fractal and fractional

MDPI

Article

Identification of the Order of the Fractional Derivative for the Fractional Wave Equation

Ravshan Ashurov [1,2] and Sergei Sitnik [3,*]

1 Institute of Mathematics, Uzbekistan Academy of Science, Student Town Str., Tashkent 100174, Uzbekistan
2 AU Engineering School, Akfa University, 264, Milliy Bog Str., Tashkent 111221, Uzbekistan
3 Department of Applied Mathematics and Computer Modelling, Belgorod State National Research University, Pobedy St. 85, Belgorod 308015, Russia
* Correspondence: mathsms@yandex.ru

Abstract: A fractional wave equation with a fractional Riemann–Liouville derivative is considered. An arbitrary self-adjoint operator A with a discrete spectrum was taken as the elliptic part. We studied the inverse problem of determining the order of the fractional time derivative. By setting the value of the projection of the solution onto the first eigenfunction at a fixed point in time as an additional condition, the order of the derivative was uniquely restored. The abstract operator A allows us to include many models. Several examples of operator A are discussed at the end of the article.

Keywords: fractional wave equation; Riemann–Liouville derivatives; inverse problem; determination of the order of derivative; Fourier method

check for updates

Citation: Ashurov, R.; Sitnik, S. Identification of the Order of the Fractional Derivative for the Fractional Wave Equation. *Fractal Fract.* **2023**, *7*, 67. https://doi.org/10.3390/fractalfract7010067

Academic Editors: Angelo B. Mingarelli, Leila Gholizadeh Zivlaei and Mohammad Dehghan

Received: 6 December 2022
Revised: 24 December 2022
Accepted: 31 December 2022
Published: 5 January 2023

1. Introduction

In applied fractional modeling, the order of the fractional derivative is often unknown, and determining this order is an important inverse problem (see, for example, the review article [1]). In this paper, we consider the inverse problem of determining the order of the fractional time derivative in the wave equation. The method proposed in the article is based on the classical Fourier method. This allows us to consider an arbitrary self-adjoint operator with a discrete spectrum as the elliptic part of the equation.

Since a precise statement of our main result requires several definitions, here (in the introduction) we formulate the corresponding result on the example of the following simple initial-boundary value problem. Let $1 < \rho < 2$ be an unknown number to be determined. Consider the time-fractional string vibration equation with the Riemann–Liouville fractional derivative (see the next Section for the definition) of order ρ:

$$\partial_t^\rho u(x,t) - u_{xx}(x,t) = f(x,t), \quad x \in (-\pi, \pi], \quad t > 0,$$

and attach the 2π-periodical boundary conditions and the following initial conditions

$$\lim_{t\to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad \lim_{t\to 0} \partial_t^{\rho-2} u(x,t) = \psi(x),$$

where φ, ψ and $f(\cdot,t)$ are 2π-periodical given functions (for the motivation to consider periodic boundary conditions, see the fundamental book by Courant and Hilbert [2] and the solution methods for the case $\rho = 2$ see the book [2], for the case $\rho \in (1,2)$ see [3,4]). Under certain conditions on these functions, there is a unique solution to this problem. Obviously, this solution depends on the choice of the order of the derivative ρ. Now let us ask a question: is there any additional information about the solution at a fixed moment of time that allows us to uniquely determine the parameter ρ?

As it follows from the main result of this paper, the answer is "yes". As for additional information at a fixed time instant t_0, one may consider the following:

$$\int_{-\pi}^{\pi} u(x,t_0)dx = d_0. \tag{1}$$

Knowledge of the value of this integral determines the parameter ρ and, moreover, if one has two pairs of solutions $\{u_1(x,t),\rho_1\}$ and $\{u_2(x,t),\rho_2\}$, then $u_1(x,t) \equiv u_2(x,t)$ and $\rho_1 = \rho_2$.

It should be noted that the first eigenfunction of the corresponding spectral problem is equal to $(2\pi)^{-1/2}$. Therefore, integral (1) is in fact, the projection of the solution onto the first eigenfunction.

This result can be interpreted as follows. The vibration of a string is usually perceived by us by the sound made by the string. The sound of a string is an overlay of simple tones corresponding to standing waves, into which vibration is decomposed. The above result states: having heard only one standing wave, one can uniquely determine the order ρ of the fractional derivative in the corresponding equation of string vibrations.

Usually, inverse problems in the theory of partial differential equations are called problems in which, along with the solution of a differential equation, it is also necessary to determine a certain coefficient of the equation or the right side or the initial or boundary function. Naturally, in this case, in order to find a new unknown function, additional information (redefinition condition) is required on the solution to the differential equation. Moreover, the redefinition condition must ensure both the existence and uniqueness of the solution to the inverse problem. Since inverse problems have important applications in many areas of modern science, including mechanics, seismology, medical tomography, geophysics, and much more (see, for example, refs. [5,6] and references therein), interest in their study is constantly growing.

As noted above, in this paper we study another inverse problem, namely, the problem of restoring the order of a fractional derivative in partial differential equations. This inverse problem has been studied in many papers ([1,7–13]). It should be noted that in all these publications the unknown order of the derivative is less than one (that is, $\rho < 1$), and the following equality was considered as a redefinition condition

$$u(x_0,t) = h(t),\ 0 < t < T, \tag{2}$$

at the observation point $x_0 \in \overline{\Omega}$. Since the goal is to find the order of the derivative in time, it seems natural to have information about the solution on a large time scale. However, this condition, as a rule (an exception is paper [13] by J. Janno, where both uniqueness and existence are proved, see below), can ensure only the uniqueness of the solution to the inverse problem. However, as the main result of this paper states, condition (1) guarantees both the existence and uniqueness of a solution.

The problem concerning the uniqueness of the solution to the inverse problem with condition (2) was studied in papers [7–10]. The authors of [7,8] considered subdiffusion equations with the Gerasimov–Caputo derivative (see next Section for definition). The problems for multi-term time-fractional diffusion equations and distributed order fractional diffusion equations were considered in papers by Li et al. [9,10], correspondingly. In the paper by J. Cheng et al. [7], the authors showed, in addition to the uniqueness of the order ρ, the uniqueness of the diffusion coefficient $p(x)$.

As far as we know, the only paper [13] by J. Janno deals with the existence problem. The author considered a subdiffusion equation with the Gerasimov–Caputo derivative. By setting an additional boundary condition $Bu(\cdot,t) = h(t), 0 < t < T$, with some functional B, the author proved the existence of an unknown order of the derivative and the kernel of the integral operator involved in the equation. The complexity of the proof of existence is due to the fact that the function $h(t)$ cannot be given arbitrarily; since t changes where the equation takes place, the function $h(t)$ must somehow be related to the equation. This circumstance is evident from the formulation of the corresponding theorem (see Theorem 7.2 of the work, which is formulated on more than one page of the journal).

In paper [14], Hatano et al., the equation $\partial_t^\rho u = \triangle u$ is considered with the Dirichlet boundary condition and the initial function $\varphi(x)$. The authors proved the following property of the parameter ρ: if $\varphi \in C_0^\infty(\Omega)$ and $\triangle\varphi(x_0) \neq 0$, then

$$\rho = \lim_{t\to 0} \left[t\partial_t u(x_0,t)[u(x_0,t) - \varphi(x_0)]^{-1}\right].$$

It should be noted that the problem considered and solved in the present article was formulated as open in the "Open Problems" section of the recent review [1] (p. 440) by Z. Li et al.: "The studies on inverse problems of the recovery of the fractional orders ... are far from satisfactory since all the publications either assumed the homogeneous boundary condition or studied this inverse problem by the measurement on $t \in (0, \infty)$. It would be interesting to investigate inverse problem by the value of the solution at a fixed time as the observation data".

In references [15–22], this problem is discussed for various equations of mathematical physics. We note right away that in these works the authors prove not only the uniqueness of the solution to the inverse problem but also its existence. The method used in this paper was first proposed in a recent paper [15], where similar questions are investigated for the subdiffusion equation with a fractional Riemann–Liouville derivative of order $0 < \rho < 1$. The elliptic part of the equation considered in [15] is a second-order differential operator. The authors of [15] instead of the redefinition condition (2), considered a condition that meets the requirements formulated in the open problem formulated above. Namely, as a redefinition condition, they took the projection of the solution onto the first eigenfunction of the elliptic part of the equation at a fixed point in time. However, note that the method of [15] requires the first eigenvalue to be zero. This limitation was lifted in recent works by Alimov and Ashurov [16,17]. The authors of these papers, taking an additional condition in the form $||u(x,t_0)||^2 = d_0$ and the boundary condition not necessarily homogeneous, proved both the existence and uniqueness of a solution to the inverse problem. In this case, the norm $||u(x,t_0)||^2$ is a part of the potential energy. Indeed, if, for example, the elliptic part of the equation has the form $Au = -\Delta u + k^2 u$, then the potential energy is equal to the sum of the norms $||\nabla u||^2 + k^2||u||^2$.

In reference [12,18], the inverse problem was studied, where it is required to determine, along with the solution to the equation, both the order of the derivative and the right-hand side of the equation. The authors of [12] proved only the uniqueness of the solution to the inverse problem, while the authors of [18] proved the existence and uniqueness theorem.

The authors of [19] have studied the subdiffusion equations, the elliptic part of which has a continuous spectrum. In this work, along with other problems, the inverse problem of determining the order of the derivative with respect to both space and time is solved.

As far as we know, the inverse problem under consideration for a mixed-type equation was first studied in [20]. The inverse problem for the fractional wave equation was studied in [21]. In this work, in contrast to the present work, the fractional derivative is taken in the Gerasimov–Caputo sense. Without additional restrictions on the spectrum of operator A, the authors present a solution to the problem posed in the review [1] for the fractional order wave equation.

We note one more paper [22], where a system of subdiffusion equations is considered, the elliptic part of which is elliptic pseudodifferential operators. The authors managed to find such additional conditions for solving the inverse problem of restoring the order of fractional derivatives, which guarantees both the uniqueness and the existence of a solution. It should be specially noted that the desired order of the fractional derivative in this work is a vector.

We also note the recent work [23], where the uniqueness of the inverse problem for the simultaneous determination of the coefficient of the equation and the order of the fractional derivative is proved.

In order to not be distracted by the technical aspects of the issue, connected with the uniform convergence of the Fourier series, we first consider an abstract statement of the problem. Then, at the end of the paper, we will make the necessary remarks for the transition to the classical setting.

This article is organized as follows. In the next section, we give the necessary definitions and formulate the main result. Note that the elliptic part of our fractional wave equation is an arbitrary self-adjoint operator A in a Hilbert space. Section 2 proves the existence and uniqueness of a solution to the direct problem. This result will be used to prove the main result in Section 3. Section 4 gives various examples of operator A for which the main result of the paper is valid. The article ends with a conclusion.

2. Main Result

Consider an arbitrary nonnegative self-adjoint operator A in a separable Hilbert space H. Let $(\cdot,\cdot)$ be a scalar product and $||\cdot||_H$ a norm in H. Assume that A has a compact inverse and denote by $\{v_k\}$ the complete system of orthonormal eigenfunctions and by $\{\lambda_k\}$ a countable set of nonnegative eigenvalues: $\lambda_k \leq \lambda_{k+1}$.

For vector functions (or just functions) $f : \mathbb{R}_+ \to H$, fractional analogs of integrals and derivatives are defined using the definition of strong integral and strong derivative (see, for example, [24]). In this case, the known formulas and properties of fractional integrals and derivatives are preserved. Thus, fractional integration in the Riemann–Liouville sense of order $\rho < 0$ is defined as

$$\partial_t^\rho f(t) = \frac{1}{\Gamma(-\rho)} \int_0^t \frac{f(\xi)}{(t-\xi)^{\rho+1}} d\xi, \quad t > 0,$$

provided the right-hand side exists as an element of H. Here the symbol $\Gamma(\rho)$ denotes the Euler gamma function. By this definition, we define the fractional derivative of order ρ, $k-1 < \rho \leq k$, $k \in \mathbb{N}$, in the Riemann–Liouville sense as

$$\partial_t^\rho f(t) = \frac{d^k}{dt^k} \partial_t^{\rho-k} f(t).$$

If in this equality the fractional integral and derivative are interchanged, then we obtain the definition of the Gerasimov–Caputo fractional derivative.

It is easy to see that for $\rho = k$ the fractional derivative coincides with the classical derivative of integer order: $\partial_t^k f(t) = \frac{d^k}{dt^k} f(t)$. For general information on fractional integro-differential operators of different classes with many applications cf. [3,25,26].

Let $\rho \in (1,2)$ be an unknown constant number and let $C((a,b);H)$ stand for a set of continuous functions $u(t)$ of $t \in (a,b)$ with values in H. Consider the Cauchy-type problem:

$$\partial_t^\rho u(t) + Au(t) = f(t), \quad 0 < t \leq T, \tag{3}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(t) = \varphi, \quad \lim_{t\to 0} \partial_t^{\rho-2} u(t) = \psi, \tag{4}$$

where the limit is taken in H norm, $f(t)$, φ, and ψ are given elements of H.

Definition 1. *If a function $u(t)$ has the properties*

1. $\partial_t^\rho u(t), Au(t) \in C((0,T];H)$,
2. $\partial_t^{\rho-1} u(t)$, $\partial_t^{\rho-2} u(t) \in C([0,T];H)$

and satisfies conditions (3) and (4), then it is called ***the (generalized) solution*** *to problems (3) and (4).*

We first prove that for any given functions $\varphi, \psi \in H$, and $t^{2-\rho}f(t) \in C([0,T];H)$, the solution of this problem exists and it is unique. This solution obviously will depend on ρ. To determine this number we use the additional condition:

$$U(\rho;t_0) \equiv (u(t_0), v_1) = d_0, \quad t_0 \geq T_0, \tag{5}$$

where T_0 is defined later.

We call problem (3) and (4) *the forward problem*. Problem (3) and (4) together with extra condition (5) is called *the inverse problem*.

Let us denote by $E_{\rho,\mu}(t)$ the Mittag-Leffler function of the form

$$E_{\rho,\mu}(t) = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(\rho k + \mu)}.$$

On Mittag-Leffler functions cf. [4,25,27,28].

Theorem 1. *For any $\varphi, \psi \in H$ and $f(t)$ with $t^{2-\rho}f(t) \in C([0,T];H)$ forward problem (3) and (4) has a unique solution and this solution has the form*

$$u(t) = \sum_{j=1}^{\infty} [\varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) + \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) + \int_0^t f_j(t-\xi)\xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi] v_j, \tag{6}$$

where the series converges in H, $f_j(t)$, φ_j and ψ_j are corresponding Fourier coefficients.

Forward problems for fractional linear wave equations and systems of such equations, involving various elliptic operators and the properties of their solutions have been studied by many authors. Since the main purpose of this article is the solution to the inverse problem, without dwelling on these papers, we refer interested readers to review papers [29,30]. We also note that in a number of papers, initial boundary value problems and the properties of their solutions for nonlinear fractional wave equations are also studied (see, for example, ref. [31] and the literature therein).

Definition 2. *Let $u(t)$ be the solution to problems (3) and (4), and the parameter $\rho \in (1,2)$. Then we call a pair $\{u(t), \rho\}$* ***the (generalized) solution*** *to the inverse problems (3)–(5).*

Let us describe the proposed method for solving the inverse problem when the following conditions are satisfied

$$\lambda_1 = 0, \quad f_1(t) \equiv 0, \quad \varphi_1^2 + \psi_1^2 \neq 0. \tag{7}$$

If these conditions are not satisfied, then the method becomes technically cumbersome. Further, let parameter T_0 in (5) be defined as

$$T_0 = \begin{cases} 2, & \varphi_1 \cdot \psi_1 \geq 0, \\ 2 \cdot \max\left\{1, \dfrac{|\psi_1|}{|\varphi_1|}\right\}, & \varphi_1 \cdot \psi_1 < 0. \end{cases}$$

Let us formulate a result on the inverse problem.

Theorem 2. *Let $\varphi, \psi \in H$ and $t^{2-\rho}f(t) \in C([0,T];H)$. Moreover, assume that the conditions (7) are satisfied and $t_0 \geq T_0$ is any fixed number. Then for the inverse problem (3)–(5) to have a unique solution $\{u(t), \rho\}$ it is necessary and sufficient that condition*

$$\min\{\varphi_1, \varphi_1 t_0 + \psi_1\} < d_0 < \max\{\varphi_1, \varphi_1 t_0 + \psi_1\}$$

be satisfied.

Remark 1. *Theorem 2 asserts the existence of a unique solution of equation (5) with respect to ρ . If we set the condition (5) at another point t_1, then we can obtain a new solution ρ_1, i.e., $U(\rho_1; t_1) = d_1$. However, then from the equality $U(\rho_1; t_0) = d_0$, by Theorem 2 we have $\rho_1 = \rho$.*

3. Forward Problem

In this section, we prove Theorem 1. In accordance with the Fourier method, we will seek the solution to the problem (3) and (4) as a series:

$$u(t) = \sum_{j=1}^{\infty} T_j(t) v_j, \tag{8}$$

where functions $T_j(t)$ are solutions to the Cauchy-type problem

$$\partial_t^\rho T_j + \lambda_j T_j = f_j(t), \quad \lim_{t\to 0} \partial_t^{\rho-1} T_j(t) = \varphi_j, \quad \lim_{t\to 0} \partial_t^{\rho-2} T_j(t) = \psi_j. \tag{9}$$

The unique solution of problem (9) has the form (see, for example, [32], p. 173)

$$T_j(t) = \varphi_j t^{\rho-1} E_{\rho,\rho}(-\lambda_j t^\rho) + \psi_j t^{\rho-2} E_{\rho,\rho-1}(-\lambda_j t^\rho) + \int_0^t f_j(t-\xi)\xi^{\rho-1} E_{\rho,\rho}(-\lambda_j \xi^\rho) d\xi. \tag{10}$$

The uniqueness of the forward problem's solution can be proved by the standard technique based on the completeness in H of the set of eigenfunctions $\{v_j\}$. For convenience, we present a proof here (see, for example [33], for the case $\rho \in (0,1)$).

Proof. Assume the opposite, i.e., let the problem (3) and (4) have two solutions $u_1(t)$ and $u_2(t)$. Let us prove that $u(t) = u_1(t) - u_2(t) \equiv 0$. Due to the linearity of the problem, to determine $u(t)$ we obtain the homogeneous problem:

$$\partial_t^\rho u(t) + Au(t) = 0, \quad t > 0; \tag{11}$$

$$\lim_{t\to 0} \partial_t^{\rho-1} u(t) = 0, \quad \lim_{t\to 0} \partial_t^{\rho-2} u(t) = 0. \tag{12}$$

Let $u(t)$ be a solution of problem (11) and (12) and v_k be an arbitrary eigenfunction with the corresponding eigenvalue λ_k. Consider the function

$$w_k(t) = (u(t), v_k). \tag{13}$$

By definition of the solution, we may write

$$\partial_t^\rho w_k(t) = (\partial_t^\rho u(t), v_k) = -(Au(t), v_k) = -(u(t), Av_k) = -\lambda_k (u(t), v_k) = -\lambda_k w_k(t), \quad t > 0.$$

Therefore, we have the Cauchy problem for $w_k(t)$:

$$\partial_t^\rho w_k(t) + \lambda_k w_k(t) = 0, \quad t > 0; \quad \lim_{t\to 0} \partial_t^{\rho-1} w_k(t) = 0, \quad \lim_{t\to 0} \partial_t^{\rho-2} w_k(t) = 0.$$

This problem has a unique null solution: $w_k(t) \equiv 0$ (see (10)). Due to the completeness of systems of eigenfunctions $\{v_k\}$, this means that $u(t) = 0$ for all $t > 0$ (see (13)). Hence the uniqueness is proved.

We turn to the proof of the existence of a solution to the forward problem. For this, we recall the following estimate for the Mittag-Leffler function with a negative argument (see, e.g., [32], p. 29)

$$|E_{\rho,\mu}(-t)| \le \frac{C}{1+t}, \quad t > 0. \tag{14}$$

Therefore, for any positive eigenvalues λ_j one has

$$|t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^\rho)| \le \frac{Ct^{\rho-1}}{1+\lambda_j t^\rho} \le \frac{C}{\lambda_j t}(t^\rho\lambda_j)^{\varepsilon/\rho}, \quad t>0, \tag{15}$$

with $0<\varepsilon<\rho$. Indeed, if $t^\rho\lambda_j<1$, then

$$\frac{1}{\lambda_j t}(t^\rho\lambda_j)^{\varepsilon/\rho} > \frac{1}{\lambda_j t}t^\rho\lambda_j > t^{\rho-1},$$

and if $t^\rho\lambda_j>1$, then

$$\frac{1}{\lambda_j t}(t^\rho\lambda_j)^{\varepsilon/\rho} > \frac{1}{\lambda_j t}.$$

The fact that function (6) formally satisfies Equation (3) follows from the definition of functions T_j (see (9)). Therefore, by Definition 1, we first need to prove that function (6) satisfies $Au(t)\in C((0,T];H)$. Consider the sum

$$S_k(t) = \sum_{j=1}^{k}[\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^\rho) + \psi_j t^{\rho-2}E_{\rho,\rho-1}(-\lambda_j t^\rho)$$

$$+\int_0^t f_j(t-\xi)\xi^{\rho-1}E_{\rho,\rho}(-\lambda_j\xi^\rho)d\xi]v_j.$$

By virtue of the Parseval equality, we may rewrite

$$||AS_k(t)||_H^2 = \sum_{j=1}^{k}\lambda_j^2[\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^\rho) + \psi_j t^{\rho-2}E_{\rho,\rho-1}(-\lambda_j t^\rho)$$

$$+\int_0^t f_j(t-\xi)\xi^{\rho-1}E_{\rho,\rho}(-\lambda_j\xi^\rho)d\xi]^2. \tag{16}$$

Using the inequality $(a+b+c)^2 \le 3(a^2+b^2+c^2)$ we have three sums on the right side.

For the first sum, one has

$$\sum_{j=1}^{k}\Big|\lambda_j\varphi_j t^{\rho-1}E_{\rho,\rho}(-\lambda_j t^\rho)\Big|^2 \le Ct^{-2}\sum_{j=1}^{k}|\varphi_j|^2 \le Ct^{-2}||\varphi||_H. \tag{17}$$

Here, we use estimate (14) and inequality $\lambda t^{\rho-1}(1+\lambda t^\rho)^{-1}<t^{-1}$.

Function $E_{\rho,\rho-1}(-\lambda_j t^\rho)$ in the second sum has the same estimate as $E_{\rho,\rho}(-\lambda_j t^\rho)$. Therefore, the second sum also has an estimate similar to (17).

Now let us consider the third sum in (16). Since operator A is nonnegative, then $\lambda_{j_0}>0$ for some $j_0\ge 1$. Further, if $f(t)$ satisfies the condition of the theorem, then $t^{2-\rho}||f(t)||_H \le C_f$. Therefore, taking into account estimate (15) and the generalized Minkowski inequality, one has

$$\sum_{j=j_0}^{k}|\int_0^t \lambda_j f_j(t-\xi)\xi^{\rho-1}E_{\rho,\rho}(-\lambda_j\xi^\rho)d\xi|^2$$

$$=\sum_{j=j_0}^{k}|\int_0^t \lambda_j(t-\xi)^{\rho-2}f_j(t-\xi)(t-\xi)^{2-\rho}\xi^{\rho-1}E_{\rho,\rho}(-\lambda_j\xi^\rho)d\xi|^2$$

$$\leq C\Big(\int_0^t \xi^{\varepsilon-1}(t-\xi)^{\rho-2}\big(\sum_{j=j_0}^{k}(t-\xi)^{2(2-\rho)}|f_j(t-\xi)|^2\big)^{1/2}d\xi\Big)^2 \leq C\cdot C_f^2\cdot(\varepsilon^{-2}+(\rho-1)^{-2}).$$

Hence, summing up the estimates of all three terms in (16), we obtain $Au(t) \in C((0,T];H)$.

Further, Equation (3) implies $\partial_t^\rho S_k(t) = -AS_k(t)$. Therefore, from the above reasoning, we finally have $\partial_t^\rho u(t) \in C((0,T];H)$.

A simple calculation shows the fulfillment of the initial conditions (4) (see (9)).

Thus, Theorem 1 is proved. □

4. Inverse Problem

First, we study some properties of the projection of the forward problem's solution onto the first eigenfunction, i.e., $U(\rho;t_0)$ (see (5)) as a function of $\rho \in (1,2)$. Let T_0 be a number, defined above.

Lemma 1. *Let conditions (7) be satisfied and $t_0 \geq T_0$. Then function $U(\rho;t_0)$ is strictly monotonic in the variable $\rho \in (1,2)$ and*

$$\lim_{\rho\to 1} U(\rho;t_0) = \varphi_1, \quad U(2;t_0) = \varphi_1 t_0 + \psi_1. \tag{18}$$

Proof. Since eigenfunctions $\{v_j\}$ are orthonormal, then from (6) by virtue of conditions (7), one may obtain

$$U(\rho;t_0) = \varphi_1 t_0^{\rho-1}E_{\rho,\rho}(0) + \psi_1 t_0^{\rho-2}E_{\rho,\rho-1}(0),$$

or, by definition of the Mittag-Leffler function,

$$U(\rho;t_0) = \varphi_1 y(\rho) + \psi_1 y(\rho-1), \quad y(\rho) = \frac{t_0^{\rho-1}}{\Gamma(\rho)}.$$

Denote by $\Psi(\rho)$ the logarithmic derivative of the gamma function $\Gamma(\rho)$ (see [34] for the definition of this function and its properties). We have $\Gamma'(\rho) = \Gamma(\rho)\Psi(\rho)$ and, then,

$$y'(\rho) = \frac{t_0^{\rho-1}}{\Gamma(\rho)}\big[\ln t_0 - \Psi(\rho)\big].$$

Let $\gamma \approx 0.57722$ be the Euler–Mascheroni constant, then $-\gamma < \Psi(\rho) < 1-\gamma$ and $\Psi(\rho-1) < 0$ for $\rho \in (1,2)$. Hence, if $t_0 \geq 2$, then $y'(\rho) > 0$ and $y'(\rho-1) > 0$. Therefore, if $\varphi_1\cdot\psi_1 \geq 0$ and $t_0 \geq 2$, then $U(\rho;t_0)$ is strictly monotonic in the variable ρ.

Let now $\varphi_1\cdot\psi_1 < 0$ and prove that t_0 can be chosen in such a way that

$$|\varphi_1 y'(\rho)| > |\psi_1 y'(\rho-1)|. \tag{19}$$

In order to show this, we will rewrite the function $U'(\rho;t_0)$, taking into account the equations

$$\frac{1}{\Gamma(\rho-1)} = \frac{\rho-1}{\Gamma(\rho)}, \quad \Psi(\rho-1) = \Psi(\rho) - \frac{1}{\rho-1},$$

in the form

$$U'(\rho;t_0) = \frac{t_0^{\rho-1}}{\Gamma(\rho)}\Big(\frac{\varphi_1}{2}[2\ln t_0 - 2\Psi(\rho)] + \frac{\psi_1}{t_0}[(\rho-1)\ln t_0 + 1 - \Psi(\rho)]\Big). \tag{20}$$

It is easy to see that

$$2\ln t_0 - 2\Psi(\rho) > (\rho-1)\ln t_0 + 1 - \Psi(\rho)$$

for all $t_0 \geq 2$. Indeed, this inequality is equivalent to the following

$$2 \ln t_0 > (3-\rho) \ln t_0 > \Psi(\rho) + 1 > -\gamma + 1 > \frac{2}{5},$$

that is $\ln t_0 > 1/5$. Therefore, if $t_0 > e^{\frac{1}{5}} > \frac{6}{5}$, or $t_0 \geq 2$, then we obtain the required estimate.

Therefore, for the validity of estimate (19), it is sufficient to simultaneously fulfill two inequalities $t_0 \geq 2$ and $t_0|\varphi_1| \geq 2|\psi_1|$ (see (20)), or which is the same, one inequality $t_0 \geq 2\max\{1, \frac{|\psi_1|}{|\varphi_1|}\}$.

Thus, if $t_0 \geq T_0$, then $U(\rho; t_0)$ is strictly monotonic in the variable ρ. The equalities (18) are easy to check. □

Now let us go to the proof of the Theorem 2.

Proof. The fact that $u(x,t)$ exists for any $\rho \in (1,2)$ follows from Theorem 1. Let the given number d_0 be such that

$$\min\{\varphi_1, \varphi_1 t_0 + \psi_1\} < d_0 < \max\{\varphi_1, \varphi_1 t_0 + \psi_1\}.$$

Then it immediately follows from Lemma 1 that there exists a unique number ρ satisfying the condition (5). Obviously, if the opposite inequalities hold, then such a number ρ does not exist.

We turn to the proof of the uniqueness of the solution to the inverse problem (3)–(5). Let there be two pairs of solutions $\{u_1, \rho_1\}$ and $\{u_2, \rho_2\}$ such that $1 < \rho_k < 2$ and

$$\partial_t^{\rho_k} u_k(t) + A u_k(t) = f(t), \quad 0 < t \leq T; \tag{21}$$

$$\lim_{t\to 0} \partial_t^{\rho_k - 1} u_k(t) = \varphi, \quad \lim_{t\to 0} \partial_t^{\rho_k - 2} u_k(t) = \psi, \tag{22}$$

where $k = 1, 2$.

Consider the following functions

$$w_k^j(t) = (u_k(t), v_j) \quad k = 1, 2; \quad j = 1, 2, \cdots$$

Then Equations (21) and (22) imply

$$\partial_t^{\rho_k} w_k^j(t) + \lambda_j w_k^j(t) = f_j(t), \quad \lim_{t\to 0} \partial_t^{\rho_k - 1} w_k^j(t) = \varphi_j, \quad \lim_{t\to 0} \partial_t^{\rho_k - 2} w_k^j(t) = \psi_j.$$

Solutions to these Cauchy-type problems can be represented as (10). Then, (5) implies $w_1^1(t_0) = w_2^1(t_0) = d_0$, or, since $f_1 = 0$,

$$\varphi_1 t_0^{\rho_1 - 1} E_{\rho_1,\rho_1}(0) + \psi_1 t_0^{\rho_1 - 2} E_{\rho_1,\rho_1 - 1}(0) = \varphi_1 t_0^{\rho_2 - 1} E_{\rho_2,\rho_2}(0) + \psi_1 t_0^{\rho_2 - 1} E_{\rho_2,\rho_2 - 1}(0) = d_0.$$

As we have seen above (see Lemma 1), it follows from these equations that $\rho_1 = \rho_2$. However, in this case, $w_1^j(t) = w_2^j(t)$ for all t and j. Hence

$$(u_1(t) - u_2(t), v_j) = 0$$

for all j. Finally, from the completeness of the set of eigenfunctions $\{v_j\}$ in H, we have $u_1(t) = u_2(t)$. Hence, Theorem 2 is completely proved. □

5. Examples of Operator A

Consideration of the abstract operator A allows us to explore many different models. In this section, we provide several examples of operator A, to which our results apply.

First, we obtain an interesting example if we take a square matrix with constant elements as the operator A: $A = \{a_{i,j}\}$ and $H = \mathbb{R}^N$. In this case, the problem (3) and (4) becomes the Cauchy problem for a linear system of differential equations of fractional order.

As an example of operator A, one can also take any of the physical examples considered in Section 6 of the article by M. Ruzhansky et al. [33]. In particular, the authors considered differential models with involution, fractional Laplacian, and fractional Sturm–Liouville operators, anharmonic and harmonic oscillators, Landau Hamiltonians, and many other operators with a discrete spectrum. If the first eigenvalue λ_1 of the operator A is not zero, then the operator $A - \lambda_1 I$ with zero first eigenvalue should be considered as required in Theorem 2. Here, I is the identity operator.

The solution to the problem in this work, as well as in our work, is understood in a generalized sense (see Definition 1).

Now, let us show how similar results as in this paper can be obtained for classical solutions (see also [15]).

Let $A(x, D) = \sum\limits_{|\alpha| \le m} a_\alpha(x) D^\alpha$ be an arbitrary non-negative formally self-adjoint elliptic differential operator of the order $m = 2l$ defined in N-dimensional bounded domain Ω with boundary $\partial\Omega$.

Assume that $1 < \rho < 2$ is an unknown parameter that needs to be determined and that the initial-boundary value problem has the form

$$\partial_t^\rho u(x,t) + A(x,D)u(x,t) = f(x,t), \quad x \in \Omega, \quad 0 < t \le T, \tag{23}$$

$$B_j u(x,t) = \sum_{|\alpha| \le m_j} b_{\alpha,j}(x) D^\alpha v(x) = 0, \; j = 1,2,...,l; \; x \in \partial\Omega, \quad 0 < t \le T, \tag{24}$$

$$\lim_{t \to 0} \partial_t^{\rho-1} u(x,t) = \varphi(x), \quad \lim_{t \to 0} \partial_t^{\rho-2} u(x,t) = \psi(x), \quad x \in \overline{\Omega} \tag{25}$$

where $f(x,t)$, $\varphi(x)$ and $\psi(x)$ are given sufficiently smooth functions from $L_2(\Omega)$.

In the paper by S. Agmon [35], it is considered the spectral problem

$$\begin{cases} A(x,D)v(x) = \lambda v(x), \quad x \in \Omega; \\ B_j v(x) = 0, \, 0 \le m_j \le m-1, \, j = 1,2,...,l; \, x \in \partial\Omega. \end{cases} \tag{26}$$

The author found sufficient conditions on the boundary of domain Ω and operators $A(x,D)$ and B_j that guarantee the compactness of the corresponding inverse operator, i.e., the existence of a complete system $\{v_k(x)\}$ of orthonormal eigenfunctions and a countable set $\{\lambda_k\}$ of non-negative eigenvalues of the spectral problem (26).

As the next example, instead of A we take operator $A(x,D)$ with boundary conditions B_j and set $H = L_2(\Omega)$. In this case, an additional condition (5) for determining ρ will have the form:

$$\int_\Omega u(x,t_0) v_1(x) dx = d_0, \quad t_0 \ge T_0, \tag{27}$$

where T_0 is defined as above. Let g_k stand for the Fourier coefficient of a function $g(x) \in L_2(\Omega)$ by the system of eigenfunctions $\{v_k(x)\}$.

Definition 3. *A pair $\{u(x,t), \rho\}$ of the function $u(x,t)$ and the parameter ρ with the properties*

1. $\rho \in (1,2)$,
2. $\partial_t^\rho u(x,t), A(x,D)u(x,t) \in C(\overline{\Omega} \times (0,\infty))$,
3. $\partial_t^{\rho-1} u(x,t), \, \partial_t^{\rho-2} u(x,t) \in C(\overline{\Omega} \times [0,\infty))$

*and satisfying all the conditions of problems (23)–(25), (27) in the classical sense is called **the classical solution** of inverse problem (23)–(25), (27).*

Theorem 3. *Let f, φ, ψ be sufficiently smooth functions. Further, let conditions (7) be satisfied and $t_0 \geq T_0$ be any fixed number. Then for the inverse problem (23)–(25), (27) to have a unique solution $\{u(x,t), \rho\}$ it is necessary and sufficient that condition*

$$\min\{\varphi_1, \varphi_1 t_0 + \psi_1\} < d_0 < \max\{\varphi_1, \varphi_1 t_0 + \psi_1\}$$

be satisfied.

The theorem is proved using similar arguments presented above (see, also [15]). In order to reduce the study of uniform convergence to the study of convergence in L_2-norm, we apply Lemma 22.1 of the monograph [36] (p. 453).

Remark 2. *Let $A_0(x,D) = \sum\limits_{0<|\alpha|\leq m} a_\alpha(x)D^\alpha$ be an elliptic operator and $B_{0,j} = \sum\limits_{0<|\alpha|\leq m_j} b_{\alpha,j}(x)D^\alpha$ be boundary operators. Then the first eigenfunction of the spectral problem (26) is a constant and $\lambda_1 = 0$.*

6. Conclusions

The problem of determining the fractional order of a model has been considered by many authors because of its importance to the application. The authors mainly considered subdiffusion equations in which the Gerasimov–Caputo fractional derivative is involved.

As far as we know, the inverse problem of determining the order of the fractional derivative for the fractional wave equation was considered only in [1]. As a fractional derivative, the authors took the Gerasimov–Caputo derivative.

In the present work, by studying the abstract wave equation with the Riemann–Liouville derivative, the open problem formulated in the review article [1] for the considered inverse problems is positively solved. Since the problem is solved on the basis of the classical Fourier method, the explicit form of the elliptic part is not fundamental. Therefore, an arbitrary non-negative self-adjoint operator A in a separable Hilbert space H is taken as the elliptic part. If $H = L_2(\Omega)$, where Ω is an N-dimensional bounded domain with a smooth boundary, then as the operator A we can take the Laplacian with the Neumann condition. In this case, the first eigenvalue is equal to zero, as required in Theorem 2.

Author Contributions: Conceptualization, R.A.; methodology, S.S.; software, R.A.; validation, S.S.; formal analysis, S.S.; investigation, R.A.; resources, S.S.; data curation, S.S.; writing—original draft preparation, R.A.; writing—review and editing, S.S.; visualization, R.A.; supervision, R.A.; project administration, S.S. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: The data presented in this study are available upon request from the corresponding author.

Acknowledgments: The authors thank Sh. A. Alimov for the discussions about the results.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Li, Z.; Liu, Y.; Yamamoto, M. Inverse problems of determining parameters of the fractional partial differential equations. In *Handbook of Fractional Calculus with Applications*; DeGruyter: Berlin, Germany, 2019; Volume 2, pp. 431–442.
2. Courant, R.; Hilbert, D. *Methods of Mathematical Physics*; John Wiley & Sons: Hoboken, NJ, USA, 1989; Volume 1.
3. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives. Theory and Applications*; Gordon and Breach Science Publishers: Philadelphia, PA, USA, 1993.
4. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogosin, S. *Mittag–Leffler Functions, Related Topics and Applications*, 2nd ed.; (Springer Monographs in Mathematics); Springer: New York, NY, USA, 2020.

5. Machado, J.A.T. (Ed.) *Handbook of Fractional Calculus with Applications*; DeGruyter: Berlin, Germany, 2019; Volume 1–2.
6. Kravchenko, V.V. *Direct and Inverse Sturm-Liouville Problems: A Method of Solution*; Birkhäuser: Cham, Switzerland; Springer Nature: Berlin, Germany, 2020.
7. Cheng, J.; Nakagawa, J.; Yamamoto, M.; Yamazaki, T. Uniqueness in an inverse problem for a one-dimensional fractional diffusion equation. *Inverse Probl.* **2009**, *25*, 115002. [CrossRef]
8. Tatar, S.; Ulusoy, S. A uniqueness result for an inverse problem in a space-time fractional diffusion equation. *Electron. J. Differ. Equ.* **2013**, *257*, 1–9.
9. Li, Z.; Yamamoto, M. Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation. *Appl. Anal.* **2015**, *94*, 570–579. [CrossRef]
10. Li, Z.; Luchko, Y.; Yamamoto, M. Analyticity of solutions to a distributed order time-fractional diffusion equation and its application to an inverse problem. *Comput. Math. Appl.* **2017**, *73*, 1041–1052. [CrossRef]
11. Zheng, X.; Cheng, J.; Wang, H. Uniqueness of determining the variable fractional order in variable-order time-fractional diffusion equations. *Inverse Probl.* **2019**, *35*, 125002. [CrossRef]
12. Yamamoto, M. Uniqueness in determining fractional orders of derivatives and initial values. *Inverse Probl.* **2021**, *37*, 57–71. [CrossRef]
13. Janno, J. Determination of the order of fractional derivative and a kernel in an inverse problem for a generalized time-fractional diffusion equation. *Electron. J. Differ.* **2016**, *216*, 1–28.
14. Hatano, Y.; Nakagawa, J.; Wang, S.; Yamamoto, M. Determination of order in fractional diffusion equation. *J. Math-for-Ind.* **2013**, *5A*, 51–57.
15. Ashurov, R.; Umarov, S. Determination of the order of fractional derivative for subdiffusion equations. *Fract. Calc. Appl. Anal.* **2020**, *23*, 1647–1662. [CrossRef] [PubMed]
16. Alimov, S.; Ashurov, R. Inverse problem of determining an order of the Caputo time-fractional derivative for a subdiffusion equation. *J. Inverse Ill-Posed Probl.* **2020**, *28*, 651–658. [CrossRef]
17. Alimov, S.; Ashurov, R.R. Inverse problem of determining an order of the Riemann-Liouville time-fractional derivative. *Progr. Fract. Differ. Appl.* **2022**, *8*, 1–8.
18. Ashurov, R.; Fayziev, Y. Determination of fractional order and source term in a fractional subdiffusion equation. *Eur. Math. J.* **2022**, *13*, 19–31. [CrossRef]
19. Ashurov, R.R.; Zunnunov, R. Initial-boundary value and inverse problems for subdiffusion equation in R^N. *Fract. Differ. Calc.* **2020**, *10*, 291–306.
20. Ashurov, R.R.; Zunnunov, R. Inverse Problem for Determining the Order of the Fractional Derivative in Mixed-Type Equations. *Lobachevskii J. Math.* **2021**, *42*, 2714–2729. [CrossRef]
21. Ashurov, R.R.; Fayziev, Y.E. Inverse problem for determining the order of a fractional derivative in a wave equation. *Math. Notes* **2021**, *110*, 824–836. [CrossRef]
22. Ashurov, R.R.; Umarov, S.R. An inverse problem of determining orders of systems of fractional pseudo-differential equations. *Fract. Calc. Appl. Anal.* **2022**, *25*, 109–127. [CrossRef]
23. Jing, X.; Yamamoto, M. Simultaneous uniqueness for multiple parameters identification in a fractional diffusion-wave equation. *Inverse Probl. Imaging* **2022**, *16*, 1199–1217. [CrossRef]
24. Lizama, C. Abstract linear fractional evolution equations. In *Handbook of Fractional Calculus with Applications*; DeGruyter: Berlin, Germany, 2019; Volume 2, pp. 465–497.
25. Shishkina, E.L.; Sitnik, S.M. *Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics*, 1st ed.; Series: Mathematics in Science and Engineering; Elsevier: Amsterdam, The Netherlands; Academic Press: Cambridge, MA, USA, 2020.
26. Urinov, A.K.; Sitnik, S.M.; Shishkina, E.L.; Karimov, S.T. *Fractional Integrals and Derivatives (Generalizations and Applications)*; Fargona Publishing: Fergana, Uzbekistan, 2020.
27. Dzhrbashian, M.M. *Integral Transforms and Representation of Functions in the Comples Domain*; Nauka: Moscow, Russia, 1966.
28. Paneva-Konovska, J. *From Bessel to Multi-Index Mittag-Leffler Functions: Enumerable Families, Series in Them and Convergence*; World Scientific Publishing: Singapore, 2016.
29. Kochubei, A.N. Fractional-hyperbolic equations and systems. Cauchy problem. In *Handbook of Fractional Calculus with Applications*; DeGruyter: Berlin, Germany, 2019; Volume 2, pp. 19–223.
30. Atanacković, T.N.; Konjik, S.; Pilipović, S. Wave equation involving fractional derivatives of real and complex fractional order. In *Handbook of Fractional Calculus with Applications*; DeGruyter: Berlin, Germany, 2019; Volume 2, pp. 327–353.
31. Ahmad, B.; Alsaedi, A.; Berbiche, M.; Kirane, M. Global Existence and Blow-up of Solutions for a System of Fractional Wave Equations. *Taiwanese J. Math.* **2022**, *26*, 103–135. [CrossRef]
32. Gorenflo, R.; Kilbas, A.A.; Mainardi, F.; Rogozin, S.V. *Mittag-Leffler Functions, Related Topics and Applications*; Springer: Berlin/Heidelberg, Germany, 2014.
33. Ruzhansky, M.; Tokmagambetov, N.; Torebek, B.T. Inverse source problems for positive operators. I: Hypoelliptic diffusion and subdiffusion equations. *J. Inverse Ill-Possed Probl.* **2019**, *27*, 891–911. [CrossRef]
34. Bateman, H. *Higher Transcendental Functions*; McGraw-Hill: New York, NY, USA, 1953.

35. Agmon, S. On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Commun. Pure Appl. Math.* **1962**, *15*, 119–141. [CrossRef]
36. Krasnoselski, M.A.; Zabreyko, P.P.; Pustilnik, E.I.; Sobolevski, P.S. *Integral Operators in the Spaces of Integrable Functions*; M. NAUKA: Moscow, Russia, 1966. (In Russian)