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# On the Solvability of Equations with a Distributed Fractional Derivative Given by the Stieltjes Integral

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**Abstract:** Linear equations in Banach spaces with a distributed fractional derivative given by the Stieltjes integral and with a closed operator $A$ in the right-hand side are considered. Unlike the previously studied classes of equations with distributed derivatives, such kinds of equations may contain a continuous and a discrete part of the integral, i.e., a standard integral of the fractional derivative with respect to its order and a linear combination of fractional derivatives with different orders. Resolving families of operators for such equations are introduced into consideration, and their properties are studied. In terms of the resolvent of the operator $A$, necessary and sufficient conditions are obtained for the existence of analytic resolving families of the equation under consideration. A perturbation theorem for such a class of operators is proved, and the Cauchy problem for the inhomogeneous equation with a distributed fractional derivative is studied. Abstract results are applied for the research of the unique solvability of initial boundary value problems for partial differential equations with a distributed derivative with respect to time.

**Keywords:** distributed fractional derivative; fractional differential equation; analytic $k$-resolving family; Cauchy problem; initial boundary value problem

**MSC:** 34G10; 35R11; 34A08; 47D99

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## 1. Introduction

Equations with fractional derivatives of various forms attract the attention of researchers both from a theoretical point of view and because of their widespread use in applied problems, see, e.g., recent papers [1–3] and many other works. The distributed derivatives (other names are continual derivatives [4], mean derivatives [5]) are used for the investigation of some real phenomena and processes when an order of a fractional derivative in a model continuously depends on the process parameters: in the theory of viscoelasticity [5], in modeling dielectric induction and diffusion [6,7], in the kinetic theory [8], and in other scientific fields [4,9–12]. This fact initiated the interest in equations with distributed derivatives of specialists in computational mathematics [13,14], of researchers in the qualitative theory of differential equations [15–21].

In the mentioned above works, researchers study specific equations or systems of them with some possible arbitrariness in the choice of parameters. The idea of the present work is to investigate the Cauchy problem for a class of abstract equations with distributed derivatives in order to be able to reduce many initial boundary value problems for partial differential equations or systems of equations of various forms to such a problem and study them through the obtained general results.

Let $\mathcal{Z}$ be a Banach space, $D^\beta$ be the fractional Gerasimov–Caputo derivative for $\beta > 0$ and the fractional Riemann–Liouville integral for $\beta \leq 0$, and $A$ be a linear closed densely defined in $\mathcal{Z}$ operator. For an unknown function $z : \mathbb{R}_+ \to \mathcal{Z}$ consider the distributed order equation

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = Az(t), \quad t > 0, \tag{1}$$

with the Cauchy conditions

$$z^{(k)}(0) = z_k, \quad k = 0, 1, \ldots, m-1. \tag{2}$$

Here $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu : [b, c] \to \mathbb{C}$ is a function with a bounded variation, $c$ is a variation point of the measure $d\mu(t)$, the integral in Equation (1) is of Riemann–Stieltjes.

The Cauchy problem for the distributed order equation

$$\int_b^c \omega(\alpha) D^\alpha z(t) d\alpha = Az(t) \tag{3}$$

with a given scalar function $\omega : (b, c) \to \mathbb{C}$ and a bounded operator $A$, or with an infinitesimal generator $A$ of an analytic semigroup was studied in [22–24]. In the case $b = 0$, $c \in (0, 1]$, necessary and sufficient conditions on a linear closed operator $A$ for the existence of an analytic resolving family of operators for Equation (3) were found in [25]. It allowed us to obtain a theorem on the existence of a unique solution for the corresponding inhomogeneous equation. In [26] these results were extended for the case $c > 1$ and a theorem on perturbations of generators of analytic resolving operators families for Equation (3) was proved. Paper [27] contains analogous results for equation with a discretely distributed Gerasimov–Caputo derivative

$$\sum_{k=1}^n \omega_k D^{\alpha_k} z(t) = Az(t).$$

In works [28,29] initial value problems for equations in Banach spaces with distributed Riemann–Liouville derivatives were studied.

Equation (1) with the Riemann–Stieltjes integral in the present work includes the listed above classes of equations as partial cases. Indeed, an arbitrary function $\mu$ with a bounded variation has the form $\mu = \mu_c + \mu_d$, where $\mu_c$ is a continuous function with a bounded variation, $\mu_d$ is a jumps function. Therefore, Equation (1) has the form

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = \int_b^c D^\alpha z(t) d\mu_c(\alpha) + \int_b^c D^\alpha z(t) d\mu_d(\alpha) =$$

$$= \int_b^c \mu_c'(\alpha) D^\alpha z(t) d\alpha + \sum_{k=1}^n \omega_k D^{\alpha_k} z(t) = Az(t),$$

if there exists an appropriate derivative $\mu_c'$, $\alpha_k$ are points of jumps of the function $\mu_d$, $\omega_k$ are values of jumps, $k = 1, 2, \ldots, n$. Moreover, here we consider $b \leq 0$ in the lower limit of integration and abandon the additional conditions of the fulfillment of some inequalities related to the integral from the equation (see [25,26]) since we prove that these inequalities follow from our general assumptions on parameters of the problem (see Lemma 1, Lemma 2).

The second section contains the study of the properties of some analytic functions, which are associated with an integral from Equation (1). Then the notion of a $k$-resolving family, $k = 0, 1, \dots, m-1$, is introduced and the properties of such families are researched. It is shown that the existence of a 0-resolving family implies the existence of $k$-resolving families, $k = 1, 2, \dots, m-1$. In the third section it is proved that inclusion $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$ is necessary and sufficient for the existence of analytic $k$-resolving families for distributed order Equation (1). This result allows us to obtain a unique solvability theorem for Cauchy problem Equations (1) and (2). In the fourth section a theorem on the perturbations for operators from the class $\mathcal{A}_W(\theta_0, a_0)$ is proved. The unique solvability of the inhomogeneous equation with the distributed derivative and with an operator $A \in \mathcal{A}_W(\theta_0, a_0)$ is studied in the fifth section. The last section concerns an application of abstract results to the investigation of an initial boundary value problem for the phase field system of equations with the distributed order time derivative.

## 2. Resolving Families of Operators and Their Properties

Let $\mathcal{Z}$ be a Banach space, denote for $\beta > 0$, $h : \mathbb{R}_+ \to \mathcal{Z}$ the Riemann–Liouville fractional derivative

$$J^\beta h(t) := \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} h(s)ds, \;\; t > 0.$$

Let $m - 1 < \alpha \leq m \in \mathbb{N}$, $D^m$ be the derivative of the $m$-th order, then the Gerasimov–Caputo derivative has the form [30–32]

$$D^\alpha h(t) := D^m J^{m-\alpha} \left( h(t) - \sum_{k=0}^{m-1} h^{(k)}(0) \frac{t^k}{k!} \right).$$

We will mean for $\alpha < 0$ that $D^\alpha h(t) := J^{-\alpha} h(t)$.

The Laplace transform of a function $h : \mathbb{R}_+ \to \mathcal{Z}$ will be denoted by $\widehat{h}$ or Lap$[h]$, if the expression for $h$ is too large. The Laplace transform of the Gerasimov–Caputo derivative of an order $\alpha \in (m-1, m]$ satisfies the equality (see, e. g., [33])

$$\widehat{D^\alpha h}(\lambda) = \lambda^\alpha \widehat{h}(\lambda) - \sum_{k=0}^{m-1} h^{(k)}(0) \lambda^{\alpha-1-k}. \tag{4}$$

Introduce the notations $S_{\theta,a} := \{\mu \in \mathbb{C} : |\arg(\mu - a)| < \theta, \, \mu \neq a\}$ for $\theta \in [\pi/2, \pi]$, $a \in \mathbb{R}$, $\Sigma_\psi := \{t \in \mathbb{C} : |\arg t| < \psi, \, t \neq 0\}$ for $\psi \in (0, \pi/2]$.

**Theorem 1** ([34], Theorem 0.1, p. 5), ([35], Theorem 2.6.1, p. 84)**.** *Let $\theta_0 \in (\pi/2, \pi]$, $a \in \mathbb{R}$, $\mathcal{X}$ be a Banach space, a map $H : (a, \infty) \to \mathcal{X}$ be set. The next assertions are equivalent.*

(i) *There exists an analytic function $F : \Sigma_{\theta_0 - \pi/2} \to \mathcal{X}$, for every $\theta \in (\pi/2, \theta_0)$ there exists $C(\theta) > 0$, such that for all $t \in \Sigma_{\theta - \pi/2}$ the inequality $\|F(t)\|_\mathcal{X} \leq C(\theta) e^{a \mathrm{Re}\, t}$ is satisfied; $\widehat{F}(\lambda) = H(\lambda)$ at $\lambda > a$.*

(ii) *The map $H$ is analytically continued on $S_{\theta_0,a}$, for every $\theta \in (\pi/2, \theta_0)$ there exists such a $K(\theta) > 0$, that for all $\lambda \in S_{\theta,a}$ $\|H(\lambda)\|_\mathcal{X} \leq K(\theta)|\lambda - a|^{-1}$.*

Let by $\mathcal{L}(\mathcal{Z})$ the Banach space of all linear continuous operators from $\mathcal{Z}$ to $\mathcal{Z}$ be denoted, denote by $\mathcal{C}l(\mathcal{Z})$ the set of all linear closed operators, densely defined in $\mathcal{Z}$, acting in the space $\mathcal{Z}$. Endow the domain $D_A$ of an operator $A \in \mathcal{C}l(\mathcal{Z})$ by the norm of its graph $\| \cdot \|_{D_A} := \| \cdot \|_\mathcal{Z} + \|A \cdot \|_\mathcal{Z}$, then $D_A$ is a Banach space.

Consider the Cauchy problem

$$z^{(k)}(0) = z_k, \quad k = 0, 1, \dots, m-1, \tag{5}$$

for the distributed order equation

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = Az(t), \quad t > 0, \tag{6}$$

where $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu : [b, c] \to \mathbb{C}$ is a function with a bounded variation, briefly $\mu \in BV([b, c]; \mathbb{C})$, $c$ is a variation point of the measure $d\mu(t)$. The integral in Equation (6) is understood in the sense of Riemann–Stieltjes.

A solution of the problem in Equations (5) and (6) is a function $z \in C^{m-1}(\overline{\mathbb{R}}_+; \mathcal{Z}) \cap C(\mathbb{R}_+; D_A)$, such that $\int_b^c \omega(\alpha) D^\alpha z(t) d\alpha \in C(\mathbb{R}_+; \mathcal{Z})$ and Equalities (5) and (6) are fulfilled. Here $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{0\}$.

It is evident, that under the conditions of this section the complex-valued functions

$$W(\lambda) := \int_b^c \lambda^\alpha d\mu(\alpha) \quad W_k(\lambda) := \int_k^c \lambda^\alpha d\mu(\alpha), \ k = 0, 1, \ldots, m-1,$$

are analytic on the set $S_{\pi,0}$. Here the Riemann–Stieltjes integrals are used also.

**Lemma 1.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b, c]; \mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$. Then for $k = 0, 1, \ldots, m-1$*

$$\forall \varepsilon \in (0, c) \quad \exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W_k(\lambda)| \geq C|\lambda|^{c-\varepsilon},$$

$$\varepsilon \in (0, c) \quad \exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W(\lambda)| \geq C|\lambda|^{c-\varepsilon}.$$

**Proof.** By the definition of the Riemann–Stieltjes integral for a small $\delta > 0$ there exists a division $\{\alpha_0 = b, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n = c\}$ of the segment $[b, c]$ with a sufficiently small radius $r := \max\{\alpha_k - \alpha_{k-1} : k = 1, 2, \ldots, n\}$ and with intermediate points $\xi_k \in [\alpha_{k-1}, \alpha_k]$, $k = 1, 2, \ldots, n$, we obtain for $|\lambda| > \varrho$

$$\left| \int_b^c \lambda^\alpha d\mu(\alpha) \right| \geq \left| \sum_{j=1}^n \lambda^{\xi_j} (\mu(\alpha_j) - \mu(\alpha_{j-1})) \right| - \delta \geq$$

$$\geq |\lambda|^{\xi_n} \left( |\mu(\alpha_n) - \mu(\alpha_{n-1})| - \left| \sum_{j=1}^{n-1} \lambda^{\xi_j - \xi_n} (\mu(\alpha_j) - \mu(\alpha_{j-1})) \right| - \delta |\lambda|^{-\xi_n} \right) \geq$$

$$\geq \frac{|\mu(\alpha_n) - \mu(\alpha_{n-1})|}{2} |\lambda|^{\xi_n}.$$

Here $\mu(\alpha_n) - \mu(\alpha_{n-1}) = \mu(c) - \mu(\alpha_{n-1}) \neq 0$ for sufficiently small $r$, since $c$ is a variation point of the measure $d\mu(t)$.

For every $\varepsilon \in (0, c)$ there exists a division of $[b, c]$, such that $r \leq \varepsilon$, then $\alpha_n - \alpha_{n-1} = c - \alpha_{n-1} \leq \varepsilon$ and $\xi_n \geq \alpha_{n-1} \geq c - \varepsilon$.

For $W_k$ the proof has the same form. □

**Lemma 2.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b, c]; \mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$. Then for all $k, l = 0, 1, \ldots, m-1$, $k > l$,*

$$\exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W_k(\lambda) - W_l(\lambda)| \leq C|\lambda|^k;$$

$$\exists C, \varrho > 0 \quad \forall \lambda \in S_{\pi,0} \setminus \{\lambda \in \mathbb{C} : |\lambda| < \varrho\} \quad |W_k(\lambda) - W(\lambda)| \leq C|\lambda|^k.$$

**Proof.** Indeed,

$$|W_k(\lambda) - W_l(\lambda)| = \left| \int_l^k \lambda^\alpha d\mu(\alpha) \right| \leq V_b^c(\mu)|\lambda|^k,$$

where $V_b^c(\mu)$ is the variation of the function $\mu$ on the segment $[b, c]$. The second inequality can be proved in the same way. □

**Definition 1.** *A family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$, $k \in \{0, 1, \ldots, m-1\}$, is called k-resolving for Equation (6), if:*

(i) *$S_k(t)$ is strongly continuous for $t \geq 0$;*
(ii) *$S_k(t)[D_A] \subset D_A$, $S_k(t)Az = AS_k(t)z$ for all $z \in D_A$, $t \geq 0$;*
(iii) *$S_k(t)z_k$ is a solution of the Cauchy problem*

$$z^{(l)}(0) = 0, \quad l \in \{0, 1, \ldots, m-1\} \setminus \{k\}, \quad z^{(k)}(0) = z_k \tag{7}$$

*to Equation (6) for any $z_k \in D_A$.*

**Remark 1.** *So, a k-resolving family $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ consists of operators, such that $S(t)$ for every fixed $t \geq 0$ maps any $z_k \in D_A$ into the value $z(t) = S_k(t)z_k$ at the point $t$ of a solution of Cauchy problem Equations (6) and (7). Thus, the totality of families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$, $k = 0, 1, \ldots, m-1$, entirely determines the solution of the complete Cauchy problem Equations (5) and (6).*

Denote by $\rho(A)$ the resolvent set $\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in \mathcal{L}(\mathcal{Z})\}$ of an operator $A \in \mathcal{C}l(\mathcal{Z})$.

**Lemma 3.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b, c]; \mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, for some $k \in \{0, 1, \ldots, m-1\}$ there exist a k-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ for Equation (6), such that for all $t \geq 0$ $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ke^{at}$ at some $K \geq 1$, $a \geq 0$. Then for* Re$\lambda > a$ *we have $W(\lambda) \in \rho(A)$ and*

$$\widehat{S}_k(\lambda) = \frac{W_k(\lambda)}{\lambda^{k+1}}(W(\lambda)I - A)^{-1}. \tag{8}$$

**Proof.** For an exponentially bounded solution $z$ of problem Equations (5) and (6), we have

$$\int_b^c D^\alpha z(t) d\mu(\alpha) = \int_b^0 D^\alpha z(t) d\mu(\alpha) + \sum_{l=1}^{m-1} \int_{l-1}^{l} D^\alpha z(t) d\mu(\alpha) + \int_{m-1}^{c} D^\alpha z(t) d\mu(\alpha),$$

hence

$$\text{Lap}\left[\int_b^c D^\alpha z(t) d\mu(\alpha)\right] = \int_b^0 \lambda^\alpha \widehat{z}(\lambda) d\mu(\alpha) + \sum_{l=1}^{m-1} \int_{l-1}^{l} \left(\lambda^\alpha \widehat{z}(\lambda) - \sum_{k=0}^{l-1} z^{(k)}(0)\lambda^{\alpha-k-1}\right) d\mu(\alpha) +$$

$$+ \int_{m-1}^{c} \left(\lambda^\alpha \widehat{z}(\lambda) - \sum_{k=0}^{m-1} z^{(k)}(0)\lambda^{\alpha-k-1}\right) d\mu(\alpha) = W(\lambda)\widehat{z}(\lambda) - \sum_{k=0}^{m-1} z^{(k)}(0) \frac{W_k(\lambda)}{\lambda^{k+1}} = A\widehat{z}(\lambda)$$

due to the closedness of $A$. Therefore, points (ii) and (iii) of Definition 1 imply that for every $z_k \in D_A$

$$W(\lambda)\widehat{S}_k(\lambda)z_k - \frac{W_k(\lambda)}{\lambda^{k+1}} z_k = A\widehat{S}_k(\lambda)z_k = \widehat{S}_k(\lambda)z_k A z_k.$$

Consequently, there exists an inverse operator for $W(\lambda)I - A : D_A \to \mathcal{Z}$ and Equality (8) holds. The right-hand side of this equation is a bounded operator by the assumptions of this lemma for $S_k$, hence $W(\lambda) \in \rho(A)$. □

**Theorem 2.** *Let $b, c \in \mathbb{R}$, $b < c$, $m-1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b,c];\mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, for some $k \in \{0, 1, \dots, m-1\}$ there exist a $k$-resolving family of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ for Equation (6), such that for all $t \geq 0$ $S_k^{(k)}(t) \in \mathcal{L}(\mathcal{Z})$, $\|S_k^{(k)}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq Ke^{at}$ at some $K \geq 1$, $a \geq 0$. Then $\{S_k^{(k)}(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ is continuous in the point $t = 0$ in the operator norm in $\mathcal{L}(\mathcal{Z})$, if and only if $A \in \mathcal{L}(\mathcal{Z})$.*

**Proof.** For $\mathrm{Re}\lambda > a$ due to Lemma 3

$$\int_0^\infty e^{-\lambda t}(S_k^{(k)}(t) - I)dt = \frac{W_k(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} - \frac{I}{\lambda}.$$

Let the function $\eta(t) := \|S_k^{(k)}(t) - I\|_{\mathcal{L}(\mathcal{Z})}$ is continuous on the segment $[0,1]$ and $\eta(0) = 0$. For $\varepsilon > 0$ take $\delta > 0$, such that $\eta(t) \leq \varepsilon$ for all $t \in [0, \delta]$, hence

$$\left\| \frac{W_k(\lambda)}{\lambda}(W(\lambda)I - A)^{-1} - \frac{I}{\lambda} \right\|_{\mathcal{L}(\mathcal{Z})} \leq \int_0^\delta e^{-\lambda t}\eta(t)dt + \int_\delta^\infty e^{-\lambda t}\eta(t)dt \leq \frac{\varepsilon}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

as $\mathrm{Re}\lambda \to +\infty$, since $\eta(t) \leq Ke^{at} + 1$ for $t \geq 0$. Therefore, for large enough $\mathrm{Re}\lambda > 0$ $\left\|W_k(\lambda)(W(\lambda)I - A)^{-1} - I\right\|_{\mathcal{L}(\mathcal{Z})} < 1$, consequently, the operator $(W(\lambda)I - A)^{-1}$ is continuously invertible, $[(W(\lambda)I - A)^{-1}]^{-1} = W(\lambda)I - A \in \mathcal{L}(\mathcal{Z})$. Thus, $A \in \mathcal{L}(\mathcal{Z})$.

Let $A \in \mathcal{L}(\mathcal{Z})$, $R > \max\{\varrho, (C^{-1}\|A\|_{\mathcal{L}(\mathcal{Z})})^{1/(c-\varepsilon)}\}$, where constants $C, \varrho > 0$, $\varepsilon \in (0, c - m + 1)$ are taken from Lemma 1. Construct the contour $\Gamma_R = \Gamma_{1,R} \cup \Gamma_{2,R} \cup \Gamma_{3,R}$, where $\Gamma_{1,R} = \{Re^{i\varphi} : \varphi \in (-\pi, \pi)\}$, $\Gamma_{2,R} = \{re^{i\pi} : r \in [R, \infty)\}$, $\Gamma_{3,R} = \{re^{-i\pi} : r \in [R, \infty)\}$. For $t \geq 0$ due to Lemma 3 by the inverse Laplace transform we obtain

$$S_k^{(k)}(t) = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{W_k(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda = \frac{1}{2\pi i}\int_{\Gamma_R} \frac{W_k(\lambda)}{\lambda}\sum_{l=0}^\infty \frac{A^l e^{\lambda t}d\lambda}{W(\lambda)^{l+1}} =$$

$$= I + \frac{1}{2\pi i}\int_{\Gamma_R} \frac{1}{\lambda}\sum_{l=1}^\infty \frac{A^l e^{\lambda t}d\lambda}{W(\lambda)^l} + \frac{1}{2\pi i}\int_{\Gamma_R} \frac{W_k(\lambda) - W(\lambda)}{\lambda}\sum_{l=0}^\infty \frac{A^l e^{\lambda t}d\lambda}{W(\lambda)^{l+1}}.$$

The series is convergent, since for $\lambda \in \Gamma_R$ by choosing $R$ $|W(\lambda)|^{-1}\|A\|_{\mathcal{L}(\mathcal{Z})} < 1$ due to Lemma 1, moreover, by Lemma 1 and Lemma 2 we obtain

$$\left\| \frac{A^l}{\lambda W(\lambda)^l} \right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{C_1\|A\|^l_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{(c-\varepsilon)l+1}}, \quad \left\| \frac{(W_k(\lambda) - W(\lambda))A^l}{\lambda W(\lambda)^{l+1}} \right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{C_2\|A\|^l_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{(c-\varepsilon)(l+1)+1-k}}.$$

For small $t > 0$ take $R = 1/t$ and obtain

$$\|S_k^{(k)}(t) - I\|_{\mathcal{L}(\mathcal{Z})} \leq C_3\sum_{k=1}^3 \left( \sum_{l=1}^\infty \int_{\Gamma_{k,R}} \frac{\|A\|^l_{\mathcal{L}(\mathcal{Z})}|d\lambda|}{|\lambda|^{(c-\varepsilon)l+1}} + \sum_{l=0}^\infty \int_{\Gamma_{k,R}} \frac{\|A\|^l_{\mathcal{L}(\mathcal{Z})}|d\lambda|}{|\lambda|^{(c-\varepsilon)(l+1)+1-k}} \right) \leq$$

$$\leq C_4\left( \sum_{l=1}^\infty \frac{\|A\|^l_{\mathcal{L}(\mathcal{Z})}}{R^{(c-\varepsilon)l}} + \sum_{l=0}^\infty \frac{\|A\|^l_{\mathcal{L}(\mathcal{Z})}}{R^{(c-\varepsilon)(l+1)-k}} \right) = \frac{C_4 t^{c-\varepsilon}\|A\|_{\mathcal{L}(\mathcal{Z})}}{1 - t^{c-\varepsilon}\|A\|_{\mathcal{L}(\mathcal{Z})}} + \frac{C_4 t^{c-\varepsilon-k}}{1 - t^{c-\varepsilon}\|A\|_{\mathcal{L}(\mathcal{Z})}} \to 0$$

as $t \to 0+$, since $c - \varepsilon - k \geq c - \varepsilon - m + 1 > 0$. □

**Theorem 3.** *Let $b, c \in \mathbb{R}$, $b < c$, $1 \le m-1 < c \le m$, $\mu \in BV([b,c];\mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, there exist a 0-resolving family of operators $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t \ge 0\}$ for Equation (6), such that for all $t \ge 0$ $\|S_0(t)\|_{\mathcal{L}(\mathcal{Z})} \le Ke^{at}$ at some $K \ge 1$, $a \ge 0$. Then there exist $k$-resolving families $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \ge 0\}$, $k = 1,2,\dots,m-1$, for Equation (6).*

**Proof.** Consider for $k = 1,2,\dots,m-1$ functions

$$\frac{W_k(\lambda)}{\lambda^k W_0(\lambda)} = \frac{1}{\lambda^k} - \frac{W_0(\lambda) - W_k(\lambda)}{\lambda^k W_0(\lambda)}.$$

Due to Lemmas 1 and 2

$$\left| \frac{W_0(\lambda) - W_k(\lambda)}{\lambda^k W_0(\lambda)} \right| \le \frac{C_1}{|\lambda|^{c-\varepsilon}}.$$

For $c > 1$ choose $\varepsilon > 0$, such that $c - \varepsilon > 1$, then there exists the inverse Laplace transform

$$w_k(t) := \int\limits_{d-i\infty}^{d+i\infty} \frac{W_k(\lambda)e^{\lambda t}}{\lambda^k W_0(\lambda)} d\lambda, \quad d > 0, \quad k = 1,2,\dots,m-1,$$

moreover, there exists $K_1 > 0$, such that for all $t \ge 0$ $|w_k(t)| \le K_1 e^{dt}$. For $k = 1$ we take into account that $\lambda^{-1} = \widehat{1}$. The functions $w_k$ are continuous for $t \ge 0$, since integrals converge uniformly with respect to $t$ on every segment. Consequently, functions of the form $S_k(t) := \int\limits_0^t w_k(t-s)S_0(s)ds$, $k = 1,2,\dots,m-1$, are strongly continuous for $t \ge 0$; point (ii) of Definition 1 holds for $S_k$ also. Moreover, $\|S_k(t)\|_{\mathcal{L}(\mathcal{Z})} \le K_2 e^{(a+d)t}$ for all $t \ge 0$.

The derivatives

$$w_k^{(l)}(t) := \int\limits_{d-i\infty}^{d+i\infty} \frac{W_k(\lambda)e^{\lambda t}}{\lambda^{k-l} W_0(\lambda)} d\lambda, \ d > 0, \ l = 1,2,\dots,k-1; \ k = 1,2,\dots,m-1,$$

are continuous for $t \ge 0$ also, since

$$\frac{W_k(\lambda)}{\lambda^{k-l} W_0(\lambda)} = \frac{1}{\lambda^{k-l}} - \frac{W_0(\lambda) - W_k(\lambda)}{\lambda^{k-l} W_0(\lambda)}, \quad \left| \frac{W_0(\lambda) - W_k(\lambda)}{\lambda^{k-l} W_0(\lambda)} \right| \le \frac{C_1}{|\lambda|^{c-l-\varepsilon}}. \tag{9}$$

From relations Equation (9) it follows that $w_k^{(l)}(0) = 0$, $l = 0,1,\dots,k-2$, $w_k^{(k-1)}(0) = 1$. Thus, for $k = 1,2,\dots,m-1$, $z_k \in D_A$

$$S_k^{(l)}(t)z_k = \int\limits_0^t w_k^{(l)}(t-s)S_0(s)z_k ds, \quad l = 1,2,\dots,k-1,$$

$$S_k^{(l)}(t)z_k = w_k^{(k-1)}(t)S_0^{(l-k)}(0)z_k + \int\limits_0^t w_k^{(k-1)}(s)S_0^{(l-k+1)}(t-s)z_k ds, \ l = k,k+1,\dots,m-1,$$

therefore, $S_k(t)z_k$ satisfies initial value conditions Equation (7).

From point (iii) of Definition 1, it follows that

$$W(\lambda)\widehat{S}_0(\lambda)z_k - \lambda^{-1}W_0(\lambda)z_k = A\widehat{S}_0(\lambda)z_k, \quad z_k \in D_A. \tag{10}$$

Since

$$\widehat{S}_k(\lambda) = \widehat{w}_k(\lambda)\widehat{S}_0(\lambda) = \frac{W_k(\lambda)}{\lambda^k W_0(\lambda)}\widehat{S}_0(\lambda),$$

after the multiplying Equation (10) by $\widehat{\omega}_k(\lambda)$ we obtain

$$\frac{W_k(\lambda)}{\lambda^k W_0(\lambda)} W(\lambda)\widehat{S}_0(\lambda)z_k - \frac{W_k(\lambda)}{\lambda^{k+1}} z_k = \frac{W_k(\lambda)}{\lambda^k W_0(\lambda)} A\widehat{S}_0(\lambda)z_k,$$

$$W(\lambda)\widehat{S}_k(\lambda)z_k - \frac{W_k(\lambda)}{\lambda^{k+1}} z_k = A\widehat{S}_k(\lambda)z_k.$$

Acting by the inverse Laplace transform, we obtain that $S_k(t)z_k$ is a solution of Equation (6). □

## 3. Analytic Resolving Families

A resolving family of operators is called *analytic*, if it has an analytic continuation to a sector $\Sigma_{\psi_0}$ at some $\psi_0 \in (0, \pi/2]$. An analytic resolving family of operators $\{S(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ has a type $(\psi_0, a_0)$ at some $\psi_0 \in (0, \pi/2]$, $a_0 \in \mathbb{R}$, if for all $\psi \in (0, \psi_0)$, $a > a_0$ there exists $C(\psi, a) > 0$, such that for all $t \in \Sigma_\psi$ the inequality $\|S(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C(\psi, a)e^{a\mathrm{Re}\, t}$ is satisfied.

**Remark 2.** *Analogous notions of analytic resolving families of operators are used in the study of integral evolution equations [34] and fractional differential equations [36]. They generalize the notion of analytic resolving semigroup of operators for the first order equation $D_t^1 z(t) = Az(t)$ (see [37–39]).*

Following the works [25,26] define a class $\mathcal{A}_W(\theta_0, a_0)$ as the set of all operators $A \in \mathcal{Cl}(\mathcal{Z})$ satisfying the following conditions:

(i) there exist $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, such that $W(\lambda) \in \rho(A)$ for every $\lambda \in S_{\theta_0, a_0}$;
(ii) for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $K(\theta, a) > 0$, such that for all $\lambda \in S_{\theta,a}$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{|\lambda|K(\theta, a)}{|W(\lambda)||\lambda - a|}.$$

**Remark 3.** *The classes $\mathcal{A}_W(\theta_0, a_0)$ in works [25–27] are partial cases of this class with the same denotation $\mathcal{A}_W(\theta_0, a_0)$ due to the more general construction of the distributed derivative in the present work. If $\mu$ is a constant, excluding a unique jump in the point $\alpha = c$, class $\mathcal{A}_W(\theta_0, a_0)$ coincides with the class $\mathcal{A}_\alpha(\theta_0, a_0)$, defined in [36]. Operators from the class $\mathcal{A}_1(\theta_0, a_0)$ are generators of an analytic semigroup of operators exactly [37–39].*

**Remark 4.** *If $A \in \mathcal{L}(\mathcal{Z})$, then for*

$$|\lambda| > (2\|A\|_{\mathcal{L}(\mathcal{Z})})^{1/(c-\varepsilon)} \tag{11}$$

*due to Lemma 1*

$$(W(\lambda)I - A)^{-1} = \sum_{n=0}^{\infty} \frac{A^n}{(W(\lambda))^{n+1}},$$

$$\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{1}{|W(\lambda)|} \sum_{n=0}^{\infty} \frac{\|A\|^n_{\mathcal{L}(\mathcal{Z})}}{|\lambda|^{(c-\varepsilon)n}} \leq \frac{2|\lambda|}{|W(\lambda)||\lambda - a|}\left(1 + \frac{a}{|\lambda|}\right).$$

*We can choose $a_0 = 2(2\|A\|_{\mathcal{L}(\mathcal{Z})})^{1/(c-\varepsilon)}$, $\theta_0 = \pi/4$, then for $\lambda \in S_{\theta_0,a_0}$ we have inequalities Equation (11) and $1 + \frac{a_0}{|\lambda|} \leq 1 + \sqrt{2}$. Thus, $A \in \mathcal{A}_W(\theta_0, a_0)$. In this reasoning we may take $a_0$ greater and $\theta_0$ closer to $\pi$, if necessary.*

For $A \in \mathcal{A}_W(\theta_0, a_0)$ the operators

$$Z_k(t) := \frac{1}{2\pi i}\int_{\Gamma} \frac{W_k(\lambda)}{\lambda^{k+1}}(W(\lambda)I - A)^{-1}e^{\lambda t}d\lambda, \quad k = 0, 1, \ldots, m-1,$$

are defined at $t > 0$. Here $\Gamma = \Gamma_+ \cup \Gamma_- \cup \Gamma_0$, $\Gamma_\pm = \{\mu \in \mathbb{C} : \mu = a + re^{\pm i\theta},\ r \in (\delta, \infty)\}$, $\Gamma_0 = \{\mu \in \mathbb{C} : \mu = a + \delta e^{i\varphi},\ \varphi \in (-\theta, \theta)\}$ for some $\delta > 0$, $a > a_0$, $\theta \in (\pi/2, \theta_0)$.

**Theorem 4.** *Let $b, c \in \mathbb{R}$, $b < c$, $m-1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b,c];\mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$. Then there exists an analytic 0-resolving family of operators of the type $(\theta_0 - \pi/2, a_0)$ for Equation (6), if and only if $A \in \mathcal{A}_W(\theta_0, a_0)$. In this case, there exists a unique $k$-resolving family of operators for every $k = 0, 1, \ldots, m-1$, it has the form $\{Z_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$.*

**Proof.** Let $A \in \mathcal{A}_W(\theta_0, a_0)$, $R > \delta$, $\Gamma_{1,R} = \Gamma_0$, $\Gamma_{2,R} = \{\lambda \in \mathbb{C} : \lambda = a + Re^{i\varphi},\ \varphi \in (-\theta, \theta)\}$, $\Gamma_{3,R} = \{\lambda \in \mathbb{C} : \lambda = a + re^{i\theta},\ r \in [\delta, R]\}$, $\Gamma_{4,R} = \{\lambda \in \mathbb{C} : \lambda = a + re^{-i\theta},\ r \in [\delta, R]\}$, $\Gamma_R = \bigcup_{k=1}^{4}\Gamma_{k,R}$ is the positively oriented closed loop, $\Gamma_{5,R} = \{\lambda \in \mathbb{C} : \lambda = a + re^{i\theta},\ r \in [R, \infty)\}$, $\Gamma_{6,R} = \{\lambda \in \mathbb{C} : \lambda = a + re^{-i\theta},\ r \in [R, \infty)\}$, then we have $\Gamma = \Gamma_{5,R} \cup \Gamma_{6,R} \cup \Gamma_R \setminus \Gamma_{2,R}$.

For $t > 0$, $z_0 \in D_A$

$$Z_0(t)z_0 = \frac{1}{2\pi i}\int_{\Gamma} \frac{W_0(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}z_0 e^{\lambda t}d\lambda = \frac{1}{2\pi i}\int_{\Gamma} \frac{e^{\lambda t}}{\lambda}z_0 d\lambda +$$

$$+\frac{1}{2\pi i}\int_{\Gamma} \frac{e^{\lambda t}}{\lambda}(W(\lambda)I - A)^{-1}Az_0 d\lambda + \frac{1}{2\pi i}\int_{\Gamma} \frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}z_0 d\lambda.$$

For $t \in [0,1]$, $\lambda \in \Gamma \setminus \{\lambda \in \mathbb{C} : |\lambda| \leq \varrho\}$

$$\left\| \frac{e^{\lambda t}}{\lambda}(W(\lambda)I - A)^{-1}Az_0 \right\|_{\mathcal{Z}} \leq \frac{e^{a+\delta}K(\theta, a)\|Az_0\|_{\mathcal{Z}}}{|W(\lambda)||\lambda - a|} \leq \frac{C_1}{|\lambda|^{c+1-\varepsilon}},$$

$$\left\| \frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}e^{\lambda t}z_0 \right\|_{\mathcal{Z}} \leq \frac{Ce^{a+\delta}K(\theta, a)\|z_0\|_{\mathcal{Z}}}{|W(\lambda)||\lambda - a|} \leq \frac{C_1}{|\lambda|^{c+1-\varepsilon}},$$

where we can take any $\varepsilon \in (0, c)$, see Lemma 1. Since $c + 1 - \varepsilon > 1$, the integral $Z_0(t)$ converges uniformly with respect to $t \in [0,1]$ and by the continuity

$$Z_0(0)z_0 = z_0 + \frac{1}{2\pi i}\int_{\Gamma} \frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0 d\lambda +$$

$$+\frac{1}{2\pi i}\int_{\Gamma} \frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}z_0 d\lambda =$$

$$= z_0 + \lim_{R\to\infty}\frac{1}{2\pi i}\left(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}}\right)\frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0 d\lambda +$$

$$+ \lim_{R\to\infty}\frac{1}{2\pi i}\left(\int_{\Gamma_R} - \int_{\Gamma_{2,R}} + \int_{\Gamma_{5,R}} + \int_{\Gamma_{6,R}}\right)\frac{W_0(\lambda) - W(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}z_0 d\lambda = z_0$$

due to Cauchy theorem and inequalities

$$\left\| \int_{\Gamma_{s,R}} \frac{1}{\lambda}(W(\lambda)I - A)^{-1}Az_0 d\lambda \right\|_{\mathcal{Z}} \leq \frac{C_2}{R^{c-\varepsilon}}, \quad s = 2, 5, 6,$$

$$\left\| \int\limits_{\Gamma_{s,R}} \frac{W_0(\lambda)-W(\lambda)}{\lambda}(W(\lambda)I-A)^{-1}z_0 d\lambda \right\|_{\mathcal{Z}} \leq \frac{C_2}{R^{c-\varepsilon}}, \quad s=2,5,6.$$

Analogously, for $t>0$, $z_0 \in D_A$, $k \in \{1,2,\dots,m-1\}$

$$Z_0^{(k)}(t)z_0 = \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{k-1}W_0(\lambda)(W(\lambda)I-A)^{-1}e^{\lambda t}z_0 d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{k-1}e^{\lambda t}d\lambda z_0 + \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{k-1}(W(\lambda)I-A)^{-1}Ae^{\lambda t}z_0 d\lambda +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{k-1}(W_0(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}e^{\lambda t}z_0 d\lambda,$$

for $t \in [0,1]$, $\lambda \in \Gamma \setminus \{\lambda \in \mathbb{C} : |\lambda| \leq \varrho\}$

$$\left\| \lambda^{k-1}(W(\lambda)I-A)^{-1}e^{\lambda t}Az_0 \right\|_{\mathcal{Z}} \leq \frac{|\lambda|^k e^{a+\delta}K(\theta,a)\|Az_0\|_{\mathcal{Z}}}{|W(\lambda)||\lambda-a|} \leq \frac{C_1}{|\lambda|^{c+1-\varepsilon-k}},$$

$$\left\| \lambda^{k-1}(W_0(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}e^{\lambda t}z_0 \right\|_{\mathcal{Z}} \leq \frac{C|\lambda|^k e^{a+\delta}K(\theta,a)\|z_0\|_{\mathcal{Z}}}{|W(\lambda)||\lambda-a|} \leq \frac{C_1}{|\lambda|^{c+1-\varepsilon-k}},$$

$$Z_0^{(k)}(0)z_0 = 0 + \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{k-1}(W(\lambda)I-A)^{-1}Az_0 d\lambda +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{k-1}(W_0(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}z_0 d\lambda = 0,$$

since $c+1-\varepsilon-k>1$ for $\varepsilon \in (0, c-m+1)$.

Consider for $t>0$, $z_k \in D_A$, $k=1,2,\dots,m-1$, $l=0,1,\dots,m-1$

$$Z_k^{(l)}(t)z_k = \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{l-k-1}W_k(\lambda)(W(\lambda)I-A)^{-1}e^{\lambda t}z_k d\lambda =$$

$$= \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{l-k-1}e^{\lambda t}d\lambda z_k + \frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{l-k-1}(W(\lambda)I-A)^{-1}e^{\lambda t}Az_k d\lambda +$$

$$+\frac{1}{2\pi i}\int\limits_{\Gamma} \lambda^{l-k-1}(W_k(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}e^{\lambda t}z_k d\lambda,$$

for $t \in [0,1]$, $\lambda \in \Gamma \setminus \{\lambda \in \mathbb{C} : |\lambda| \leq \varrho\}$

$$\left\| \lambda^{l-k-1}(W(\lambda)I-A)^{-1}e^{\lambda t}Az_k \right\|_{\mathcal{Z}} \leq \frac{|\lambda|^{l-k} e^{a+\delta}K(\theta,a)\|Az_k\|_{\mathcal{Z}}}{|W(\lambda)||\lambda-a|} \leq \frac{C_1}{|\lambda|^{c+1-\varepsilon-l+k}},$$

$$\left\| \lambda^{l-k-1}(W_k(\lambda)-W(\lambda))(W(\lambda)I-A)^{-1}e^{\lambda t}z_k \right\|_{\mathcal{Z}} \leq \frac{C_2|\lambda|^l e^{a+\delta}K(\theta,a)}{|W(\lambda)||\lambda-a|} \leq \frac{C_1}{|\lambda|^{c+1-\varepsilon-l}}.$$

Then for $k \neq l$ $Z_k^{(l)}(0)z_k = 0$, $Z_k^{(k)}(0)z_k = z_k$.

Thus, the functions $Z_k(t)z_k \in C^{m-1}(\overline{\mathbb{R}}_+;\mathcal{Z})$ for $z_k \in D_A$ satisfy Cauchy conditions Equation (7) with the correponding $k \in \{0,1,\dots,m-1\}$. Since the operator $A$ is closed and commutes with $(W(\lambda)I-A)^{-1}$ on $D_A$, at $z_k \in D_A$ $AZ_k(t)z_k \in C(\mathbb{R}_+;\mathcal{Z})$, so, $Z_k(t)z_k \in C(\mathbb{R}_+;D_A)$, $k=0,1,\dots,m-1$.

For $\mathrm{Re}\,\nu > a$ we have

$$\widehat{Z}_k(\nu) = \frac{1}{2\pi i}\int\limits_{\Gamma} \frac{W_k(\lambda)}{\lambda^{k+1}(\nu-\lambda)}(W(\lambda)I - A)^{-1}d\lambda.$$

For $\lambda \in S_{\theta,a}$

$$\left\|\frac{W_k(\lambda)}{\lambda^{k+1}(\nu-\lambda)}(W(\lambda)I - A)^{-1}\right\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta,a)}{|\lambda|^k|\lambda-a||\lambda-\nu|},$$

hence,

$$\lim_{R\to\infty}\frac{1}{2\pi i}\int\limits_{\Gamma_{s,R}} \frac{W_k(\lambda)}{\lambda^{k+1}(\nu-\lambda)}(W(\lambda)I - A)^{-1}d\lambda = 0, \quad s = 2,5,6,$$

and by the Cauchy integral formula

$$\widehat{Z}_k(\nu) = \lim_{R\to\infty}\frac{1}{2\pi i}\int\limits_{\Gamma_R} \frac{W_k(\lambda)}{\lambda^{k+1}(\nu-\lambda)}(W(\lambda)I - A)^{-1}d\lambda = \frac{W_k(\nu)}{\nu^{k+1}}(W(\nu)I - A)^{-1}.$$

Take in Theorem 1

$$H_k(\lambda) = \frac{W_k(\lambda)}{\lambda^{k+1}}(W(\lambda)I - A)^{-1}, \quad F_k = Z_k, \quad k = 0,1,\ldots,m-1,$$

then due to the inclusion $A \in \mathcal{A}_W(\theta_0, a_0)$ and Lemma 1 for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$

$$\|H_k(\lambda)\|_{\mathcal{L}(\mathcal{Z})} = \frac{1}{|\lambda|^{k+1}}\Big(\|(W_k(\lambda) - W(\lambda))(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} +$$

$$+\|W(\lambda)(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})}\Big) \leq \frac{K(\theta,a)}{|\lambda-a|}\left(\frac{1}{|W(\lambda)|}+1\right) \leq \frac{K_1(\theta,a)}{|\lambda-a|}$$

and by Theorem 1 the mappings $Z_k : \Sigma_{\theta_0-\pi/2} \to \mathcal{L}(\mathcal{Z})$ are analytic and for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$ there exists $C_k(\theta,a) > 0$ such that for all $t \in \Sigma_{\theta-\pi/2}$ $\|Z_k(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_k(\theta,a)e^{a\mathrm{Re}\,t}$, $k = 0,1,\ldots,m-1$.

For $z_k \in D_A$ put $x_k(t) := Z_k(t)z_k$, then $\widehat{Ax}_k(\lambda) = \lambda^{-k-1}W_k(\lambda)(W(\lambda)I - A)^{-1}Az_k$. Hence $\widehat{x}_k(\lambda) \in D_A$, $A\widehat{x}_k(\lambda) = \widehat{Ax}_k(\lambda)$, $\widehat{x}_k(\lambda)$ and $\widehat{Ax}_k(\lambda)$ have analytic extensions on $S_{\theta_0,a_0}$, since $A \in \mathcal{A}_W(\theta_0, a_0)$. By Formula (4) of the Laplace transform

$$\mathrm{Lap}\left[\int\limits_b^c D^\alpha x_k(t)d\mu(\alpha)\right](\lambda) = \frac{W(\lambda)W_k(\lambda)}{\lambda^{k+1}}(W(\lambda)I - A)^{-1}z_k - \frac{W_k(\lambda)}{\lambda^{k+1}}z_k =$$

$$= \frac{W_k(\lambda)}{\lambda^{k+1}}(W(\lambda)I - A)^{-1}Az_k = \widehat{Ax}_k(\lambda).$$

Apply the inverse Laplace transform to both sides of the obtained equality and get equality Equation (6) for all points of the function $Ax_k$ continuity, hence for all $t > 0$. Therefore, $x_k$ is a solution of problem Equations (6) and (7) and $\{Z_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ is an analytic $k$-resolving family of operators of the type $(\theta_0 - \pi/2, a_0)$ for Equation (6), $k = 0,1,\ldots,m-1$.

If there exists an analytic 0-resolving family of operators $\{S_0(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ of the type $(\theta_0 - \pi/2, a_0)$ for Equation (6), by Lemma 3

$$\widehat{S}_0(\lambda) = \frac{W_0(\lambda)}{\lambda}(W(\lambda)I - A)^{-1}, \quad \mathrm{Re}\lambda > a_0.$$

Theorem 1 implies that $A \in \mathcal{A}_W(\theta_0, a_0)$, $S_0(t) \equiv Z_0(t)$ by virtue of the uniqueness of the inverse Laplace transform.

By Theorem 3 there exist $k$-resolving families of operators $\{S_k(t) \in \mathcal{L}(\mathcal{Z}) : t \geq 0\}$ for Equation (6), $k = 1, 2, \ldots, m-1$, such that

$$\widehat{S}_k(\lambda) = \frac{W_k(\lambda)}{\lambda^{k+1}}(W(\lambda)I - A)^{-1}, \quad \mathrm{Re}\lambda > a_0.$$

This equality implies that $S_k(t) \equiv Z_k(t)$, $k = 1, 2, \ldots, m-1$. □

**Theorem 5.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b,c];\mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, $A \in \mathcal{A}_W(\theta_0, a_0)$. Then for any $z_k \in D_A$, $k = 0, 1, \ldots, m-1$, the function $z(t) = \sum\limits_{k=0}^{m-1} Z_k(t)z_k$ is a unique solution of problems (5) and (6). In this case, the solution is analytic in the sector $\Sigma_{\theta_0 - \pi/2}$.*

**Proof.** From Theorem 4 and linearity of problems (5) and (6) it follows that this funtion $z$ is an analytic in the sector $\Sigma_{\theta_0-\pi/2}$ solution of the problem.

If there exist two solutions $y_1, y_2$ of problems (5) and (6), then their difference $y = y_1 - y_2$ is a solution of Equation (6), which satisfies the initial value conditions $y^{(k)}(0) = 0$, $k = 0, 1, \ldots, m-1$. Take $T > 0$ and redefine the function $y$ on $[T, \infty)$ by zero, denote the obtained function by $y_T$. It satisfies Equation (6) on $\mathbb{R}_+$, excluding, possibly, the point $t = T$, where the function $y_T$ may be discontinuous. Acting by the Laplace transform on both parts of Equation (6) and due to the initial conditions, we get the equality $W(\lambda)\widehat{y}_T(\lambda) = A\widehat{y}_T(\lambda)$. Since $A \in \mathcal{A}_W(\theta_0, a_0)$, for $\lambda \in S_{\theta_0, a_0}$ we get the identity $\widehat{y}_T(\lambda) = 0$. This means that $y_T \equiv 0$ and $y_1(t) = y_2(t)$ for $t \in [0, T)$. We can choose arbitrary large $T > 0$, therefore, $y_1 \equiv y_2$ on $\overline{\mathbb{R}}_+$. □

**Theorem 6.** *Let $b, c \in \mathbb{R}$, $b < c$, $2 < c$, $\mu \in BV([b,c];\mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, $\mu(t) \in \mathbb{R}$ for all $t$ from some left neighbourhood of $c$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$. Then $A \in \mathcal{L}(\mathcal{Z})$.*

**Proof.** Due to the definition of the Riemann–Stieltjes integral for a small $\delta_1 > 0$ there exists a division $\{\alpha_0 = b, \alpha_1, \ldots, \alpha_{n-1}, \alpha_n = c\}$ of the segment $[b, c]$ with a sufficiently small radius $r := \max\{\alpha_k - \alpha_{k-1} : k = 1, 2, \ldots, n\}$ and with any intermediate points $\xi_k \in [\alpha_{k-1}, \alpha_k]$, $k = 1, 2, \ldots, n$, such that

$$\left| W(\lambda) - \sum_{j=1}^{n} \lambda^{\xi_j}(\mu(\alpha_j) - \mu(\alpha_{j-1})) \right| < \delta_1,$$

for all sufficiently large $|\lambda|$ due to Lemma 1 $|W(\lambda)| > 1$, therefore, for some $d > 0$

$$\left| \arg W(\lambda) - \arg\left( \sum_{j=1}^{n} \lambda^{\xi_j}(\mu(\alpha_j) - \mu(\alpha_{j-1})) \right) \right| < d\delta_1.$$

Since $\lim\limits_{|\lambda| \to \infty} \sum\limits_{j=1}^{n-1} \lambda^{\xi_j - \xi_n}(\mu(\alpha_j) - \mu(\alpha_{j-1})) = 0$ and $\mu(\alpha_n) - \mu(\alpha_{n-1}) \in \mathbb{R}$ for a sufficiently small radius $r > 0$, for every $\gamma > 0$ there exists $\delta > 0$, such that for all $|\lambda| > \delta$

$$\xi_n \arg\lambda - \frac{\gamma}{2} \leq \arg\left( \sum_{j=1}^{n} \lambda^{\xi_j}(\mu(\alpha_j) - \mu(\alpha_{j-1})) \right) =$$

$$= \arg\lambda^{\xi_n} + \arg\left( \sum_{j=1}^{n-1} \lambda^{\xi_j - \xi_n}(\mu(\alpha_j) - \mu(\alpha_{j-1})) + \mu(\alpha_n) - \mu(\alpha_{n-1}) \right) \leq \xi_n \arg\lambda + \frac{\gamma}{2}.$$

Choosing $\xi_n = c$ and sufficiently small $\delta_1 < \gamma/(2d)$ we obtain

$$|\arg W(\lambda) - c \arg \lambda| < \gamma. \tag{12}$$

Since $c > 2$, there exist $\lambda \in S_{\theta_0,a_0}$, such that $|\arg W(\lambda)| > \pi$. We have also

$$C|\lambda|^{c-\varepsilon} \leq |W(\lambda)| \leq C_1|\lambda|^c. \tag{13}$$

Further, without limitation of generality, we can assume that $C_1 \geq 1$.

For arbitrary $\nu_0 \in \mathbb{C}$, such that $|\nu_0| > (C_1\delta)^c + 1$, take $\lambda_0 = \nu_0^{1/c}$, then $|\lambda_0| > \delta$, $\arg \lambda_0 = \arg \nu_0/c \in (-\pi/c, \pi/c)$, $\lambda_0 \in S_{\theta_0,a_0}$ for sufficiently large $\delta > 0$. The boundary of the region

$$\Omega_{\lambda_0} := \left\{ \lambda \in \mathbb{C} : \left( \frac{|\nu_0| - 1}{C} \right)^{1/(c-\varepsilon)} < |\lambda| < \left( \frac{|\nu_0| + 1}{C_1} \right)^{1/c}, \left| \arg \lambda - \frac{\arg \nu_0}{c} \right| < \frac{2\gamma}{c} \right\},$$

which belongs to $S_{\theta_0,a_0}$ for small enough $\gamma > 0$, is mapped by the function $\nu = W(\lambda)$ into the contour, for the point of which due to inequalities (12) and (13) $||\nu| - |\nu_0|| \geq 1$, $|\arg \nu - \arg \nu_0| \geq \gamma$. Therefore, $\nu_0$ lies inside the contour $W[\partial\Omega_{\lambda_0}]$ and is the image of some point from $\Omega_{\lambda_0}$. Thus, $\{\nu \in \mathbb{C} : |\nu| > (C_1\delta)^c + 1\} \subset W[S_{\theta_0,a_0}] \subset \rho(A)$, since $A \in \mathcal{A}_W(\theta_0, a_0)$. Moreover, for sufficiently large $|\nu|$, where $\nu = W(\lambda)$,

$$\|\nu R_\nu(A)\|_{\mathcal{L}(\mathcal{Z})} \leq \frac{K(\theta, a)|\lambda|}{|\lambda - a|} \leq C_2,$$

hence Lemma 5.2 [40] implies the boundedness of the operator $A$. $\square$

## 4. Perturbations of Operators of the Class $\mathcal{A}_W(\theta_0, a_0)$

The result of this section generalizes the perturbation theorem for analytic semigroups of operators [39] and a similar result for generators of resolving families of the distributed fractional derivative of the partial form in [26].

**Theorem 7.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b, c]; \mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, $A \in \mathcal{A}_W(\theta_0, a_0)$ for some $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, $B \in \mathcal{C}l(\mathcal{Z})$, for all $x \in D_A \subset D_B$*

$$\|Bx\|_{\mathcal{Z}} \leq \beta\|Ax\|_{\mathcal{Z}} + \gamma\|x\|_{\mathcal{Z}}, \tag{14}$$

*where $\beta, \gamma \geq 0$, there exists $q \in (0, 1)$ such that $\beta(1 + K(\theta, a)) < q$ for every $\theta \in (\pi/2, \theta_0)$, $a > a_0$. Then $A + B \in \mathcal{A}_W(\theta_0, a_1)$ for sufficiently large $a_1 > a_0$.*

**Proof.** Take $k > \sin^{-1}\theta_0$, $\lambda \in S_{\theta,ka} \subset S_{\theta,a}$ for some $\theta \in (\pi/2, \theta_0)$, $a > a_0$, then inequality (14) implies that

$$\|B(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq \beta\|A(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} + \gamma\|(W(\lambda)I - A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \leq$$

$$\leq \beta\left(1 + \frac{|\lambda|K_A(\theta, a)}{|\lambda - a|}\right) + \frac{\gamma|\lambda|K_A(\theta, a)}{|\lambda - a||W(\lambda)|},$$

where $K_A$ is the constant from the definition of the class $\mathcal{A}_W(\theta_0, a_0)$. Note that the value

$$\frac{|\lambda|}{|\lambda - a|} \leq \frac{1}{1 - \frac{a}{|\lambda|}} \leq \frac{1}{1 - \frac{1}{k \sin \theta_0}}$$

is close to 1 and

$$\frac{|\lambda|}{|\lambda - a||W(\lambda)|} \leq \frac{1}{\left(1 - \frac{1}{k \sin \theta_0}\right) C(ka_0 \sin \theta_0)^{c-\varepsilon}}$$

is close to 0 for a large enough number $k$. Here $C$ is the constant from Lemma 1. Hence for such a $k$ the inequality

$$\|B(W(\lambda)I-A)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \le \beta\left(1+\frac{K_A(\theta,a)}{1-\frac{1}{k\sin\theta_0}}\right)+\frac{\gamma K_A(\theta,a)}{\left(1-\frac{1}{k\sin\theta_0}\right)C(ka_0\sin\theta_0)^{c-\varepsilon}} \le q < 1$$

holds. Further, we have

$$(W(\lambda)I-A-B)^{-1} \le (W(\lambda)I-A)^{-1}(I-B(W(\lambda)I-A)^{-1})^{-1} =$$

$$= (W(\lambda)I-A)^{-1}\sum_{n=0}^{\infty}[B(W(\lambda)I-A)^{-1}]^n,$$

$$\frac{|\lambda-ka|}{|\lambda-a|} = \left|1-\frac{(k-1)a}{\lambda-a}\right| \le 1+\frac{(k-1)a}{|\lambda-a|} \le 1+\frac{1}{\sin\theta_0}.$$

Therefore,

$$\|(W(\lambda)I-A-B)^{-1}\|_{\mathcal{L}(\mathcal{Z})} \le \frac{|\lambda|K_A(\theta,a)}{(1-q)|\lambda-a||W(\lambda)|} \le \frac{|\lambda|K_A(\theta,a)\left(1+\frac{1}{\sin\theta_0}\right)}{(1-q)|\lambda-ka||W(\lambda)|}.$$

Thus, $A+B \in \mathcal{A}_W(\theta_0,a_1)$, $a_1 = ka_0$, we can take for all $\theta \in (\pi/2,\theta_0)$, $a > a_1$

$$K_{A+B}(\theta,a) = \frac{K_A(\theta,a/k)}{1-q}\left(1+\frac{1}{\sin\theta_0}\right).$$

□

**Remark 5.** *Any bounded operator $B \in \mathcal{L}(\mathcal{Z})$ satisfies (14) with $\beta = 0$, $\gamma = \|B\|_{\mathcal{L}(\mathcal{Z})}$.*

## 5. Inhomogeneous Equation

A solution to the Cauchy problem

$$z^{(k)}(0) = z_k, \quad k = 0,1,\ldots,m-1, \tag{15}$$

for the inhomogeneous equation

$$\int_b^c D^\alpha z(t)d\mu(\alpha) = Az(t)+g(t), \quad t \in (0,T], \tag{16}$$

where $b < c$, $m-1 < c \le m \in \mathbb{N}$, $\mu \in BV([b,c];\mathbb{C})$, $T > 0$, $g \in C([0,T];\mathcal{Z})$, is a function $z \in C^{m-1}([0,T];\mathcal{Z}) \cap C((0,T];D_A)$, such that there exists $\int_b^c D^\alpha z(t)d\mu(\alpha) \in C((0,T];\mathcal{Z})$ and equalities (15) and (16) are fulfilled.

Denote

$$Z(t) := \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(W(\lambda)I-A)^{-1}d\lambda.$$

**Lemma 4.** *Let $b,c \in \mathbb{R}$, $b < c$, $m-1 < c \le m \in \mathbb{N}$, $\mu \in BV([b,c];\mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, $\theta_0 \in (\pi/2,\pi]$, $a_0 \ge 0$, $A \in \mathcal{A}_W(\theta_0,a_0)$, $g \in C([0,T];D_A)$. Then the function*

$$z_g(t) = \int_0^t Z(t-s)g(s)ds \tag{17}$$

*is a unique solution of problems (15) and (16) with $z_k = 0$, $k = 0,1,\ldots,m-1$.*

**Proof.** Due to Lemma 1 and the inclusion $A \in \mathcal{A}_W(\theta_0, a_0)$, we have for $\varepsilon \in (0, c - m + 1)$

$$\|Z^{(k)}(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_1 \int\limits_{\Gamma} \frac{e^{t \operatorname{Re} \lambda} |d\lambda|}{|\lambda|^{c-\varepsilon-k}}, \quad k = 0, 1, \ldots, m-1.$$

Since $c - \varepsilon - k \geq c - \varepsilon - m + 2 > 1$ for $k = 0, 1, \ldots, m-2$, we have $Z^{(k)}(0) = 0$. For $k = m - 1$

$$\int\limits_{\Gamma_0} \frac{e^{t \operatorname{Re} \lambda} |d\lambda|}{|\lambda|^{c-\varepsilon-m+1}} \leq 2\pi \delta^{m-c+\varepsilon} e^{a+\delta},$$

$$\int\limits_{\Gamma_\pm} \frac{e^{t \operatorname{Re} \lambda} |d\lambda|}{|\lambda|^{c-\varepsilon-m+1}} \leq \int\limits_{\delta}^{\infty} \frac{e^{rt\cos\theta} dr}{r^{c-\varepsilon-m+1}} = (-t\cos\theta)^{c-\varepsilon-m} \Gamma(m - c + \varepsilon).$$

Thus, $\|Z^{(m-1)}(t)\|_{\mathcal{L}(\mathcal{Z})} = O(t^{c-\varepsilon-m})$ as $t \to 0+$,

$$z_g^{(k)}(t) = \int\limits_0^t Z^{(k)}(t-s) g(s) ds,$$

$z_g^{(k)}(0) = 0$, $k = 0, 1, \ldots, m-2$, $\|z_g^{(m-1)}(t)\| \leq C_1 t^{c-\varepsilon-m+1} \to 0$ as $t \to 0+$. Therefore, conditions (15) with $z_k = 0$, $k = 0, 1, \ldots, m-1$, are fulfilled.

Define $g(t) \equiv 0$ at $t > T$, then $z_g$ is defined on $\mathbb{R}_+$, $z_g = Z * g$, $\widehat{z}_g = \widehat{Z}\widehat{g}$. Besides, $\widehat{Z}(\lambda) = (W(\lambda)I - A)^{-1}$, since due Lemma 1 and the inclusion $A \in \mathcal{A}_W(\theta_0, a_0)$ at $\lambda \in \Gamma \setminus \{\kappa \in \mathbb{C} : |\kappa| < \varrho\}$ for $\operatorname{Re}\nu > a$

$$\left\| \frac{1}{\nu - \lambda} (W(\lambda)I - A)^{-1} \right\| \leq \frac{C_3}{|\lambda|^{c-\varepsilon+1}}.$$

Since $c - \varepsilon + 1 > 1$,

$$\operatorname{Lap}\left[\int\limits_0^b D^\alpha z_g d\mu(\alpha)\right](\nu) = W(\nu)(W(\nu)I - A)^{-1}\widehat{g}(\nu) = \widehat{g}(\nu) + A(W(\nu)I - A)^{-1}\widehat{g}(\nu).$$

Acting by the inverse Laplace transform, we get

$$\int\limits_0^b D^\alpha z_g(t) d\mu(\alpha) = g(t) + A(Z * g)(t) = g(t) + A z_g(t).$$

Here due to the inclusion $g \in C([0, T]; D_A)$, the closedness of the operator $A$ and the commutation of operators $Z(t)$, $t \in \mathbb{R}_+$, with $A$ by their construction, the values $A(Z * g)(t) = Z * Ag(t)$, $t \in (0, T]$, are defined.

The proof of the uniqueness of problems (15) and (16) solution is the same as in the proof of Theorem 5. □

Denote by $C^\gamma([0, T]; \mathcal{Z})$ for $\gamma \in (0, 1]$ the class of functions $f : [0, T] \to \mathcal{Z}$, such that for all $t, s \in [0, T]$ the inequality $\|f(t) - f(s)\|_{\mathcal{Z}} \leq C|t - s|^\gamma$ is satisfied with some $C > 0$.

**Lemma 5.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b, c]; \mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_W(\theta_0, a_0)$, $\gamma \in (0, 1]$, $g \in C^\gamma([0, T]; \mathcal{Z})$. Then function (17) is a unique solution to problems (15) and (16) with $z_k = 0$, $k = 0, 1, \ldots, m-1$.*

**Proof.** Denote for $t > 0$

$$Y(t) = \frac{1}{2\pi i} \int\limits_{\Gamma} (W(\lambda) - W_0(\lambda))(W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda.$$

Due to the closedness of $A$ we have for $t > 0$

$$AZ(t) = \frac{1}{2\pi i} \int\limits_{\Gamma} A(W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda = \frac{1}{2\pi i} \int\limits_{\Gamma} W(\lambda)(W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda =$$

$$= Y(t) + D^1 Z_0(t), \quad \|Y(t)\|_{\mathcal{L}(\mathcal{Z})} \leq C_1 \int\limits_{\Gamma} \frac{e^{t\mathrm{Re}\lambda} |d\lambda|}{|\lambda|^{c-\varepsilon}},$$

hence $\mathrm{im} Z(t) \subset D_A$. Arguing as in the previous proof, we obtain that $\|Y(t)\|_{\mathcal{L}(\mathcal{Z})} = O(t^{c-\varepsilon-1})$ as $t \to 0+$.

Since

$$\left\| \int\limits_{\Gamma} W(\lambda)(W(\lambda)I - A)^{-1} e^{\lambda t} d\lambda \right\|_{\mathcal{L}(\mathcal{Z})} \leq C_2 + K(\theta, a) \int\limits_{\Gamma_\pm} \frac{|\lambda| e^{t\mathrm{Re}\,\lambda}}{|\lambda - a|} ds \leq \frac{C_3}{-t\cos\theta},$$

we have $\|AZ(t)\|_{\mathcal{L}(\mathcal{Z})} = O(t^{-1})$ as $t \to 0+$, therefore,

$$\|AZ(t-s)(g(s) - g(t))\|_{\mathcal{Z}} \leq C_3 |t-s|^{\gamma - 1}.$$

Then the integral

$$\int\limits_0^t AZ(t-s)g(s)ds = \int\limits_0^t AZ(t-s)(g(s) - g(t))ds + \int\limits_0^t Y(t-s)g(t)ds + (Z_0(t) - I)g(t)$$

converges, since

$$\left\| \int\limits_0^t AZ(t-s)g(s)ds \right\|_{\mathcal{Z}} \leq C_4(t^\gamma + t^{c-\varepsilon}) + \|(Z_0(t) - I)g(t)\|_{\mathcal{Z}}.$$

Therefore, $z_g(t) \in D_A$, $z_g \in C([0,T]; D_A)$.

The rest of the proof is the same as for Lemma 4. □

Theorem 5, Lemmas 4 and 5 imply the following assertion.

**Theorem 8.** *Let $b, c \in \mathbb{R}$, $b < c$, $m - 1 < c \leq m \in \mathbb{N}$, $\mu \in BV([b,c]; \mathbb{C})$, $c$ be a variation point of the measure $d\mu(t)$, $\theta_0 \in (\pi/2, \pi]$, $a_0 \geq 0$, $A \in \mathcal{A}_W(\theta_0, a_0)$, $g \in C([0,T]; D_A) \cup C^\gamma([0,T]; \mathcal{Z})$, $\gamma \in (0,1]$, $z_k \in D_A$, $k = 0, 1, \ldots, m-1$. Then the function*

$$z(t) = \sum_{k=0}^{m-1} Z_k(t) z_k + \int\limits_0^t Z(t-s)g(s)ds$$

*is a unique solution of problems (15) and (16).*

## 6. Application to an Initial-Boundary Value Problem

Let $\Omega \subset \mathbb{R}^d$ be a bounded region with a boundary $\partial\Omega$ of the class $C^\infty$, $\beta, \gamma, \delta, \nu \in \mathbb{R}$, $b \leq 0$, $c \in (1,2)$, $\alpha_1 < \alpha_2 < \cdots < \alpha_n \leq c$, $\omega_j \in \mathbb{R} \setminus \{0\}$, $j = 1, 2, \ldots, n$, $\omega \in C([b,c];\mathbb{R})$, $\omega(c) \neq 0$ in a some left vicinity of $c$. Consider the initial-boundary value problem

$$u(s,0) = u_0(s), \quad v(s,0) = v_0(s), \quad s \in \Omega, \tag{18}$$

$$\frac{\partial u}{\partial t}(s,0) = u_1(s), \quad \frac{\partial v}{\partial t}(s,0) = v_1(s), \quad s \in \Omega, \tag{19}$$

$$u(s,t) = v(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T], \tag{20}$$

for the system of equations in $\Omega \times (0,T]$

$$\sum_{j=1}^{n} \omega_j D_t^{\alpha_j} u(s,t) + \int_b^c \omega(\alpha) D_t^\alpha u(s,t) d\alpha = \Delta u(s,t) - \Delta v(s,t) + f_1(s,t), \tag{21}$$

$$\sum_{j=1}^{n} \omega_j D_t^{\alpha_j} v(s,t) + \int_b^c \omega(\alpha) D_t^\alpha v(s,t) d\alpha = \nu \Delta v(s,t) + \beta u(s,t) + \gamma v(s,t) + f_2(s,t). \tag{22}$$

The system at $\omega_2 = \omega_3 = \cdots = \omega_n = 0$, $\alpha_1 = 1$, $\omega(\alpha) \equiv 0$ for all $\alpha \in (b,c)$ up to linear replacement of unknown functions $u(s,t) = \widetilde{u}(s,t) + \frac{l}{2}\widetilde{v}(s,t)$, $v(s,t) = \frac{l}{2}\widetilde{v}(s,t)$, $l \in \mathbb{R}$, coincides with the linearization of the phase field system of equations, describing phase transitions of the first kind within the framework of mesoscopic theory [41,42].

Set

$$\mathcal{Z} = (L_2(\Omega))^2, \quad A = \begin{pmatrix} \Delta & -\Delta \\ \beta I & \gamma I + \nu\Delta \end{pmatrix}, \quad D_A = (H_0^2(\Omega))^2,$$

where $H_0^2(\Omega) := \{z \in H^2(\Omega) : z(s) = 0,\ s \in \partial\Omega\}$. Hence $A \in \mathcal{C}l(\mathcal{Z})$.

Denote $\Lambda_1 z = \Delta z$, $D_{\Lambda_1} = H_0^2(\Omega) \subset L_2(\Omega)$. By $\{\varphi_k : k \in \mathbb{N}\}$ denote an orthonormal in the sense of the inner product $\langle \cdot, \cdot \rangle$ in $L_2(\Omega)$ eigenfunctions of the operator $\Lambda_1$, which are enumerated in the non-increasing order of the corresponding eigenvalues $\{\lambda_k : k \in \mathbb{N}\}$ taking in account their multiplicities.

**Theorem 9.** *Let $c \in (1,2)$, $\nu > 0$, $\beta, \gamma, \delta \in \mathbb{R}$, then there exist $\theta_0 \in (\pi/2, \pi)$, $a_0 \geq 0$, such that $A \in \mathcal{A}_W(\theta_0, a_0)$,*

$$\sigma(A) = \left\{ \lambda = \frac{1}{2}(\lambda_k(1+\nu) + \gamma \pm \sqrt{(\gamma + \lambda_k(\nu - 1))^2 - 4\beta\lambda_k}) \in \mathbb{C} : k \in \mathbb{N} \right\}.$$

**Proof.** Using decomposition by the basis $\{\varphi_k : k \in \mathbb{N}\}$ in the space $L_2(\Omega)$ and the denotations $\mu_k^\pm := \frac{1}{2}(\lambda_k(1+\nu) + \gamma \pm \sqrt{(\gamma + \lambda_k(\nu-1))^2 - 4\beta\lambda_k})$, for

$$W(\lambda) = \sum_{j=1}^{n} \omega_j \lambda^{\alpha_j} + \int_b^c \omega(\alpha) \lambda^\alpha d\alpha \neq \mu_k^\pm, \quad k \in \mathbb{N},$$

obtain the operators

$$W(\lambda)I - A = \begin{pmatrix} W(\lambda)I - \Delta & \Delta \\ -\beta I & (W(\lambda) - \gamma)I - \nu\Delta \end{pmatrix},$$

$$(W(\lambda)I - A)^{-1} = \sum_{k=1}^{\infty} \begin{pmatrix} W(\lambda) - \gamma - \nu\lambda_k & -\lambda_k \\ \beta & W(\lambda) - \lambda_k \end{pmatrix} \frac{\langle \cdot, \varphi_k \rangle \varphi_k}{(W(\lambda) - \mu_k^+)(W(\lambda) - \mu_k^-)}.$$

Since $\mu_k^+ \sim \nu\lambda_k$ and $\mu_k^- \sim \lambda_k$ as $k \to \infty$ (or inversely, depending on the condition $\nu \geq 1$ or $\nu \in (0,1)$), for $\nu > 0$ $\lim\limits_{k\to\infty} \arg \mu_k^\pm = \pi$, therefore, there exists $\mu_0 = \max\limits_{k \in \mathbb{N}} \mathrm{Re}\mu_k^\pm$.

In the proof of Theorem 6 it was shown that for every $\varepsilon > 0$ there exists $\delta \geq 1$ such that for $|\lambda| > \delta$ we have $|\arg W(\lambda) - c \arg \lambda| < \varepsilon$. Therefore, we can choose sufficiently close to $\pi/2$ $\theta_0 \in (\pi/2, \pi)$ and large enough $a_0 > \delta \sin^{-1}\theta_0$, such that for all $k \in \mathbb{N}$ $\mu_k^\pm \notin S_{c\theta_0+2\varepsilon,a_0}$, where $c\theta_0 + 2\varepsilon < \pi$ for small enough $\varepsilon > 0$. Then due to Lemma 1 $|W(\lambda)| > C|\lambda|^{c-\varepsilon} > |\lambda|^{c-\varepsilon/2}$, where $\varepsilon \leq 2(c-1)$, hence $W(\lambda) \in S_{c\theta_0+\varepsilon,a_0}$. Then for $\lambda \in S_{\theta_0,a_0}$

$$\left|\frac{W(\lambda) - \gamma - \nu\lambda_k}{W(\lambda) - \mu_k^+}\right| \leq 1 + \frac{|\gamma|}{|W(\lambda)| \sin\varepsilon} \leq 1 + \frac{|\gamma|}{(a_0 \sin\theta_0)^{c-\varepsilon/2} \sin\varepsilon} = C_1,$$

$$\left|\frac{\lambda_k}{W(\lambda) - \mu_k^-}\right| \leq 1 + \frac{1}{\sin\varepsilon} = C_2,$$

$$\left|\frac{\beta}{W(\lambda) - \mu_k^+}\right| \leq \frac{|\beta|}{(a_0 \sin\theta_0)^{c-\varepsilon/2} \sin\varepsilon} = C_3, \quad \left|\frac{W(\lambda) - \lambda_k}{W(\lambda) - \mu_k^-}\right| = 1.$$

Hence for all $\lambda \in S_{\theta_0,a_0}$, $z \in (L_2(\Omega))^2$

$$\|R_{W(\lambda)}(A)z\|^2_{\mathcal{L}(\mathcal{Z})} \leq C_4 \sup_{k\in\mathbb{N}} \left(\frac{1}{|W(\lambda) - \mu_k^-|} + \frac{1}{|W(\lambda) - \mu_k^+|}\right) \leq \frac{2C_4 \sin^{-2}\varepsilon}{|W(\lambda) - a_0|}.$$

Thus, $A \in \mathcal{A}_W(\theta_0, a_0)$. $\square$

By Theorems 8 and 9 we obtain the corollary.

**Corollary 1.** *Let $c \in (1,2)$, $\nu > 0$, $\beta, \gamma, \delta \in \mathbb{R}$, $\kappa \in (0,1]$, $f_i \in C([0,T]; H^2(\Omega)) \cup C^\kappa([0,T]; L_2(\Omega))$, $i = 1,2$. Then for all $u_0, u_1, v_0, v_1 \in H_0^2(\Omega)$ there exists a unique solution of problem (18)–(22).*

**Remark 6.** *Analogously, but essentially simpler, we can study the initial boundary value problem*

$$u(s,0) = u_0(s), \quad \frac{\partial u}{\partial t}(s,0) = u_1(s), \quad s \in \Omega, \tag{23}$$

$$u(s,t) = 0, \quad (s,t) \in \partial\Omega \times (0,T], \tag{24}$$

*for the ultraslow diffusion equation*

$$\sum_{j=1}^{n} \omega_j D_t^{\alpha_j} u(s,t) + \int_b^c \omega(\alpha) D_t^\alpha u(s,t) d\alpha = \Delta u(s,t) + f(s,t), \quad \Omega \times (0,T]. \tag{25}$$

*For this aim, put $\mathcal{Z} = L_2(\Omega)$, $A = \Lambda_1$. A similar equation without the sum $\sum\limits_{j=1}^{n} \omega_j D_t^{\alpha_j} u(s,t)$ was investigated in [7].*

## 7. Conclusions

The Cauchy problem for equations in Banach spaces with a distributed fractional derivative and with a linear closed operator $A$ at the unknown function is studied. The derivative is given by the Riemann–Stieltjes integral with respect to the order of the fractional differentiation, therefore, the considered class of equations includes equations with a distributed derivative, defined by a standard integral, or with a discretely distributed derivative, which was researched earlier. The notion of a $k$-resolving family of operators for the equation is introduced, and properties of such families are studied. It is shown that the existence of a 0-resolving family implies the existence of other $k$-resolving families, $k = 1, 2, \ldots, m-1$. Necessary and sufficient conditions for the existence of an analytic

0-resolving family of operators in terms of the resolvent of the operator $A$ is the key result of this work. The corresponding class of the operators is denoted by $\mathcal{A}_W(\theta_0, a_0)$. The properties of analytic resolving families, generated by operators from this class, are investigated, and a perturbutaion theorem for such class of operators is proved. The unique solvability theorem for the inhomogeneous equation with a distributed fractional derivative and with $A \in \mathcal{A}_W(\theta_0, a_0)$ is obtained. Results of the work are applied to the research of an initial boundary value problem for a system of partial differential equations with a distributed time derivative in a general form.

There are a large number of various types of fractional derivatives, and in recent decades, new constructions of them have appeared: Riemann–Liouville derivative, Hadamard derivative, Marchot derivative, Dzhrbashyan–Nersesyan derivative, Prabhakar derivative, Caputo–Fabrizio and Atangana–Baleanu integro-differential operators, etc. Every construction of a fractional derivative corresponds to certain features of considered problems. The Gerasimov–Caputo derivative is the most studied (along with the Riemann–Liouville derivative) and mathematically simplest among such derivatives. The results obtained in this paper allow us to understand the features of equations with distributed derivatives given by the Stieltjes integral. This will allow us to move on to the study of equations with integrals with respect to the differentiation order of fractional derivatives of other types.

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## References


1. Rezapour, S.; Addas, M.I.; Etemad, S.; Dien, N.M. On a multi-point $p$-Laplacian fractional differential equation with generalized fractional derivatives. *Math. Methods Appl. Sci.* **2022**. [CrossRef]
2. Rezapour, S.; Souid, M.S.; Bouazza, Z.; Hussain, A.; Etemad, S. On the fractional variable order thermostat model: Existence theory on cones via piece-wise constant functions. *Adv. Nonlinear Anal. Appl.* **2022**, *2022*, 8053620. [CrossRef]
3. Sintunavarat, W.; Turab, A. Mathematical analysis of an extended SEIR model of COVID-19 using the ABC-fractional operator. *Math. Comput. Simul.* **2022**, *198*, 65–84. [CrossRef]
4. Nakhushev, A.M. On continual differential equations and their difference analogues. *Sov. Math. Dokl.* **1988**, *37*, 729–732.
5. Caputo, M. Mean fractional order derivatives. Differential equations and filters. *Ann. Dell'Universita Ferrara Sez. VII Sci. Mat.* **1995**, *41*, 73–84. [CrossRef]
6. Caputo, M. Distributed order differential equations modeling dielectric induction and diffusion. *Fract. Calc. Appl. Anal.* **2001**, *4*, 421–442
7. Kochubei, A.N. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.* **2008**, *340*, 252–280. [CrossRef]
8. Sokolov, I.M.; Chechkin, A.V.; Klafter, J. Distributed-order fractional kinetics. *Acta Phys. Pol. B* **2004**, *35*, 1323–1341.
9. Nakhushev, A.M. Positiveness of the operators of continual and discrete differentiation and integration, which are quite important in the fractional calculus and in the theory of mixed-type equations. *Differ. Equ.* **1998**, *34*, 103–112.
10. Bagley, R.L.; Torvik, P.J. On the existence of the order domain and the solution of distributed order equations. Part 1. *Int. J. Appl. Math.* **2000**, *2*, 865–882.
11. Lorenzo, C.F.; Hartley, T.T. Variable order and distributed order fractional operators. *Nonlinear Dyn.* **2002**, *29*, 57–98. [CrossRef]

12. Jiao, Z.; Chen, Y.; Podlubny, I. *Distributed-Order Dynamic System. Stability, Simulations, Applications and Perspectives*; Springer: London, UK, 2012.
13. Diethelm, K.; Ford, N.J. Numerical solution methods for distributed order time fractional diffusion equation. *Fract. Calc. Appl. Anal.* **2001**, *4*, 531–542.
14. Diethelm, K.; Ford, N.; Freed, A.D.; Luchko, Y. Algorithms for the fractional calculus: A selection of numerical methods. *Comput. Methods Appl. Mech. Eng.* **2003**, *194*, 743–773. [CrossRef]
15. Pskhu, A.V. On the theory of the continual and integro-differentiation operator. *Differ. Equ.* **2004**, *40*, 128–136. [CrossRef]
16. Pskhu, A.V. *Partial Differential Equations of Fractional Order*; Nauka Publ.: Moscow, Russia, 2005. (In Russian)
17. Pskhu, A.V. Fractional diffusion equation with a discretely distributed differentiation operator. *Sib. Elektron. Math. Rep.* **2016**, *13*, 1078–1098.
18. Umarov, S.; Gorenflo, R. Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations. *Z. Für Anal. Ihre Anwend.* **2005**, *24*, 449–466.
19. Atanacković, T.M.; Oparnica, L.; Pilipović, S. On a nonlinear distributed order fractional differential equation. *J. Math. Anal. Appl.* **2007**, *328*, 590–608. [CrossRef]
20. Efendiev, B.I. Steklov problem for a second-order ordinary differential equation with a continual derivative. *Differ. Equ.* **2013**, *49*, 450–456. [CrossRef]
21. Efendiev, B.I. Lagrange formula for ordinary continual second-order differential equations. *Differ. Equ.* **2017**, *53*, 736–744. [CrossRef]
22. Streletskaya, E.M.; Fedorov, V.E.; Debbouche, A. The Cauchy problem for distributed order equations in Banach spaces. *Math. Notes NEFU* **2018**, *25*, 63–72. (In Russian)
23. Fedorov, V.E.; Streletskaya, E.M. Initial-value problems for linear distributed-order differential equations in Banach spaces. *Electron. J. Differ. Equ.* **2018**, *2018*, 176.
24. Fedorov, V.E.; Phuong, T.D.; Kien, B.T.; Boyko, K.V.; Izhberdeeva, E.M. A class of semilinear distributed order equations in Banach spaces. *Chelyabinsk Phys. Math. J.* **2020**, *5*, 343–351.
25. Fedorov, V.E. On generation of an analytic in a sector resolving operators family for a distributed order equation. *Zap. POMI* **2020**, *489*, 113–129. [CrossRef]
26. Fedorov, V.E. Generators of analytic resolving families for distributed order equations and perturbations. *Mathematics* **2020**, *8*, 1306. [CrossRef]
27. Fedorov, V.E.; Filin, N.V. Linear equations with discretely distributed fractional derivative in Banach spaces. *Tr. Instituta Mat. I Mekhaniki UrO RAN* **2021**, *27*, 264–280.
28. Fedorov, V.E.; Abdrakhmanova, A.A. Distributed order equations in Banach spaces with sectorial operators. In *Transmutation Operators and Applications*; Kravchenko, V.F., Sitnik, S.M., Eds.; Springer Nature: Cham, Switzerland, 2020; pp. 509–538.
29. Fedorov, V.E.; Du W.-S.; Kostić M.; Abdrakhmanova, A.A. Analytic resolving families for equations with distributed Riemann—Liouville derivatives. *Mathematics* **2022**, *10*, 681. [CrossRef]
30. Gerasimov, A.N. A generalization of linear laws of deformation and its application to the problems of internal friction. *Prikl. Mat. I Mekhanika* **1948**, *12*, 251–260. (In Russian)
31. Caputo, M. Linear model of dissipation whose *Q* is almost frequancy independent. II. *Geophys. J. R. Astron. Soc.* **1967**, *13*, 529–539. [CrossRef]
32. Novozhenova, O.G. Life and science of Alexey Gerasimov, one of the pioneers of fractional calculus in Soviet Union. *Fract. Calc. Appl. Anal.* **2017**, *20*, 790–809. [CrossRef]
33. Podlubny, I. *Fractional Differential Equations*; Academic Press: Boston, MA, USA, 1999.
34. Prüss, J. *Evolutionary Integral Equations and Applications*; Springer: Basel, Switzerland, 1993.
35. Arendt, W.; Batty, C.J.K.; Hieber, M.; Neubrander, F. *Vector-Valued Laplace Transforms and Cauchy Problems*; Springer: Basel, Switzerland, 2011.
36. Bajlekova, E.G. Fractional Evolution Equations in Banach Spaces. Ph.D. Thesis, University Press Facilities, Eindhoven University of Technology, Eindhoven, The Netherlands, 2001.
37. Hille, E.; Phillips, R.S. *Functional Analysis and Semi-Groups*; American Mathematical Society: Providence, RI, USA, 1957.
38. Yosida, K. *Functional Analysis*; Springer: Berlin/Heidelberg, Germany, 1965.
39. Kato, K. *Perturbation Theory for Linear Operators*; Springer: Berlin/Heidelberg, Germany, 1966.
40. Goldstein, J.A. Semigroups and second-order differential equations. *J. Funct. Anal.* **1969**, *4*, 50–70. [CrossRef]
41. Caginalp, G. An analysis of a phase field model of a free boundary. *Arch. Ration. Mech. Anal.* **1986**, *92*, 205–245. [CrossRef]
42. Caginalp, G. Stefan and Hele—Shaw type models as asymptotic limits of the phase-field equations. *Phys. Rev. A* **1989**, *39*, 5887–5896. [CrossRef]