf{t}$ means that $x_1 > t_1, \ldots, x_n > t_n$, the nonstrict inequality $\geq$ has similar meaning; $\int\limits_{\mathbf{0}}^{\mathbf{x}} = \int\limits_{\mathbf{0}}^{x_1} \int\limits_{\mathbf{0}}^{x_2} \cdots \int\limits_{\mathbf{0}}^{x_n}$; by $\mathrm{N} = \{1, 2, \ldots\}$ we denote the set of positive integers, $\mathrm{N}_0 = \mathrm{N} \bigcup \{0\}$,$\mathrm{N}^n_0 = \mathrm{N}_0 \times \mathrm{N}_0 \times \ldots \times \mathrm{N}_0$, $\mathrm{R}^n_+ = \{\mathbf{x} \in \mathrm{R}^n,\ \mathbf{x} > 0\}$;

$\mathbf{m} = (m_1, m_2, \ldots, m_n) \in \mathrm{N}^n_0$ and $m_1 = m_2 = \ldots = m_n$; $\mathbf{n} = (\overline{n}_1, \overline{n}_2, \ldots, \overline{n}_n) \in \mathrm{N}^n_0$ and $\overline{n}_1 = \overline{n}_2 = \ldots = \overline{n}_n$; $\mathbf{p} = (p_1, p_2, \ldots, p_n) \in \mathrm{N}_0$ and $p_1 = p_2 = \ldots = p_n$; $\mathbf{q} = (q_1, q_2, \ldots, q_n) \in \mathrm{N}_0$ and $q_1 = q_2 = \ldots = q_n$) $(0 \leq \mathbf{m} \leq \mathbf{q}, 0 \leq \mathbf{n} \leq \mathbf{p})$;

$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathrm{C}^n$; $\kappa = (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \mathrm{C}^n$;

$\delta = (\delta_1, \delta_2, \ldots, \delta_n) \in \mathrm{R}^n$; $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n) \in \mathrm{R}^n$; $0 < \gamma < 1$;

$\mathbf{a}_i = (a_{i_1}, a_{i_2}, \ldots, a_{i_n})$, $1 \leq i \leq p$, $a_{i_1}, a_{i_2}, \ldots, a_{i_n} \in \mathrm{C}$ $(1 \leq i_1 \leq p_1, \ldots, 1 \leq i_n \leq p_n)$;

$\mathbf{b}_j = (b_{j_1}, b_{j_2}, \ldots, b_{j_n})$, $1 \leq j \leq q$, $b_{j_1}, b_{j_2}, \ldots, b_{j_n} \in \mathrm{C}$ $(1 \leq j_1 \leq q_1, \ldots, 1 \leq j_n \leq q_n)$;

$\alpha_i = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n})$, $1 \leq i \leq p$, $\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n} \in \mathrm{R}^+_1$ $(1 \leq i_1 \leq p_1, \ldots, 1 \leq i_n \leq p_n)$;

$\beta_j = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_n})$, $1 \leq j \leq q$, $\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_n} \in \mathrm{R}^+_1$ $(1 \leq j_1 \leq q_1, \ldots, 1 \leq j_n \leq q_n)$;

$\mathbf{k}=(k_1,k_2,\ldots,k_n)\in N_0^n=N_0\times\ldots\times N_0$ $(k_i\in N_0,\ i=1,2,\ldots,n)$ is a multi-index with $\mathbf{k}!=k_1!\cdots k_n!$ and $|\mathbf{k}|=k_1+k_2+\ldots+k_n$; for $l=(l_1,l_2,\ldots,l_n)\in R_+^n$
$\mathbf{D}^l=\frac{\partial^{|l|}}{(\partial x_1)^{l_1}\cdots(\partial x_n)^{l_n}}$, $\mathbf{dt}=dt_1\cdot dt_2\cdots dt_n$; $\mathbf{t}^l=t^{l_1}\cdots t^{l_n}$;
$\mathbf{x}^2-\mathbf{t}^2=(x_1^2-t_1^2)\cdots(x_n^2-t_n^2)$; $f(\mathbf{t})=f(t_1,t_2,\ldots,t_n)$; we introduce the function

$$\mathrm{H}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\,\mathbf{n}}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle|\begin{matrix}(\mathbf{a}_i,\alpha_i)_{1,p}\\(\mathbf{b}_j,\beta_j)_{1,q}\end{matrix}\right]=\prod_{k=1}^{n}\mathrm{H}_{p_k,\,q_k}^{m_k,\,\overline{n}_k}\left[\frac{x_k}{t_k}\middle|\begin{matrix}(a_{i_k},\alpha_{i_k})_{1,p_k}\\(b_{j_k},\beta_{j_k})_{1,q_k}\end{matrix}\right],\tag{43}$$

which is the product of the H-functions $\mathrm{H}_{p,\,q}^{m,\,n}[z]$. Such a function is defined by

$$\mathrm{H}_{p,\,q}^{m,n}[z]\equiv\mathrm{H}_{p,q}^{m,n}\left[z\middle|\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\right]=\frac{1}{2\pi i}\int\limits_L\mathcal{H}_{p,q}^{m,n}(s)z^{-s}ds,\ \ z\neq 0,\tag{44}$$

where

$$\mathcal{H}_{p,\,q}^{m,n}(s)\equiv\mathcal{H}_{p,\,q}^{m,n}\left[\begin{matrix}(a_i,\alpha_i)_{1,p}\\(b_j,\beta_j)_{1,q}\end{matrix}\middle|s\right]=\frac{\prod\limits_{j=1}^{m}\Gamma(b_j+\beta_j s)\prod\limits_{i=1}^{n}\Gamma(1-a_i-\alpha_i s)}{\prod\limits_{i=n+1}^{p}\Gamma(a_i+\alpha_i s)\prod\limits_{j=m+1}^{q}\Gamma(1-b_j-\beta_j s)}.\tag{45}$$

Here L—is a specially chosen infinite contour and empty product, if it occurs, being taken to be one. Note that most of the elementary and special functions are special cases of the H-function (44), and one may find its properties in the books by Mathai and Saxena [41, Chapter 2], Srivastava et al. [58, Chapter 1], Prudnikov et al. [42, Section 8.3] and Kilbas and Saigo [25, Chapters 1 and 2].

We introduce the function

$$\mathrm{P}_{\delta}^{\gamma}[\mathbf{z}]=\prod_{k=1}^{n}\mathrm{P}_{\delta_k}^{\gamma_k}[z_k],\tag{46}$$

which is the product of the Legendre functions $\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(z)$ of the first kind. For complex $\overline{\gamma}$, $Re(\overline{\gamma})<1$, and $\overline{\delta}$, $z\in C$ this function is defined by

$$\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(z)=\frac{1}{\Gamma(1-\overline{\gamma})}\left(\frac{z+1}{z-1}\right)^{\frac{\overline{\gamma}}{2}}{}_2\mathrm{F}_1\left(-\overline{\delta},\ 1+\overline{\delta};\ 1-\overline{\gamma};\ \frac{1-z}{2}\right),\ \ |arg(z-1)|<\pi,\tag{47}$$

$$\mathrm{P}_{\overline{\delta}}^{\overline{\gamma}}(x)=\frac{1}{\overline{\Gamma}(1-\overline{\gamma})}\left(\frac{1+x}{1-x}\right)^{\frac{\overline{\gamma}}{2}}{}_2\mathrm{F}_1\left(-\overline{\delta},\ 1+\overline{\delta};\ 1-\overline{\gamma};\ \frac{1-x}{2}\right),\quad -1<x<1,\tag{48}$$

see ([[16], Formulas 3.2(3) and 3.4(6)], [[42], Section 11.18]), where ${}_2F_1(-\overline{\delta},\ 1+\overline{\delta};\ 1-\overline{\gamma};\ z)$—is the Gauss hypergeometric function [[16], Section 2.1].

Our paper is devoted to the study of transforms $P^{\gamma}_{\delta,k} f$ $(k = 1, 2)$ in the weighted spaces $\mathfrak{L}_{\overline{\nu},\,\overline{2}}$ summable functions $f(\mathbf{x}) = f(x_1, \ldots, x_n)$ on $R^{\mathbf{n}}_{+}$, such that:

$$\|f\|_{\overline{\nu},\overline{2}} = \{\int_{R^1_+} x_n^{\nu_n\cdot 2-1}\{\cdots\{\int_{R^1_+} x_2^{\nu_2\cdot 2-1}\times$$

$$\times [\int_{R^1_+} x_1^{\nu_1\cdot 2-1}|f(x_1,\ldots,x_n)|^2 dx_1]dx_2\}\cdots\}dx_n\}^{1/2} < \infty \qquad (49)$$

$(\overline{2} = (2, \ldots, 2), \overline{\nu} = (\nu_1, \ldots, \nu_n) \in R^n, \nu_1 = \nu_2 = \ldots = \nu_n)$.

Our investigations are based on representations of Eqs. (41) and (42) via the modified H-transform of the form (40). Mapping properties such as the boundedness the range, the representation and the inversion of the considered transforms are established.

Preliminaries

Denote by $[X, Y]$ a set of bounded linear operators acting from a Banach space X into a Banach space Y.

The n-dimensional Mellin transform $(\mathfrak{M}f)(\mathbf{x})$ of a function $f(\mathbf{x}) = f(x_1, x_2, \ldots, x_n)$, $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in R^n_+$, is defined by

$$(\mathfrak{M}f)(\mathbf{s}) = \int_0^{\infty} f(\mathbf{t})\mathbf{t}^{\mathbf{s}-1}\mathbf{dt}, \quad Re(\mathbf{s}) = \overline{\nu}, \qquad (50)$$

$\mathbf{s} = (s_1, s_2, \ldots, s_n) \in C^n$; while the inverse Mellin transform is given for $\mathbf{x} \in R^n_+$ by the formula

$$(\mathfrak{M}^{-1}g)(\mathbf{x}) = \mathfrak{M}^{-1}[g(\mathbf{p})](\mathbf{x}) = \frac{1}{(2\pi i)^n}\int_{\gamma_1-i\infty}^{\gamma_1+i\infty}\cdots\int_{\gamma_n-i\infty}^{\gamma_n+i\infty} \mathbf{x}^{-\mathbf{s}} g(\mathbf{s})\mathbf{ds}, \qquad (51)$$

with $\gamma_j = Re(s_j)$ $(j = 1, \cdots, n)$. The theory for these multidimensional Mellin transforms appears in the book by Brychkov [3], see also [29, Chapter 1].

Let $\mathbf{M}_{\zeta}$, $\mathbf{R}$ be elementary operators (see [29, Chapter 1]):

$$(\mathbf{M}_{\zeta} f)(\mathbf{x}) = \mathbf{x}^{\zeta} f(\mathbf{x}) \ (\zeta = (\zeta_1, \zeta_2, \ldots, \zeta_n) \in C^n), \quad (\mathbf{R}f)(\mathbf{x}) = \frac{1}{\mathbf{x}} f\left(\frac{1}{\mathbf{x}}\right). \qquad (52)$$

There holds the following assertion, which follows from [29] formulas (1.4.44), (1.4.45), (1.4.46)] [25, Lemma 3.2].

Lemma 1 *Let* $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in R^n$ $(\nu_1 = \nu_2 = \ldots = \nu_n)$ *and* $1 \le \overline{r} < \infty$.

(a) $\mathbf{M}_\zeta$ *is isometric isomorphism of* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *onto* $\mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\zeta),\overline{r}}$ *and if* $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$*, then*

$$(\mathfrak{M}\mathbf{M}_\zeta f)(\mathbf{s}) = (\mathfrak{M} f)(\mathbf{s} + \zeta) \ \ (\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(\zeta)). \tag{53}$$

(b) $\mathbf{R}$ *is an isometric isomorphism of* $\mathfrak{L}_{\overline{\nu},\overline{r}}$ *onto* $\mathfrak{L}_{1-\overline{\nu},\overline{r}}$ *and if* $f \in \mathfrak{L}_{\overline{\nu},\overline{r}}$ $(1 \le \overline{r} \le 2)$*, then*

$$(\mathfrak{M}\mathbf{R} f)(\mathbf{s}) = (\mathfrak{M} f)(1 - \mathbf{s}) \ \ (\mathrm{Re}(\mathbf{s}) = \overline{\nu}). \tag{54}$$

Let $\mathrm{I}^{\alpha}_{0+;\,\sigma,\eta}$ and $\mathrm{I}^{\alpha}_{-;\,\sigma,\eta}$ be the Erdelyi-Kober operators of fractional integration, defined for $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathrm{C}^n$ $(\mathrm{Re}(\alpha) > 0)$, $\sigma > 0$, $\eta \in C^n$ by:

$$\left(\mathrm{I}^{\alpha}_{0+;\,\sigma,\,\eta} f\right)(\mathbf{x}) = \frac{\sigma \mathbf{x}^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int\limits_0^{\mathbf{x}} \left(\mathbf{x}^\sigma - \mathbf{t}^\sigma\right)^{\alpha-1} \mathbf{t}^{\sigma\eta+\sigma-1} f(\mathbf{t})\mathbf{dt} \ (\mathbf{x} > 0), \tag{55}$$

$$\left(\mathrm{I}^{\alpha}_{-;\,\sigma,\,\eta} f\right)(\mathbf{x}) = \frac{\sigma \mathbf{x}^{\sigma\eta}}{\Gamma(\alpha)} \int\limits_{\mathbf{x}}^{\infty} \left(\mathbf{t}^\sigma - \mathbf{x}^\sigma\right)^{\alpha-1} \mathbf{t}^{\sigma(1-\alpha-\eta)-1} f(\mathbf{t})\mathbf{dt} \ (\mathbf{x} > 0). \tag{56}$$

2.1 $\mathfrak{L}_{\overline{\nu},\overline{2}}$–*Theory and the Inversion Formulas for the Modified* **H**-*Transform*

To formulate the results presented $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory and the inversion formulas for the modified H-transform (40) we need the following constants, analogical for one-dimensional case defined via the parameters of the H-function (44) [[25], (3.4.1), (3.4.2), (1.1.7), (1.1.8), (1.1.10)]:

$$\alpha_1 = \begin{cases} -\min\limits_{1\le j_1 \le m_1}\left[\frac{\mathrm{Re}(b_{j_1})}{\beta_{j_1}}\right], & m_1 > 0, \\ 0, & m_1 = 0; \end{cases} \qquad \beta_1 = \begin{cases} \min\limits_{1\le i_1 \le \overline{n}_1}\left[\frac{1-\mathrm{Re}(a_{i_1})}{\alpha_{i_1}}\right], & \overline{n}_1 > 0, \\ 0, & \overline{n}_1 = 0; \end{cases}$$

$$\alpha_2 = \begin{cases} -\min\limits_{1\le j_2 \le m_2}\left[\frac{\mathrm{Re}(b_{j_2})}{\beta_{j_2}}\right], & m_2 > 0, \\ 0, & m_2 = 0; \end{cases} \qquad \beta_2 = \begin{cases} \min\limits_{1\le i_2 \le \overline{n}_2}\left[\frac{1-\mathrm{Re}(a_{i_2})}{\alpha_{i_2}}\right], & \overline{n}_2 > 0, \\ 0, & \overline{n}_2 = 0; \end{cases}$$

and so on

$$\alpha_n = \begin{cases} -\min\limits_{1\le j_n \le m_n}\left[\frac{\mathrm{Re}(b_{j_n})}{\beta_{j_n}}\right], & m_n > 0, \\ 0, & m_2 = 0; \end{cases} \qquad \beta_n = \begin{cases} \min\limits_{1\le i_n \le \overline{n}_n}\left[\frac{1-\mathrm{Re}(a_{i_n})}{\alpha_{i_n}}\right], & \overline{n}_n > 0, \\ 0, & \overline{n}_n = 0; \end{cases} \tag{57}$$

$$a_1^* = \sum_{i=1}^{\overline{n}_1} \alpha_{i_1} - \sum_{i=\overline{n}_1+1}^{p_1} \alpha_{i_1} + \sum_{j=1}^{m_1} \beta_{j_1} - \sum_{j=m_1+1}^{q_1} \beta_{j_1}, \;\; \Delta_1 = \sum_{j=1}^{q_1} \beta_{j_1} - \sum_{i=1}^{p_1} \alpha_{i_1},$$

$$a_2^* = \sum_{i=1}^{\overline{n}_2} \alpha_{i_2} - \sum_{i=\overline{n}_2+1}^{p_2} \alpha_{i_2} + \sum_{j=1}^{m_2} \beta_{j_2} - \sum_{j=m_1+1}^{q_2} \beta_{j_2}, \;\; \Delta_2 = \sum_{j=1}^{q_2} \beta_{j_2} - \sum_{i=1}^{p_2} \alpha_{i_2},$$

and so on

$$a_n^* = \sum_{i=1}^{\overline{n}_n} \alpha_{i_n} - \sum_{i=\overline{n}_n+1}^{p_n} \alpha_{i_n} + \sum_{j=1}^{m_n} \beta_{j_n} - \sum_{j=m_n+1}^{q_n} \beta_{j_n}, \;\; \Delta_n = \sum_{j=1}^{q_n} \beta_{j_n} - \sum_{i=1}^{p_n} \alpha_{i_n}; \tag{58}$$

$$\mu_1 = \sum_{j=1}^{q_1} b_{j_1} - \sum_{i=1}^{p_1} a_{i_1} + \frac{p_1 - q_1}{2}, \mu_2 = \sum_{j=1}^{q_2} b_{j_2} - \sum_{i=1}^{p_2} a_{i_2} + \frac{p_2 - q_2}{2}, \ldots,$$

$$\mu_n = \sum_{j=1}^{q_n} b_{j_n} - \sum_{i=1}^{p_n} a_{i_n} + \frac{p_n - q_n}{2}; \tag{59}$$

$$\alpha_0^1 = \begin{cases} 1 + \max\limits_{m_1+1 \le j_1 \le q_1} \left[\frac{\mathrm{Re}(b_{j_1})-1}{\beta_{j_1}} \right], & q_1 > m_1, \\ \infty, & q_1 = m_1, \end{cases} \quad \beta_0^1 = \begin{cases} 1 + \min\limits_{\overline{n}_1+1 \le i_1 \le p_1} \left[\frac{\mathrm{Re}(a_{i_1})}{\alpha_{i_1}} \right], & p_1 > \overline{n}_1, \\ \infty, & p_1 = \overline{n}_1; \end{cases}$$

$$\alpha_0^2 = \begin{cases} 1 + \max\limits_{m_2+1 \le j_2 \le q_2} \left[\frac{\mathrm{Re}(b_{j_2})-1}{\beta_{j_2}} \right], & q_2 > m_2, \\ \infty, & q_2 = m_2, \end{cases} \quad \beta_0^2 = \begin{cases} 1 + \min\limits_{\overline{n}_2+1 \le i_2 \le p_2} \left[\frac{\mathrm{Re}(a_{i_2})}{\alpha_{i_2}} \right], & p_2 > \overline{n}_2, \\ \infty, & p_2 = \overline{n}_2; \end{cases} \ldots$$

$$\alpha_0^n = \begin{cases} 1 + \max\limits_{m_n+1 \le j_n \le q_n} \left[\frac{\mathrm{Re}(b_{j_n})-1}{\beta_{j_n}} \right], & q_n > m_n, \\ \infty, & q_n = m_n, \end{cases} \quad \beta_0^n = \begin{cases} 1 + \min\limits_{\overline{n}_n+1 \le i_n \le p_2} \left[\frac{\mathrm{Re}(a_{i_n})}{\alpha_{i_n}} \right], & p_n > \overline{n}_n, \\ \infty, & p_n = \overline{n}_n. \end{cases} \tag{60}$$

The exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of a function $\overline{\mathcal{H}}_{\mathbf{p},\mathbf{q}}^{\mathbf{m},\mathbf{n}}(\mathbf{s})$:

$$\overline{\mathcal{H}}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\mathbf{n}}(\mathbf{s}) \equiv \overline{\mathcal{H}}_{\mathbf{p},\,\mathbf{q}}^{\mathbf{m},\mathbf{n}} \left[\begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix} \middle| \mathbf{s} \right] = \prod_{k=1}^{n} \mathcal{H}_{p_k,\,q_k}^{m_k,\overline{n}_k} \left[\begin{matrix} (a_{i_k}, \alpha_{i_k})_{1,p_k} \\ (b_{j_k}, \beta_{j_k})_{1,q_k} \end{matrix} \middle| s \right], \tag{61}$$

is called a set of vectors $\overline{\nu} = (\nu_1, \nu_2, \ldots, \nu_n) \in R^n$ $(\nu_1 = \nu_2 = \ldots = \nu_n)$, such that $\alpha_1 < 1 - \nu_1 < \beta_1$, $\alpha_2 < 1 - \nu_2 < \beta_2, \ldots, \alpha_n < 1 - \nu_n < \beta_n$, and functions $\mathcal{H}_{p_1,\,q_1}^{m_1,\overline{n}_1}(s_1)$, $\mathcal{H}_{p_2,\,q_2}^{m_2,\overline{n}_2}(s_2),\ldots,\mathcal{H}_{p_n,\,q_n}^{m_n,\overline{n}_n}(s_n)$, have zeros on lines $Re(s_1) < 1 - \nu_1$, $Re(s_2) < 1 - \nu_2, \ldots, Re(s_n) < 1 - \nu_n$, respectively.

Applying multidimensional Mellin transform (50) to (40), taking into account the results for the one-dimensional case [25, Formulae (5.1.14)], we obtain:

$$(\mathfrak{M}\mathrm{H}^1_{\sigma,\kappa} f)(\mathbf{s}) = \overline{\mathcal{H}}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\begin{array}{c|c} (\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}} & \\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}} & \end{array}\mathbf{s}+\sigma\right](\mathfrak{M}f)(\mathbf{s}+\sigma+\kappa). \tag{62}$$

The following assertion presents the $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory of the modified H-transform (40). One dimensional case see in [25, Theorem 5.37].

Theorem 9 *Let*

$$\alpha_1 < \nu_1 - \mathrm{Re}(\kappa_1) < \beta_1, \alpha_2 < \nu_2 - \mathrm{Re}(\kappa_2) < \beta_1, \ldots, \alpha_n$$
$$< \nu_n - \mathrm{Re}(\kappa_1) < \beta_n, \nu_1 = \nu_2 = \ldots = \nu_n;$$

$$a_1^* = 0, a_2^* = 0, \ldots, \; a_n^* = 0; \; \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) \le 0,$$

$$\Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) \le 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) \le 0. \tag{63}$$

There hold the following assertions:

(a) There exists a one-to-one map $\mathrm{H}^1_{\sigma,\kappa} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\kappa+\sigma),\overline{2}}]$ *such the relation (62) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(\kappa+\sigma)$.
If $a_1^* = 0,\ a_2^* = 0, \ldots, a_n^* = 0;\ \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) = 0,\ \Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) = 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) = 0$ *and* $1 - \overline{\nu} + \mathrm{Re}(\kappa) \notin \mathcal{E}_{\overline{\mathcal{H}}}$, *then* $\mathrm{H}^1_{\sigma,\kappa}$ *maps* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ *onto* $\mathfrak{L}_{\overline{\nu}-\mathrm{Re}(\kappa+\sigma),\overline{2}}$.

(b) The transform $\mathrm{H}^1_{\sigma,\kappa}$ *does not depend on* $\overline{\nu}$ *in the sense if* $\overline{\nu}$ *and* $\widetilde{\overline{\nu}}$ *satisfy Eq. (63) and if the transforms* $\mathrm{H}^1_{\sigma,\kappa}$ *and* $\widetilde{\mathrm{H}}^1_{\sigma,\kappa}$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu},\overline{2}}$ и $\mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}}$ *by Eq. (62), then* $\mathrm{H}^1_{\sigma,\kappa} f = \widetilde{\mathrm{H}}^1_{\sigma,\kappa} f$ *for* $f \in \mathfrak{L}_{\widetilde{\overline{\nu}},\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu},\overline{2}}$.

(c) If $a_1^* = 0,\ a_2^* = 0, \ldots, a_n^* = 0;\ \Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) < 0,\ \Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) < 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) < 0$*; then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ $\mathrm{H}^1_{\sigma,\kappa} f$ *is given by Eq. (40).*

(d) Let $\overline{\lambda} = (\lambda_1, \ldots, \lambda_n) \in \mathrm{C}^n$, $\overline{h} = (h_1, \ldots, h_n) > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(\kappa))\overline{h} - 1$, *then* $\mathrm{H}^1_{\sigma,\kappa} f$ *is represented in the form*

$$\left(\mathrm{H}^1_{\sigma,\kappa} f\right)(\mathbf{x}) = \overline{h}\mathbf{x}^{\sigma+1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$
$$\times \int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{x}}{\mathbf{t}}\,\middle|\, \begin{array}{c} (-\overline{\lambda},\overline{h}),\,(\mathbf{a}_i,\alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j,\beta_j)_{1,\mathbf{q}},\ (-\overline{\lambda}-1,\overline{h}) \end{array}\right]\mathbf{t}^{\kappa-1} f(\mathbf{t})\mathbf{dt}. \tag{64}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(k))\overline{h} - 1$ *is given by*

$$\left(\mathrm{H}^1_{\sigma,\kappa} f\right)(\mathbf{x}) = -\overline{h}\mathbf{x}^{\sigma+1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times \\ \times \int_0^\infty \mathrm{H}^{\mathbf{m}+1,\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{x}}{\mathbf{t}} \middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}}, (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda}-1, \overline{h}), (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix}\right] \mathbf{t}^{\kappa-1} f(\mathbf{t})\mathbf{dx}. \tag{65}$$

(e) If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(\kappa+\sigma),\overline{2}}$, *then there holds the relation:*

$$\int_0^\infty f(\mathbf{x})\left(\mathrm{H}^1_{\sigma,\kappa} g\right)(\mathbf{x})\mathbf{dx} = \int_0^\infty \left(\mathrm{H}^2_{\sigma,\kappa} f\right)(\mathbf{x}) g(\mathbf{x}) d\mathbf{x}, \tag{66}$$

where

$$\left(\mathrm{H}^2_{\sigma,\kappa} f\right)(\mathbf{x}) = \mathbf{x}^\sigma \int_0^\infty \mathrm{H}^{\mathbf{m},\mathbf{n}}_{\mathbf{p},\mathbf{q}}\left[\frac{\mathbf{t}}{\mathbf{x}} \middle| \begin{matrix} (\mathbf{a}_i, \alpha_i)_{1,\mathbf{p}} \\ (\mathbf{b}_j, \beta_j)_{1,\mathbf{q}} \end{matrix}\right] \mathbf{t}^\kappa f(\mathbf{t}) \frac{\mathbf{dt}}{\mathbf{x}}. \tag{67}$$

Inversion formulas for the transform $\mathrm{H}^1_{\sigma,\kappa}$ *are given by the following equalities (one-dimensional case see in [[25], (5.5.23) and (5.5.24)]):*

$$f(\mathbf{x}) = -\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-\kappa} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}} \times \\ \times \int_0^\infty \mathrm{H}^{\mathbf{q}-\mathbf{m},\mathbf{p}-\mathbf{n}+1}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{t}}{\mathbf{x}} \middle| \begin{matrix} (-\overline{\lambda}, \overline{h}), (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{\mathbf{n}+1,\mathbf{p}}, (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{1,\mathbf{n}} \\ (1-\mathbf{b}_j-\beta_j, \beta_j)_{\mathbf{m}+1,\mathbf{q}}, (1-\mathbf{b}_j-\beta_j, \beta_j)_{1,\mathbf{m}}\, (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right] \\ \times \mathbf{t}^{-\sigma}(\mathrm{H}^1_{\sigma,\kappa} f)(\mathbf{t})\mathbf{dt} \tag{68}$$

or

$$f(\mathbf{x}) = \overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}} \times \\ \times \int_0^\infty \mathrm{H}^{\mathbf{q}-\mathbf{m}+1,\mathbf{p}-\mathbf{n}}_{\mathbf{p}+1,\mathbf{q}+1}\left[\frac{\mathbf{t}}{\mathbf{x}} \middle| \begin{matrix} (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{\mathbf{n}+1,\mathbf{p}}, (1-\mathbf{a}_i-\alpha_i, \alpha_i)_{1,\mathbf{n}},\ (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda}-1, \overline{h}), (1-\mathbf{b}_j-\beta_j, \beta_j)_{\mathbf{m}+1,\mathbf{q}}, (1-\mathbf{b}_j-\beta_j, \beta_j)_{1,\mathbf{m}} \end{matrix}\right] \\ \times \mathbf{t}^{-\sigma}(\mathrm{H}^1_{\sigma,\kappa} f)(\mathbf{t})\mathbf{dt}. \tag{69}$$

Condition for the validity of these formulas are given by the following assertion (one-dimensional case see in [25, Theorem 5.47]).

Theorem 10 *Let* $a_1^* = 0, a_2^* = 0, \ldots, a_n^* = 0$; $\alpha_1 < \nu_1 - \mathrm{Re}(\kappa_1) < \beta_1$, $\alpha_2 < \nu_2 - \mathrm{Re}(\kappa_2) < \beta_2, \ldots, \alpha_n < \nu_n - \mathrm{Re}(\kappa_n) < \beta_n$; $\alpha_0^1 < 1 - \nu_1 + \mathrm{Re}(\kappa_1) < \beta_0^1$, $\alpha_0^2 < 1-\nu_2+\mathrm{Re}(\kappa_2) < \beta_0^2, \ldots, \alpha_0^n < 1-\nu_n+\mathrm{Re}(\kappa_n) < \beta_0^n$; *and let* $\overline{\lambda} \in C^n$, $\overline{h} > 0$.

If $\Delta_1[\nu_1 - \mathrm{Re}(\kappa_1)] + \mathrm{Re}(\mu_1) = 0$, $\Delta_2[\nu_2 - \mathrm{Re}(\kappa_2)] + \mathrm{Re}(\mu_2) = 0, \ldots, \Delta_n[\nu_n - \mathrm{Re}(\kappa_n)] + \mathrm{Re}(\mu_n) = 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}(\nu_1, \nu_2, \ldots, \nu_n)$, *then the inversion formulas (68) and (69) are valid for* $\mathrm{Re}(\overline{\lambda}) > (1 - \overline{\nu} + \mathrm{Re}(\kappa))\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1 - \overline{\nu} + \mathrm{Re}(\kappa))\overline{h} - 1$, *respectively.*

2.2 Representations in the Form of Modified H-Transform

Introduce so-called one-sided functions

$$\mathrm{K}_1(\mathbf{x}) = (\mathbf{x}^2 - 1)_+^{-\gamma/2}\, \mathrm{P}_\gamma^\delta(\mathbf{x}) = \begin{cases} (\mathbf{x}^2 - 1)^{-\gamma/2}\mathrm{P}_\gamma^\delta(\mathbf{x}), & \mathbf{x} > 1, \\ 0, & 0 < \mathbf{x} < 1; \end{cases} \tag{70}$$

$$\mathrm{K}_2(\mathbf{x}) = (1 - \mathbf{x}^2)_+^{-\gamma/2}\, \mathrm{P}_\gamma^\delta(\mathbf{x}) = \begin{cases} (1 - \mathbf{x}^2)^{-\gamma/2}\mathrm{P}_\gamma^\delta(\mathbf{x}), & 0 < \mathbf{x} < 1, \\ 0, & \mathbf{x} > 1. \end{cases} \tag{71}$$

Using notations in (52) and (70), (71), present transforms (41) and (42) in respective forms

$$\left(\mathrm{P}_{\delta,1}^\gamma f\right)(\mathbf{x}) = \int_0^\infty \mathrm{K}_1\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{-\gamma} f\right)(\mathbf{t})d\mathbf{t}; \tag{72}$$

$$\left(\mathrm{P}_{\delta,2}^\gamma f\right)(\mathbf{x}) = \mathbf{x}^{1-\gamma} \int_0^\infty \left(\mathrm{R}\mathrm{K}_2\right)\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{-1} f\right)(\mathbf{t})d\mathbf{t}. \tag{73}$$

The following assertion yields the Mellin transform formulas (50) of $\mathrm{K}_1(\mathbf{x})$ and $\mathrm{K}_2(\mathbf{x})$ in (70) and (71).

Lemma 2 *Let* $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$, $\delta = (\delta_1, \delta_2, \ldots, \delta_n)$, $\mathbf{s} = (s_1, s_2, \ldots, s_n) \in \mathrm{C}^n$.

(a) If $\mathrm{Re}(\gamma) < 1$, $\mathrm{Re}(\mathbf{s}) < 1 + \mathrm{Re}(\gamma + \delta)$, $\mathrm{Re}(\mathbf{s}) < \mathrm{Re}(\gamma - \delta)$, *then*

$$\left(\mathfrak{M}\mathrm{K}_1\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\left(\frac{1+\gamma+\delta-\mathbf{s}}{2}\right)\Gamma\left(\frac{\gamma-\delta-\mathbf{s}}{2}\right)}{\Gamma\left(1-\frac{\mathbf{s}}{2}\right)\Gamma\left(\frac{1-\mathbf{s}}{2}\right)}. \tag{74}$$

(b) If $\mathrm{Re}(\gamma) < 1,\ \mathrm{Re}(\mathbf{s}) > 0$, *then*

$$\left(\mathfrak{M}\mathrm{K}_2\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\left(\frac{\mathbf{s}}{2}\right)\Gamma\left(\frac{\mathbf{s}+1}{2}\right)}{\Gamma\left(\frac{1-\gamma-\delta+\mathbf{s}}{2}\right)\Gamma\left(1+\frac{\delta-\gamma+\mathbf{s}}{2}\right)}. \tag{75}$$

Proof By [42, 2.172.9], under conditions in (a) there holds the formula

$$\left(\mathfrak{M}\mathrm{K}_1\right)(\mathbf{s}) = \frac{2^{\gamma_1-s_1-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma_1+\delta_1-s_1}{2}\right)\Gamma\left(\frac{\gamma_1-\delta_1-s_1}{2}\right)}{\Gamma(1-s_1)}\frac{2^{\gamma_2-s_2-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma_2+\delta_2-s_2}{2}\right)\Gamma\left(\frac{\gamma_2-\delta_2-s_2}{2}\right)}{\Gamma(1-s_2)}\cdots$$

$$\frac{2^{\gamma_1-s_n-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma_n+\delta_1-s_n}{2}\right)\Gamma\left(\frac{\gamma_n-\delta_n-s_n}{2}\right)}{\Gamma(1-s_n)} = \frac{2^{\gamma-\mathbf{s}-1}}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1+\gamma+\delta-\mathbf{s}}{2}\right)\Gamma\left(\frac{\gamma-\delta-\mathbf{s}}{2}\right)}{\Gamma(1-\mathbf{s})}. \tag{76}$$

Using the duplication formula for the gamma function

$$\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) \tag{77}$$

with $z = \frac{1-s}{2}$, from Eq. (76) we deduce Eq. (74).

If conditions in (b) are satisfied, then according to [42, 2.172].

$$\left(\mathfrak{M}\mathrm{K}_2\right)(s) = 2^{\gamma-s}\sqrt{\pi}\frac{\Gamma(s)}{\Gamma\left(\frac{1-\gamma-\delta+s}{2}\right)\Gamma\left(1+\frac{\delta-\gamma+s}{2}\right)}. \tag{78}$$

Applying Eq. (77) with $z = \frac{s}{2}$, from Eq. (78) we deduce Eq. (75). Lemma is proved. □

Applying the convolution Mellin formula [29, (1.4.56)]

$$\left(\mathfrak{M}\int_0^\infty \mathrm{K}\left(\frac{\mathbf{x}}{\mathbf{t}}\right)y(\mathbf{t})\frac{d\mathbf{t}}{\mathbf{t}}\right)(\mathbf{s}) = \left(\mathfrak{M}\mathrm{K}\right)(\mathbf{s})\left(\mathfrak{M}f\right)(\mathbf{s}),\ (\mathbf{x}\in\mathrm{R}^n_+), \tag{79}$$

being valid for suitable $\mathrm{K}\left(\frac{\mathbf{x}}{\mathbf{t}}\right) = \mathrm{K}\left(\frac{x_1}{t_1}, \frac{x_2}{t_2}, \ldots, \frac{x_n}{t_n}\right)$ and $y(\mathbf{x})$, and formulas (53) and (54) for Mellin transform of $\mathrm{M}_\zeta f$, $\mathrm{R}f$, we find the Mellin transform of Eqs. (72) and (73) for suitable f.

Applying (74), we have for $\left(\mathrm{P}^\gamma_{\delta,1}f\right)(\mathbf{x})$:

$$(\mathfrak{M}\mathrm{P}^\gamma_{\delta,1}f)(\mathbf{s}) = \left(\mathfrak{M}\int_0^\infty \mathrm{K}_1\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{1-\gamma}f\right)(\mathbf{t})\frac{d\mathbf{t}}{\mathbf{t}}\right)(\mathbf{s}) = \left(\mathfrak{M}\mathrm{K}_1\right)(\mathbf{s})\left(\mathfrak{M}\mathrm{M}_{1-\gamma}f\right)(\mathbf{s}) =$$

$$= 2^{\gamma-1}\frac{\Gamma\Big((1+\gamma+\delta-\mathbf{s})/2\Big)\Gamma\Big((\gamma-\delta-\mathbf{s})/2\Big)}{\Gamma\Big(1-\mathbf{s}/2\Big)\Gamma\Big((1-\mathbf{s})/2\Big)}\Big(\mathfrak{M}f\Big)(1-\gamma+\mathbf{s}). \tag{80}$$

In accordance with (61), relation (80) takes the form

$$\left(\mathfrak{M}\mathrm{P}_{\delta,1}^{\gamma} f\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\Big((1+\gamma+\delta-\mathbf{s})/2\Big)\Gamma\Big((\gamma-\delta-\mathbf{s})/2\Big)}{\Gamma\Big(1-\mathbf{s}/2\Big)\Gamma\Big((1-\mathbf{s})/2\Big)}\Big(\mathfrak{M}f\Big)(1-\gamma+\mathbf{s}) =$$

$$2^{\gamma-1}\overline{\mathcal{H}}_{2,2}^{0,2}\left[\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right), & \left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right), & \left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\Bigg|\mathbf{s}\right]\Big(\mathfrak{M}f\Big)(\mathbf{s}+1-\gamma). \tag{81}$$

Therefore, by (62), the initial integral transform (41) is modified H-transform (40) with $\sigma = 0,\ \kappa = 1-\gamma$:

$$\left(\mathrm{P}_{\delta,1}^{\gamma} f\right)(\mathbf{s}) = 2^{\gamma-1}\int\limits_0^\infty \mathrm{H}_{2,2}^{0,2}\left[\frac{\mathbf{x}}{\mathbf{t}}\Bigg|\begin{matrix}\left(\frac{1-\gamma-\delta}{2},\frac{1}{2}\right) & \left(1+\frac{\delta-\gamma}{2},\frac{1}{2}\right)\\ \left(0,\frac{1}{2}\right) & \left(\frac{1}{2},\frac{1}{2}\right)\end{matrix}\right]\mathbf{t}^{-\gamma} f(\mathbf{t})d\mathbf{t}. \tag{82}$$

Similarly to the above, using Eq. (75) we have for $\left(\mathrm{P}_{\delta,2}^{\gamma} f\right)(\mathbf{x})$:

$$\left(\mathfrak{M}\mathrm{P}_{\delta,2}^{\gamma} f\right)(\mathbf{s}) = \left(\mathfrak{M}\left(\mathbf{x}^{1-\gamma}\int\limits_0^\infty (\mathrm{RK}_2)\left(\frac{\mathbf{x}}{\mathbf{t}}\right)\left(\mathrm{M}_{-1} f\right)(\mathbf{t})d\mathbf{t}\right)\right)(\mathbf{s})$$

$$= \left(\mathfrak{M}\int\limits_0^\infty (\mathrm{RK}_2)\left(\frac{\mathbf{x}}{\mathbf{t}}\right) f(\mathbf{t})\frac{d\mathbf{t}}{\mathbf{t}}\right)(\mathbf{s}+1-\gamma) =$$

$$= \left(\mathfrak{M}(\mathrm{RK}_2)\right)(\mathbf{s}+1-\gamma)\left(\mathfrak{M}f\right)(\mathbf{s}+1-\gamma) = \left(\mathfrak{M}\mathrm{K}_2\right)(\gamma-\mathbf{s})\left(\mathfrak{M}f\right)(\mathbf{s}+1-\gamma) =$$

$$= 2^{\gamma-1}\frac{\Gamma\big((\gamma-\mathbf{s})/2\big)\Gamma\big((\gamma-\mathbf{s}+1)/2\big)}{\Gamma\big((1-\delta-\mathbf{s})/2\big)\Gamma\big(1+(\delta-\mathbf{s})/2\big)}\left(\mathfrak{M}f\right)(1-\gamma+\mathbf{s}). \tag{83}$$

According to Eq. (61), relation (83) takes the form:

$$\left(\mathfrak{M}\mathrm{P}_{\delta,2}^{\gamma} f\right)(\mathbf{s}) = 2^{\gamma-1}\frac{\Gamma\big((\gamma-\mathbf{s})/2\big)\Gamma\big((\gamma-\mathbf{s}+1)/2\big)}{\Gamma\big((1-\delta-\mathbf{s})/2\big)\Gamma\big(1+(\delta-\mathbf{s})/2\big)}\left(\mathfrak{M}f\right)(1-\gamma+\mathbf{s})$$

$$= 2^{\gamma-1}\overline{\mathcal{H}}_{2,2}^{0,2}\left[\begin{matrix}\left(\frac{1-\gamma}{2},\frac{1}{2}\right), & \left(1-\frac{\gamma}{2},\frac{1}{2}\right)\\ \left(\frac{1+\delta}{2},\frac{1}{2}\right), & \left(-\frac{\delta}{2},\frac{1}{2}\right)\end{matrix}\Bigg|\mathbf{s}\right]\Big(\mathfrak{M}f\Big)(\mathbf{s}+1-\gamma), \tag{84}$$

and hence, in accordance with Eq. (62), the initial transform $\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(\mathbf{x})$ is also modified H-transform (40), with $\sigma = 0$, $\kappa = 1-\gamma$:

$$\left(\mathrm{P}^{\gamma}_{\delta,2}f\right)(\mathbf{s}) = 2^{\gamma-1}\int\limits_0^{\infty} \mathrm{H}^{0,2}_{2,2}\left[\frac{\mathbf{x}}{\mathbf{t}}\middle| \begin{matrix} \left(1-\frac{\gamma}{2},\frac{1}{2}\right), & \left(\frac{1-\gamma}{2},\frac{1}{2}\right) \\ \left(\frac{1+\delta}{2},\frac{1}{2}\right), & \left(-\frac{\delta}{2},\frac{1}{2}\right) \end{matrix}\right]\mathbf{t}^{-\gamma}f(\mathbf{t})d\mathbf{t}. \tag{85}$$

$\mathfrak{L}_{\overline{\nu},\overline{2}}$-Theory of Transforms $\mathbf{P}^{\gamma}_{\delta,k}f$ $(k = 1, 2)$

$\mathfrak{L}_{\nu,\overline{2}}$-theory of transforms (41)–(42) follows from Eqs. (82) and (85) with using Theorem 9 for the $\mathrm{H}^{1}_{\sigma,\kappa}$-transform.

By Eqs. (82), (85), and (40), $a_1^* = a_2^* = \ldots = a_n^* = 0$; $\Delta_1 = \Delta_2 = \ldots = \Delta_n = 0$; $\mathbf{p} = (p_1, p_2, \ldots, p_n) = (2, 2, \ldots, 2)$; $\mathbf{q} = (q_1, q_2, \ldots, q_n) = (2, 2, \ldots, 2)$, $\alpha_i = (\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_n}) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, $\beta_j = (\beta_{j_1}, \beta_{j_2}, \ldots, \beta_{j_n}) = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$ $(i = 1, \ldots, p;\ j = 1, \ldots, q)$; $\mu = \gamma - 1$.

As for $\mathbf{m}$, $\mathbf{n}$ and other parameters in Eqs. (57) and (59), we have:

$$\mathbf{m} = 0,\ \mathbf{n} = 2,\ \alpha = -\infty,\ \beta = \min[\mathrm{Re}(1+\gamma+\delta),\ \mathrm{Re}(\gamma-\delta)]; \tag{86}$$

$$\mathbf{m} = 0,\ \mathbf{n} = 2,\ \alpha = -\infty,\ \beta = \mathrm{Re}(\gamma); \tag{87}$$

respectively for the operators (41) and (42).

According to (80), $1-\overline{\nu}$ does not belong to exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of the $\overline{\mathcal{H}}^{0,2}_{2,2}$-function in the right-hand side of (81), if:

$$\mathbf{s} \neq 2m+1,\ \mathbf{s} \neq 2l+2\ (l = (l_1, l_2, \ldots, l_n);\ m = (\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_n) \in N_0^n), \tag{88}$$

for $\mathrm{R}e(\mathbf{s}) = 1-\overline{\nu}$.

According to (83), $1-\overline{\nu}$ does not belong to exceptional set $\mathcal{E}_{\overline{\mathcal{H}}}$ of the $\overline{\mathcal{H}}^{2,0}_{2,2}$-function in the right-hand side of (84), if:

$$\mathbf{s} \neq -\delta+2m+1,\ \mathbf{s} \neq \delta+2l+2\ (l = (l_1, l_2, \ldots, l_n);\ m = (\overline{m}_1, \overline{m}_2, \ldots, \overline{m}_n) \in N_0^n), \tag{89}$$

for $\mathrm{R}e(\mathbf{s}) = 1-\overline{\nu}$.

By Eqs. (82), (85) and (86), (87), from Theorem 9 we deduce $\mathfrak{L}_{\overline{\nu},\overline{2}}$-theory of the transforms $\mathrm{P}^{\gamma}_{\delta,k}f$ $(k = 1, 2)$.

Theorem 11 *Let*

$$-\infty < \nu_1 - \mathrm{Re}(1-\gamma_1) < \min[\mathrm{Re}(1+\gamma_1+\delta_1),\ \mathrm{Re}(\gamma_1-\delta_1)],\ \mathrm{Re}(\gamma_1-1) \leq 0;$$

$$-\infty < \nu_2 - \mathrm{Re}(1-\gamma_2) < \min[\mathrm{Re}(1+\gamma_2+\delta_2),\ \mathrm{Re}(\gamma_2-\delta_2)],\ \mathrm{Re}(\gamma_2-1) \leq 0; \ldots;$$

$$-\infty < \nu_n - \mathrm{Re}(1-\gamma_n) < \min[\mathrm{Re}(1+\gamma_n+\delta_n),\ \mathrm{Re}(\gamma_n-\delta_n)],\ \mathrm{Re}(\gamma_n-1) \leq 0. \tag{90}$$

There hold the following assertions:

(a) *There exists a one-to-one map* $\mathrm{P}^{\gamma}_{\delta,1} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(1-\gamma),\overline{2}}]$ *such that the relation (81) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(1-\gamma)$. *If* $\mathrm{Re}(\gamma - 1) = 0$ *and Eq. (88) holds, then* $\mathrm{P}^{\gamma}_{\delta,1}$ *is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$.

(b) *The transform* $\mathrm{P}^{\gamma}_{\delta,1} f$ *does not depend on* $\overline{\nu}$ *in the sense if* $\overline{\nu}_1$ *and* $\overline{\nu}_2$ *satisfy Eq. (90) and if the transforms* $\mathrm{P}^{\gamma}_{\delta,1} f$ *and* $\widetilde{\mathrm{P}}^{\gamma}_{\delta,1} f$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu}_1,\overline{2}}$ *and* $\mathfrak{L}_{\overline{\nu}_2,\overline{2}}$ *by Eq. (81), then* $\mathrm{P}^{\gamma}_{\delta,1} f = \widetilde{\mathrm{P}}^{\gamma}_{\delta,1} f$ *for* $f \in \mathfrak{L}_{\overline{\nu}_1,\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu}_2,\overline{2}}$.

(c) *If* $\mathrm{Re}(\gamma - 1) < 0$, *then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ $\mathrm{P}^{\gamma}_{\delta,1} f$ *is given by Eqs. (41) and (82).*

(d) *Let* $\overline{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathrm{C}^n$, $\overline{h} = (h_1, \ldots, h_n) > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$, *then* $\mathrm{P}^{\gamma}_{\delta,1} f$ *is represented in the form*

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{x}) = 2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$

$$\times \int_0^\infty \mathrm{H}^{0,3}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}} \middle| \begin{matrix} (-\overline{\lambda}, h), \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right), & \left(1 + \frac{\delta-\gamma}{2}, \frac{1}{2}\right) \\ \left(0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}\right), & (-\overline{\lambda} - 1, \overline{h}) \end{matrix}\right] \mathbf{t}^{-\gamma} f(\mathbf{t})\mathbf{dt}, \tag{91}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$ *is given by*

$$\left(\mathrm{P}^{\gamma}_{\delta,1} f\right)(\mathbf{x}) = -2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$

$$\times \int_0^\infty \mathrm{H}^{1,2}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}} \middle| \begin{matrix} \left(\frac{1-\gamma-\delta}{2}, \frac{1}{2}\right), \quad \left(1 + \frac{\delta-\gamma}{2}, \frac{1}{2}\right), (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda} - 1, \overline{h}), \left(0, \frac{1}{2}\right), \quad \left(\frac{1}{2}, \frac{1}{2}\right) \end{matrix}\right] \mathbf{t}^{-\gamma} f(\mathbf{t})\mathbf{dt}. \tag{92}$$

(e) *If* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(1-\gamma),\overline{2}}$, *then there holds the relation:*

$$\int_0^\infty f(\mathbf{x})\left(\mathrm{P}^{\gamma}_{\delta,1} g\right)(\mathbf{x})\mathbf{dx} = \int_0^\infty 2^{\gamma-1}\left(\mathrm{P}^{*\gamma}_{\delta,2} f\right)(\mathbf{x}) g(\mathbf{x})\mathbf{dx}, \tag{93}$$

where $\left(\mathrm{P}^{*\gamma}_{\delta,2} f\right)(\mathbf{x})$ *is the transform*

$$\left(\mathrm{P}^{*\gamma}_{\delta,2} f\right)(\mathbf{x}) = \int_{\mathbf{x}}^\infty \left(\mathbf{t}^2 - \mathbf{x}^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{t}}{\mathbf{x}}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0). \tag{94}$$

Theorem 12 *Let*

$$-\infty < \nu_1 - \mathrm{Re}(1-\gamma_1) < \mathrm{Re}(\gamma_1), \ \mathrm{Re}(\gamma_1 - 1) \le 0;$$

$$-\infty < \nu_2 - \mathrm{Re}(1-\gamma_2) < \mathrm{Re}(\gamma_2), \ \mathrm{Re}(\gamma_2 - 1) \le 0; \ldots;$$

$$-\infty < \nu_n - \mathrm{Re}(1-\gamma_n) < \mathrm{Re}(\gamma_n), \ \mathrm{Re}(\gamma_n - 1) \le 0. \tag{95}$$

There hold the following assertions:

(a) *There exists a one-to-one map* $\mathrm{P}^{\gamma}_{\delta,2} \in [\mathfrak{L}_{\overline{\nu},\overline{2}}, \mathfrak{L}_{\overline{\nu}-\mathrm{Re}(1-\gamma),2}]$ *such that the relation (84) holds for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $\mathrm{Re}(\mathbf{s}) = \overline{\nu} - \mathrm{Re}(1-\gamma)$. *If* $\mathrm{Re}(\gamma - 1) = 0$ *and Eq. (89) holds, then* $\mathrm{P}^{\gamma}_{\delta,2}$ *is one-to-one on* $\mathfrak{L}_{\overline{\nu},\overline{2}}$.

(b) *The transform* $\mathrm{P}^{\gamma}_{\delta,2} f$ *does not depend on* $\overline{\nu}$ *in the sense if* $\overline{\nu}_1$ *and* $\overline{\nu}_2$ *satisfy Eq. (95) and if the transforms* $\mathrm{P}^{\gamma}_{\delta,2} f$ *and* $\widetilde{\mathrm{P}}^{\gamma}_{\delta,2} f$ *are defined in respective spaces* $\mathfrak{L}_{\overline{\nu}_1,\overline{2}}$ *and* $\mathfrak{L}_{\overline{\nu}_2,\overline{2}}$ *by Eq. (84), then* $\mathrm{P}^{\gamma}_{\delta,2} f = \widetilde{\mathrm{P}}^{\gamma}_{\delta,2} f$ *for* $f \in \mathfrak{L}_{\overline{\nu}_1,\overline{2}} \bigcap \mathfrak{L}_{\overline{\nu}_2,\overline{2}}$.

(c) *If* $\mathrm{Re}(\gamma - 1) < 0$, *then for* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ $\mathrm{P}^{\gamma}_{\delta,2} f$ *is given by Eqs. (42) and (85).*

(d) *Let* $\overline{\lambda} \in \mathrm{C}^n$, $\overline{h} > 0$, *and* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$. *If* $\mathrm{Re}(\overline{\lambda}) > (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$, *then* $\mathrm{P}^{\gamma}_{\delta,2} f$ *is represented in the form*

$$\left(\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{x}) = 2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{d}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$
$$\times \int_0^\infty \mathrm{H}^{0,3}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}} \,\middle|\, \begin{matrix} (-\overline{\lambda}, \overline{h}), \left(1-\frac{\gamma}{2}, \frac{1}{2}\right), & \left(\frac{1-\gamma}{2}, \frac{1}{2}\right) \\ \left(\frac{1+\delta}{2}, \frac{1}{2}\right), \quad \left(-\frac{\delta}{2}, \frac{1}{2}\right), & (-\overline{\lambda}-1, \overline{h}) \end{matrix}\right] \mathbf{t}^{-\gamma} f(\mathbf{t})\mathbf{dt}, \tag{96}$$

while for $\mathrm{Re}(\overline{\lambda}) < (\overline{\nu} - \mathrm{Re}(1-\gamma))\overline{h} - 1$ *is given by*

$$\left(\mathrm{P}^{\gamma}_{\delta,2} f\right)(\mathbf{x}) = -2^{\gamma-1}\overline{h}\mathbf{x}^{1-(\overline{\lambda}+1)/\overline{h}} \frac{\mathbf{x}}{\mathbf{dx}} \mathbf{x}^{(\overline{\lambda}+1)/\overline{h}} \times$$
$$\times \int_0^\infty \mathrm{H}^{1,2}_{3,3}\left[\frac{\mathbf{x}}{\mathbf{t}} \,\middle|\, \begin{matrix} \left(1-\frac{\gamma}{2}, \frac{1}{2}\right), & \left(\frac{1-\gamma}{2}, \frac{1}{2}\right), (-\overline{\lambda}, \overline{h}) \\ (-\overline{\lambda}-1, \overline{h}), \left(\frac{1+\delta}{2}, \frac{1}{2}\right), & \left(-\frac{\delta}{2}, \frac{1}{2}\right) \end{matrix}\right] \mathbf{t}^{-\gamma} f(\mathbf{t})\mathbf{dt}. \tag{97}$$

(e) *If* $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$ *and* $g \in \mathfrak{L}_{1-\overline{\nu}+\mathrm{Re}(1-\gamma),\overline{2}}$, *then there holds the relation:*

$$\int_0^\infty f(\mathbf{x})\left(\mathrm{P}^{\gamma}_{\delta,2} g\right)(\mathbf{x})\mathbf{dx} = \int_0^\infty 2^{\gamma-1}\left(\mathrm{P}^{*\gamma}_{\delta,2} f\right)(\mathbf{x}) g(\mathbf{x})\mathbf{dx}, \tag{98}$$

where $\left(\mathrm{P}^{*\gamma}_{\delta,2} f\right)$ *is given by*

$$\left(\mathrm{P}^{*\gamma}_{\delta,2} f\right)(\mathbf{x}) = \int_{\mathbf{x}}^\infty \left(\mathbf{t}^2 - \mathbf{x}^2\right)^{-\gamma/2} \mathrm{P}^{\gamma}_{\delta}\left(\frac{\mathbf{x}}{t}\right) f(\mathbf{t})\mathbf{dt} = g(\mathbf{x}) \ \ (\mathbf{x} > 0). \tag{99}$$

Inversion Formulas of Transforms $\mathrm{P}^{\gamma}_{\delta,k} f$ $(k = 1, 2)$

By substitution Eqs. (82), (85), and (40) parameters in Eq. (60) leads to

$$\alpha_0 = 0, \ \beta_0 = \infty; \tag{100}$$

$$\alpha_0 = 1 + \max[\mathrm{Re}(\delta - 1), \ \mathrm{Re}(-\delta - 2)], \ \beta_0 = \infty; \tag{101}$$

respectively for the operators (41), (42).

According to Eq. (82) the relation formulas (68) and (69) for $\mathrm{P}^{\gamma}_{\delta,1}f$ take the forms:

$$f(\mathbf{x}) = -2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1+\gamma}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int_0^\infty \mathrm{H}^{2,1}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{array}{l} -(\overline{\lambda},\overline{h}),\ \left(\frac{\gamma+\delta}{2},\frac{1}{2}\right),\ \left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right) \\ \left(\frac{1}{2},\frac{1}{2}\right),\ \left(0,\frac{1}{2}\right),\ (-\overline{\lambda}-1,\overline{h}) \end{array}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,1}f\right)(\mathbf{t})\mathbf{dt}, \tag{102}$$

or

$$f(\mathbf{x}) = 2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int_0^\infty \mathrm{H}^{3,0}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{array}{l} \left(\frac{\gamma+\delta}{2},\frac{1}{2}\right),\ \left(\frac{\gamma-\delta-1}{2},\frac{1}{2}\right),\ (-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}),\ \left(\frac{1}{2},\frac{1}{2}\right),\ \left(0,\frac{1}{2}\right) \end{array}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,1}f\right)(\mathbf{t})\mathbf{dt}. \tag{103}$$

According to Eq. (85) the relation formulas (68) and (69) for $\mathrm{P}^{\gamma}_{\delta,4}f$ take the forms:

$$f(\mathbf{x}) = -2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1+\gamma}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int_0^\infty \mathrm{H}^{2,1}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{array}{l} (-\overline{\lambda},\overline{h}),\ \left(\frac{\gamma-1}{2},\frac{1}{2}\right),\ \left(\frac{\gamma}{2},\frac{1}{2}\right) \\ \left(-\frac{\delta}{2},\frac{1}{2}\right),\ \left(\frac{\delta+1}{2},\frac{1}{2}\right),\ (-\overline{\lambda}-1,\overline{h}) \end{array}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,4}f\right)(\mathbf{t})\mathbf{dt}, \tag{104}$$

or

$$f(\mathbf{x}) = 2^{1-\gamma}\overline{h}\mathbf{x}^{(\overline{\lambda}+1)/\overline{h}-1}\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x}^{-(\overline{\lambda}+1)/\overline{h}}\times$$

$$\times\int_0^\infty \mathrm{H}^{3,0}_{3,3}\left[\frac{\mathbf{t}}{\mathbf{x}}\left|\begin{array}{l} \left(\frac{\gamma-1}{2},\frac{1}{2}\right),\ \left(\frac{\gamma}{2},\frac{1}{2}\right),\ (-\overline{\lambda},\overline{h}) \\ (-\overline{\lambda}-1,\overline{h}),\ \left(-\frac{\delta}{2},\frac{1}{2}\right),\ \left(\frac{\delta+1}{2},\frac{1}{2}\right) \end{array}\right.\right]\left(\mathrm{P}^{\gamma}_{\delta,4}f\right)(\mathbf{t})\mathbf{dt}. \tag{105}$$

Theorem 13 *Let* $\mathrm{Re}(\gamma) = 1,\ -\infty < \overline{\nu} < \min[1, \mathrm{Re}(2+\delta), \mathrm{Re}(1-\delta)]$ *and let* $\overline{\lambda} \in \mathrm{C}^n,\ \overline{h} > 0$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then the inversion formulas (102) and (103) are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, *respectively.*

Theorem 14 *Let* $\mathrm{Re}(\gamma) = 1,\ -\infty < \overline{\nu} < \min[1, \mathrm{Re}(1-\delta), \mathrm{Re}(2+\delta)]$ *and let* $\overline{\lambda} \in \mathrm{C}^n,\ \overline{h} > 0$.

If $f \in \mathfrak{L}_{\overline{\nu},\overline{2}}$, *then the inversion formulas (104) and (105) are valid for* $\mathrm{Re}(\overline{\lambda}) > (1-\overline{\nu})\overline{h} - 1$ *and* $\mathrm{Re}(\overline{\lambda}) < (1-\overline{\nu})\overline{h} - 1$, *respectively.*

In the second part of the paper we summarize the corresponding results for the one-dimensional case, obtained in [28].

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