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LINEAR CONJUGATION PROBLEM WITH A TRIANGULAR MATRIX COEFFICIENT

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Abstract. We consider a classical linear conjugation problem for analytic vector-valued functions on a piecewise smooth curve with a triangular matrix coefficient in weighted Hölder spaces. In the two-dimensional case, conditions for the existence of a solution are found, a solution of this problem is given, and the construction of the canonical matrix function is analyzed in detail.

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Let us consider the classic linear conjugation problem

$$\phi^+ - G\phi^- = g \tag{1}$$

for analytic vector functions $\phi = (\phi_1, \dots, \phi_l)$ with a triangular matrix coefficient G given on a piecewise smooth curve Γ. This curve consists of a finite number of oriented smooth arcs, which can pairwise intersect only at their ends. The boundary values of $\phi^\pm$ are considered with respect to this orientation. The endpoints of these arcs form the set F of the angular points of the curve.

For a sufficiently small $\rho > 0$, the curve $\Gamma_\tau = \Gamma \cap \{|z - \tau| \le \rho\}$ consists of several arcs $\Gamma_{\tau,j}$, $1 \le j \le n_\tau$, with the common end τ. For definiteness, the numbering of these points is chosen in the order of going around the point τ counterclockwise. With respect to the orientation of the curve Γ, the arc $\Gamma_{\tau,j}$ can either start or terminate at the point τ; therefore, we assume that $\sigma_{\tau,j} = 1$ or $\sigma_{\tau,j} = -1$, respectively. The curve Γ divides the open circle $|z - \tau| < \rho$ into curvilinear sectors $S_{\tau,j}$, $1 \le j \le n_\tau$, whose lateral sides are the arcs $\Gamma_{\tau,j}$ and $\Gamma_{\tau,j+1}$. For $n_\tau = 1$, these sides coincide, that is, the set $S_\tau = S_{\tau,1}$ is the circle with the cut along $\Gamma_\tau = \Gamma_{\tau,1}$.

We use the notation used in [1] for weighted Hölder classes. As in [1], we assume that the matrix function G is piecewise continuous and belongs to the class $C^\mu_{(+0)}(\Gamma, F)$, and its determinant $\det G$ is nonzero everywhere, including limit values

$$(\det G)(\tau, j) = \lim_{\substack{t\in\Gamma_{\tau,j}\\ t\to\tau}} (\det G)(t), \quad 1 \le j \le n_\tau,$$

at the angular points of the curve $\tau \in F$.

The problem (1) is considered in the weight class $C^\mu_\lambda(\hat{D}, F)$ of functions that are analytic in the open set $D = \mathbb{C} \setminus F$ whose components ϕ_k have finite orders at infinity satisfying the conditions

$$\deg \phi_k \le n_k - 1, \quad 1 \le k \le l, \tag{2}$$

with given integers n_k. In other words, in a neighborhood of ∞, they behave as $O(|z|^{n_k-1})$ or, equivalently, can be decomposed as follows:

$$\phi_k(z) = \sum_{s\le n_k-1} c_{j,s} z^s.$$

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The Riemann–Hilbert problem was exhaustively studied in the well-known monographs [2, 4, 6] in the class H^* of integrable functions ϕ belonging to $C^\mu_\lambda(\hat{D}, F)$ with some $\lambda > -1$ and $0 < \mu < 1$, in the class H_ε of almost bounded functions, and in the class $H(\hat{D}, F)$ of bounded functions that belong, respectively, to $C^\mu_\lambda(\hat{D}, F)$ for all $\lambda < 1$, and $C^\mu_{(+0)}(\hat{D}, F)$ with some $0 < \mu < 1$. However, various applications of this problem require the study of this problem in the space C^μ_λ for all weighted orders. For example, a similar situation occurs when considering the Riemann–Hilbert problem in simply connected domains with a piecewise smooth boundary using conformal mappings (see [3]), as well as when studying the problem of linear conjugation for polyanalytic functions.

Further, for simplicity, we restrict ourselves to the case $l = 2$ where

$$G = \begin{pmatrix} G_1 & G_0 \\ 0 & G_2 \end{pmatrix}. \tag{3}$$

Since this matrix is triangular, the problem (1) is reduced to the successive solution of two scalar conjugation problems

$$\psi^+ - G_k \psi^- = g, \quad k = 1, 2, \tag{4}$$

in the class of functions $\psi \in C^\mu_\lambda(\hat{D}, F)$ satisfying the condition (2) at infinity.

For brevity, we write the Cauchy integral in the following form:

$$(I\varphi)(z) = \frac{1}{2\pi i} \int\limits_\Gamma \frac{\varphi(t)dt}{t - z}, \quad z \notin \Gamma,$$

which for $-1 < \lambda < 0$ defines a bounded operator $I : C^\mu_\lambda(\Gamma, F) \to C^\mu_\lambda(\hat{D}, F)$. Considering the branch of the logarithm $\ln G_k$ continuous on $\Gamma \setminus F$, which along with G_k belongs to the class $C^\mu_{(+0)}(\Gamma, F)$, we introduce the function $h_k = I(\ln G_k)$, which vanishes at infinity, and its associated function

$$X_k(z) = e^{h_k(z)} \prod_\tau (z - \tau)^{-s_\tau}, \tag{5}$$

with some integer s_τ, which is canonical for the problem (4).

In the sectors $S_{\tau,j}$, the function h_k can be represented as follows:

$$h_k(z) = \frac{1}{2\pi i} \left[\sum_{s=1}^{n_\tau} \sigma_{\tau,s} (\ln G_k)(\tau, s) \right] \ln(z - \tau) + h_{k,\tau,j}(z), \quad h_{k,\tau,j} \in C^\mu_{(+0)}(S_{\tau,j}, \tau).$$

Assume that

$$\frac{1}{2\pi} \arg \prod_{j=1}^{n_\tau} \left[G_k(\tau, j) \right]^{\sigma_{\tau,j}} = \alpha_{k,\tau} + i\beta_{k,\tau}, \quad 0 \le \alpha_{k,\tau} < 1,$$

so that

$$\frac{1}{2\pi i} \left[\sum_{s=1}^{n_\tau} \sigma_{\tau,s} (\ln G_k)(\tau, s) \right] = \alpha_{k,\tau} + i\beta_{k,\tau} + s_{k,\tau}$$

with some integer $s_{k,\tau}$. Note that the sum on the left-hand side of τ coincides with the Cauchy index $\operatorname{Ind} G_k$ of the function G_k, that is, with the sum of the increments of $\ln G_k$ on arcs that form the curve $\Gamma \setminus F$, which are taken in accordance with their orientation, divided by $2\pi i$. Thus,

$$\operatorname{Ind} G_k = \sum_\tau (\alpha_{k,\tau} + i\beta_{k,\tau} + s_{k,\tau}). \tag{6}$$

Obviously, the function X_k in the sectors $S_{\tau,j}$ can be represented as follows:

$$X_k(z) = A_{k,\tau,j}(z)(z - \tau)^{\delta_{k,\tau} + i\beta_{k,\tau}}, \quad \delta_{k,\tau} = -s_\tau + s_{k,\tau} + \alpha_{k,\tau}, \tag{7}$$

where $A, 1/A \in C^\mu_{(+0)}(\hat{S}_{\tau,j}, \tau)$.

The integers s_τ in the definition (5) can be chosen arbitrarily; we choose them so that

$$\lambda \le \delta_k < \lambda + 1 \tag{8}$$

with respect to the weighted order $\delta_k = (\delta_{k,\tau}, \tau \in F)$. Then

$$[\alpha_{k,\tau} - \lambda_\tau] + s_{k,\tau} = s_\tau,$$

where $[x]$ means the integer part of x. Taking into account (6), we obtain

$$\lim_{z\to\infty} z^{\varkappa_k} X_k(z) = 1, \quad \varkappa_k = \sum_\tau \bigl[\alpha_{k,\tau} - \lambda_\tau\bigr] + \operatorname{Ind} G_k - \sum_\tau \bigl(\alpha_{k,\tau} + i\beta_{k,\tau}\bigr). \tag{9}$$

Using the canonical function, it is easy to describe (see [1]) the solvability of the problem (4). For this, we denote the class of polynomials of degree $\le \varkappa_k + n_k - 1$ (for $\varkappa_k + n_k \le 0$ by P_k, assume that $P_k = 0$), and let Q_k have a similar meaning with respect to polynomials of degree at most $-(\varkappa_k + n_k) - 1$. Thus,

$$\begin{aligned} &\dim P_k = \varkappa_k + n_k, \quad &&\dim Q_k = 0 &&\text{for } \varkappa_k + n_k \ge 0,\\ &\dim P_k = 0, &&\dim Q_k = -(\varkappa_k + n_k) &&\text{for } \varkappa_k + n_k \le 0. \end{aligned}$$

In all cases,

$$\dim P_k - \dim Q_k = \varkappa_k + n_k.$$

Theorem 1. *Under the conditions*

$$\lambda_\tau - \alpha_{k,\tau} \notin \mathbb{Z}, \quad \tau \in F,$$

the problem (4) *is solvable in the class of functions* $\psi \in C^\mu_\lambda(\hat D, F)$ *satisfying the condition* $\deg\psi \le n_k - 1$ *at infinity if and only if*

$$\bigl\langle (X^+)^{-1} g, q \bigr\rangle = 0, \quad q \in Q_k,$$

where the following notation is introduced:

$$\psi = X_k I[(X_k^+)^{-1} g] + X_k p, \quad p \in P_k.$$

If this condition is fulfilled, then the general solution is given by the formula

$$\psi = X_k I[(X_k^+)^{-1} g] + X_k p, \quad p \in P_k.$$

Note that, according to (7), the multiplication operator $g \to (X_k^+)^{-1} g$ determines an isomorphism $C^\mu_\lambda \to C^\mu_{\lambda-\delta_k}$. Due to (9), the weighted order $\lambda_k = \lambda - \delta_k$ satisfies the condition $-1 < \lambda_k < 0$, so the operator $g \to X_k I[(X_k^+)^{-1} g]$ acting from $C^\mu_\lambda(\Gamma, F)$ to $C^\mu_\lambda(\hat D, F)$ is bounded. If the condition (9) is violated for some τ, then we can only say that the function $\psi = X_k I[(X_k^+)^{-1} g]$ belongs to the class $C^\mu_{\lambda-0}$ (i.e., the class $C^\mu_{\lambda-\varepsilon}$ for any $\varepsilon > 0$).

The theorem also implies that the index of the problem is $\dim P_k - \dim Q_k = \varkappa_k + n_k$.

Let us turn to the problem (1)–(3), for which we set

$$a = G_0 X_2^- (X_1^+)^{-1}. \tag{10}$$

Due to (7), this function belongs to the class $C^\mu_{\delta_2-\delta_1}(\Gamma, F)$. According to (8), the weighted order $\delta_2 - \delta_1$ lies strictly between -1 and 1, so the function a is integrable on Γ. Moreover, in this notation, we can introduce the polynomial classes

$$P_2^0 = \{p \in P_2 \mid \langle ap, q\rangle = 0,\ q \in Q_1\}, \quad Q_1^0 = \{q \in Q_1 \mid \langle ap, q\rangle = 0,\ p \in P_2\}. \tag{11}$$

Lemma 1. *In the decompositions*

$$P_2 = P_2^0 \oplus P_2^1, \quad Q_1 = Q_1^0 \oplus Q_1^1 \tag{12}$$

the subspaces P_2^1 and Q_1^1 have a common dimension $r = \dim P_2^1 = \dim Q_1^1$. Moreover, there exists a unique linear operator R, which to any integrable function $g \in L(\Gamma)$ on Γ assign a polynomial $p = Rg \in P_2^1$ with the following property:

$$\langle ap, q_i \rangle = \langle g, q_i \rangle, \quad 1 \le i \le r, \tag{13}$$

where $q_1, \dots, q_r$ is some basis in Q_1^1.

Proof. According to the definition (11), the bilinear form $\langle ap, q\rangle$ is nondegenerate on the product $P_2^1 \times Q_1^1$ in the sense that the equations $\langle ap, q\rangle = 0$, $q \in Q_1^1$, imply $p = 0$ and, conversely, the equations $\langle ap, q\rangle = 0$, $p \in P_2^1$, imply $q = 0$. Hence the equality $\dim P_2^1 = \dim Q_1^1$ is obtained directly.

Indeed, let the elements $p_1, \dots, p_s$ and $q_1, \dots, q_r$ form bases in P_2^1 and Q_1^1, respectively. Then, by virtue of the nondegeneracy property indicated above, rows and columns of the $(s \times r)$-matrix A with elements $\langle ap_i, q_j\rangle$ are linearly independent, so this matrix is a square matrix.

Assuming that $p = \xi_1 p_1 + \dots \xi_r p_r \in P_2^1$, we can write the system (13) as follows:

$$\sum_{i=1}^{r} \xi_i \langle ap_i, q_j\rangle = \langle g, q_j\rangle, \quad j = 1, \dots, r.$$

Since the matrix A of this system is invertible, with respect to the inverse matrix $B = (B_{ij})_1^r$ we arrive at the equation

$$\xi_i = \sum_{j=1}^{r} B_{ij} \langle g, q_j\rangle,$$

so we can set

$$Rg = \sum_{1 \le i,j \le r} B_{ij} p_i \langle g, q_j\rangle.$$

The uniqueness of the operator R with the property (13) is almost obvious. Indeed, let $p \in P_2^1$ and $\langle ap, q_i\rangle = 0$, $1 \le i \le r$. Then $\langle ap, q\rangle = 0$ for all $q \in Q_1$ and, therefore, $p \in P_1^0$, which, according to the decomposition (12), is possible only for $p = 0$. □

Theorem 2. *Let the conditions* (9) *hold for both values $k = 1, 2$ and let the function $g_1, g_2 \in L(\Gamma)$ be determined by the equations*

$$2X_1^+ g_1 = 2f_1 + G_0 G_2^{-1}\big[-f_2 + X_2^+ S (X_2^+)^{-1} f g_2\big], \quad X_2^+ g_2 = f_2 \tag{14}$$

with the singular Cauchy operator S. The problem (1)–(3) *is solvable in the class $C_\lambda^\mu(\hat{D}, F)$ of vector-valued functions $\phi = (\phi_1, \phi_2)$ analytic in $D = \mathbb{C} \setminus \Gamma$ if and only if*

$$\langle g_1, q\rangle = 0, \quad q \in Q_1^0; \qquad \langle g_2, q\rangle = 0, \quad q \in Q_2. \tag{15}$$

If these conditions are fulfilled , then, in the notation of Lemma 1*, the general solution of the problem is given by the formula*

$$\phi_1 = X_1\big[I(g_1 - Rg_1 + p_2^0) + p_1\big], \quad \phi_2 = X_2(Ig_2 - Rg_1 + p_2^0), \quad p_1 \in P_1, \; p_2^0 \in P_2^0, \tag{16}$$

with the operator R from Lemma 1*.*

Proof. We write the boundary condition (3) in the component-wise form

$$\phi_1^+ - G_1\phi_1^- = f_1 + G_0\phi_2^-, \quad \phi_2^+ - G_2\phi_2^- = f_2,$$

and consecutively apply Theorem 1 to the second and first equations. Then the necessary and sufficient conditions for the solvability of the problem take the following form:

$$\langle (X_1^+)^{-1}(f_1 - G_0\phi_2^-), q\rangle = 0, \quad q \in Q_1; \qquad \langle (X_2^+)^{-1} f_2, q\rangle = 0, \quad q \in Q_2, \tag{17}$$

If these conditions are fulfilled, then the solution is given by the formulas

$$\begin{aligned} \phi_1 &= X_1 I\big[(X_1^+)^{-1}(f_1 - G_0\phi_2^-)\big] + X_1 p_1, \qquad && p_1 \in P_1, \\ \phi_2 &= X_2 I\big[(X_2^+)^{-1} f_2\big] + X_2 p_2, && p_2 \in P_2. \end{aligned} \tag{18}$$

From the last equation, according to the Sokhotski—Plemelj formula we have

$$2\phi_2^- = X_2^-\big[-(X_2^+)^{-1} f_2 + S(X_2^+)^{-1} f_2\big] + 2X_2^- p_2 = G_2^{-1}\big[-f_2 + X_2^+ S(X_2^+)^{-1} f_2\big] + 2X_2^- p_2,$$

so in the notation (10), (14) we get

$$(X_1^+)^{-1}(f_1 - G_0\phi_2^-) = g_1 + a p_2. \tag{19}$$

As a result, the first relation in (17) takes the form

$$\langle g_1 + a p_2, q\rangle = 0, \quad q \in Q_1.$$

Obviously, it is equivalent to the pair of relations $\langle g_1, q\rangle = 0$, $q \in Q_1^0$, and $\langle g_1 + a p_2, q\rangle = 0$, $q \in Q_1^1$. Assuming that $p_2 = p_2^0 + p_2^1$, $p_2^j \in P_2^j$, in the last equation we can replace p_2 by p_2^1. Due to Lemma 1, we obtain that $p_2^1 = -Rg_1$. Together with the first relation, we arrive at the solvability condition (15) for g_1. At the same time, (19) turns into

$$(X_1^+)^{-1}(f_1 - G_0\phi_2^-) = g_1 - Rg_1 + a p_2^0, \quad p_2^0 \in P_2^0.$$

Substituting this expression into (18), we obtain (16), which completes the proof of the theorem. □

Note that the number of linearly independent orthogonality conditions in (15) is equal to $\dim Q_1^0 + \dim Q_2$. On the other hand, the formula (16) shows that the space of solutions of the homogeneous problem has the dimension $\dim P_1 + \dim P_2^0$. Therefore, the index of the problem is equal to

$$\varkappa(G) = \dim P_1 + \dim P_2^0 - \dim Q_1^0 - \dim Q_2.$$

According to (12) we have

$$\dim P_2^0 = \dim P_2 - \dim P_2^1, \quad \dim Q_2^0 = \dim Q_2 - \dim Q_2^1.$$

Substituting these expressions into the previous equality and taking into account the relation $\dim P_2^1 = \dim Q_1^1$, by virtue of Lemma 1, we obtain the equation

$$\varkappa(G) = \sum_{k=1,2} (\dim P_k - \dim Q_k) = \varkappa_1 + \varkappa_2 - n_1 - n_2$$

for the index of the problem, similar to the scalar case.

The linear conjugation problem (1) in the space C_λ^μ with any weight order λ can be solved with the help of the canonical matrix-valued function $X(z)$. The problem on the existence and asymptotics at the points $\tau \in F$ was examined in [5]. However, in the case (3) with a triangular matrix, this question is solved elementarily.

We search for the canonical matrix for this coefficient in a similar form:

$$X = \begin{pmatrix} X_1 & X_0 \\ 0 & X_2 \end{pmatrix};$$

then the relation $X^+ = GX^-$ for these matrices is reduced to the equation

$$X_0^+ = G_1 X_0^- + G_0 X_2^-$$

for the unknown function X_0. This is an inhomogeneous conjugation problem; its solution can be written by the formula $X_0 = X_1 I\big[(X_1^+)^{-1}X_2^- G_0\big]$. Since $(X_1^+)^{-1}X_2^- G_0 \in C^\mu_{\delta_2-\delta_1}(\Gamma, F)$ and by virtue of (8) and (9) the weight orders δ_k satisfy the condition $-1 < \delta_2 - \delta_1 < 1$, we conclude that the function X_0 in the sectors $S_{\tau,j}$ belongs to the classes

$$X_0(z) \in \begin{cases} C^\mu_{\delta_2}(\hat S_{\tau,j}, \tau), & \delta_{2,\tau} < \delta_{1,\tau}, \\ C^\mu_{\delta_1-0}(\hat S_{\tau,j}, \tau), & \delta_{2,\tau} \geq \delta_{1,\tau}, \end{cases}$$

Consequently,

$$X_0 \in C^\mu_{\delta'-0}(\hat D, F), \quad \delta'_\tau = \min(\delta_{1,\tau}, \delta_{2,\tau}), \tag{20}$$

and

$$X^{-1} = \begin{pmatrix} X_1^{-1} & -X_1^{-1}X_2^{-1}X_0 \\ 0 & X_2^{-1} \end{pmatrix} \in C^\mu_{\delta''-0}(\hat D, F), \quad \delta''_\tau = \max(\delta_{1,\tau}, \delta_{2,\tau}). \tag{21}$$

As in the scalar case, we can write the general solution of the problem (1), (3) in the class C^μ_λ. Indeed, for the vector-valued function $\psi = X^{-1}\phi \in C^\mu_{\lambda-\delta''}(\hat D, F)$, we have the boundary-value problem $\psi^+ - \psi^- = g$ with the right-hand side $g = (X^+)^{-1}f \in C^\mu_{\lambda-\delta''}(\Gamma, F)$. Since $-1 < \lambda - \delta'' < 0$, its general solution has the form $\psi = Ig + p$ with some polynomial vector $p = (p_1, p_2)$. It follows that

$$\phi = XI\big[(X^+)^{-1}f\big] + Xp. \tag{22}$$

Note that in the class C^μ_λ, where λ satisfies the condition (9), the problem (1), (3) is always solvable and its solution is given by the formula

$$\phi_1 = X_1 I\big[(X_1^+)^{-1}(f_1 + G_0\phi_2^-)\big], \quad \phi_2 = If_2.$$

At the same time, this solution can be represented in the form (22), although the right-hand side of this formula, by virtue of (20), (21), belongs only to the class $C^\mu_{\lambda+\delta'-\delta''}$.

The right-hand side of (22) satisfies the condition (2) if we impose the corresponding conditions on the function f and the polynomial p; this leads to the description of the kernel and cokernel of the problem appearing in Theorem 2. According to [5], we can write the asymptotic representation for the matrix $X(z)$ in the sectors $S_{\tau,k}$ based on the spectral characteristics of the matrix

$$G_\tau = \prod_{j=1}^{n_\tau} \big[G(\tau, j)\big]^{\sigma_{\tau,j}},$$

where the order of the product corresponds to the order of arcs $\Gamma_{\tau,j}$ with a common start point τ passing counterclockwise. Using this asymptotics, we can show that in fact the formula (22) defines a solution in the class C^μ_λ.

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