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GRAPHS AND ALGEBRAS OF SYMMETRIC FUNCTIONS

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Abstract. We describe an algebraic technique for operating with power series whose coefficients are represented by integrals of symmetric functions f_n defined on the Cartesian powers Ω^n of a set Ω with a measure μ. Moreover, each of the coefficient functions f_n is obtained by means of a special mapping from graphs with n labeled vertices belonging to a fixed class. This technique has application to equilibrium statistical mechanics and to problems of enumeration of graphs.

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1. Introduction. In 1938, J. Mayer proposed to describe the structure of the coefficients of power series arising in problems of statistical mechanics of gases by means of graphs (see [6, 7]). The construction used in these papers can be easily implemented in the case of the so-called *group expansions* in terms of activity degrees (see [5]). The formulas determining these coefficients have complete mathematical proofs (see, e.g., [4, 10]). However, this algebraic technique can be generalized and hence can be applied to a much wider range of problems in mathematical physics. In particular, it is suitable for calculating the coefficients of the so-called virial expansions of statistical mechanics, which remain little known (see [1]); these expansions have no rigorous proofs in the mathematical literature. This paper is devoted to filling this gap. The outline is the following. In Sec. 2, we present necessary information (basic notions and facts) from the theory of graphs with labeled vertices; here we omit proofs (the reader the reader is referred to well-known monographs on graph theory, e.g., [2, 8]). In Sec. 3, we briefly recall basic information about infinite-dimensional commutative algebras of sequences of symmetric functions. In Sec. 4, the relationship between graphs with labeled vertices and symmetric functions is established. The last section is devoted to the proof of the formula that plays the main role in the construction of virial expansions.

2. Graphs with labeled vertices. Let V be a finite set of elements that are called *vertices*; we denote them by lowercase Latin letters. We denote by $V^{(2)}$ the set of all pairs $\{x, y\} \subset V$. A *graph with labeled vertices* (in the sequel, we use the term "graph") over the set V is an ordered pair $\mathfrak{G} = \langle V, \Psi\rangle$, where the subset $\Psi \subset V^{(2)}$ is called the *adjacency set* of the graph; its elements are called *edges* of the graph $\langle V, \Psi\rangle$. The graph theory also considers graphs whose vertices are not labeled; they are defined as the factor set $\langle V, \Psi\rangle/\mathbb{P}_{|V|}$ by the permutation group $\mathbb{P}_{|V|}$. A graph $\mathfrak{G}' = \langle V', \Psi'\rangle$ such that $V' \subset V$ and $\Psi' = \Psi \cap V'^{(2)}$ is called a *subgraph* of the graph $\mathfrak{G}$.

The symmetric binary relation Ψ (the *adjacency* relation) on V determines a binary relation on V, namely, the *connectedness* relation. Its construction is based on the notion of a *path* on the graph $\mathfrak{G} = \langle V, \Psi\rangle$. A sequence $\gamma(x, y) = \langle x, x_1, x_2, \dots, x_{n-1}, y\rangle$ of vertices of V such that $\{x_j, x_{j+1}\} \in \Psi$, $j = 0, 1, \dots, n-1$, $x_0 = x$, $x_n = y$, is called a path with the set of vertices $\{\gamma(x, y)\}$. A pair $\{x, y\} \subset V$ is said to be *connected* on the graph $\mathfrak{G}$ if there exists a path $\langle x, x_1, x_2, \dots, x_{n-1}, y\rangle$ containing these vertices. The subset of pairs of connected vertices generates a binary relation, which is symmetric and transitive; thus, it is an *equivalence* relation. Therefore, it generates the decomposition of the graph into connected, pairwise disjoint components.

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Let $\mathfrak{G}_1 = \langle V_1, \Psi_1 \rangle$ and $\mathfrak{G}_2 = \langle V_2, \Psi_2 \rangle$ be connected subgraphs of a connected graph $\mathfrak{G} = \langle V, \Psi \rangle$ and

$$V_1 \cap V_2 = \{x\}, \quad V = V_1 \cup V_2; \quad \Psi = \Psi_1 \cap \Psi_2 = \varnothing, \quad \Psi = \Psi_1 \cup \Psi_2. \tag{2.1}$$

Then x is called a *cut vertex* of the graph $\mathfrak{G}$. Graphs without cut vertices are called *blocks*. By definition, each graph $\{x\}$ with one vertex has no cut vertices. If x is a cut vertex of a graph $\mathfrak{G}$ (i.e., Eqs. (2.1) hold), then we say that the graph $\mathfrak{G}$ is *glued* at this vertex and denote this fact as follows: $\mathfrak{G} = \mathfrak{G}_1 \vee \mathfrak{G}_2$ (the symbol $\vee$ denotes the gluing operation). A cut vertex in a connected graph $\mathfrak{G}$ is characterized by the following property.

Theorem 2.1. *A vertex x of a connected graph $\mathfrak{G} = \langle V, \Psi \rangle$ is a cut vertex if and only if there exists a pair of vertices $y_1 \in V$, $y_2 \in V$, $y_j \neq x$, $j = 1, 2$, such that any path $\gamma(y_1, y_2)$ from y_1 to y_2 contains the vertex x.*

If a vertex x is a cut vertex in a graph $\mathfrak{G}$ and this graph can be represented in the form

$$\mathfrak{G} = \bigvee_{j=1}^{p} \mathfrak{G}_j,$$

where $\mathfrak{G}_j = \langle V_j, \Psi_j \rangle$, $j = 1, \dots, p$, are connected graphs such that

$$V_j \cap V_k = \{x\}, \quad \Psi_j \cap \Psi_k = \varnothing, \quad j \neq k; \quad j, k = 1, \dots, p$$

and x is not a cut vertex for all these graphs, then the number p is called the *degree* of the cut vertex x in the graph $\mathfrak{G}$. The graphs $\mathfrak{G}_j$ are called the *components corresponding to the cut vertex x*. If a vertex x is not a cut vertex, then, by definition, we assume that its degree is equal to 1. Each cut vertex has a degree.

Theorem 2.2. *Let x be a cut vertex of a connected graph $\mathfrak{G} = \langle V, \Psi \rangle$. Then there exist a number $s \geq 2$ and a unique set of connected graphs $\mathfrak{G}_j = \langle V_j, \Psi_j \rangle$, $j = 1, \dots, s$, in which the vertex x is not a cut vertex, such that the following relations hold*:

$$V = V_1 \cup V_2 \cup \dots \cup V_s, \qquad V_i \cap V_j = \{x\}, \quad i \neq j, \quad i, j = 1, \dots, s;$$
$$\Psi = \Psi_1 \cup \Psi_2 \cup \dots \cup \Psi_s, \qquad \Psi_i \cap \Psi_j = \varnothing, \quad i \neq j, \quad i, j = 1, \dots, s.$$

Corollary 2.1. *Any finite non-one-vertex graph contains at least two vertices that are not cut vertices.*

Introduce a more general notion of the gluing of two graphs. Let two graphs $\mathfrak{G}_j = \langle V_j, \Psi_j \rangle$, $j = 1, 2$, be such that $V_1 \cap V_2 \neq \varnothing$. The *gluing* $\mathfrak{G}_1 \vee \mathfrak{G}_2$ of these graphs is the graph $\langle V_1 \cup V_2, \Psi_1 \cup \Psi_2 \rangle$. This notion allows one to introduce the following construction.

Assume that the graph $\mathfrak{G} = \langle V, \Psi \rangle$ contains a subgraph $\mathfrak{G}_B = \langle B, \Psi_B \rangle$, $\Psi_B = \Psi \cap B^{(2)}$, without cut vertices such that there exists a set $\{\mathfrak{G}(z) = \langle V(z), \Psi(z) \rangle; z \in B\}$ of connected, pairwise disjoint subgraphs of the graph $\mathfrak{G}$ that possess the following properties: $B \cap V(z) = \{z\}$, $\Psi_B \cap \Psi(z) = \varnothing$, $z \in B$, and the graph $\mathfrak{G}$ can be represented in the form

$$\mathfrak{G} = \bigvee_{z \in B} \Big[\mathfrak{G}_B \vee \mathfrak{G}(z) \Big]; \tag{2.2}$$

some of the subgraphs $\mathfrak{G}(z)$, $z \in B$, may be empty. Then we say that the subgraph $\mathfrak{G}_B$ is a block of the graph $\mathfrak{G}$.

Since each graph has a vertex, which is not a cut vertex, the following assertion guarantees the existence of a block in each graph.

Theorem 2.3. *Let x be a vertex of a graph $\mathfrak{G} = \langle V, \Psi \rangle$, which is not a cut vertex. Then the graph $\mathfrak{G}$ contains a unique block $\mathfrak{G}_B = \langle B, \Psi_B \rangle$ satisfying the representation* (2.2) *and containing x.*

The following assertion clarifies the formula (2.2).

Theorem 2.4. *Let x be a vertex of a connected graph $\mathfrak{G} = \langle V, \Psi \rangle$, which is not a cut vertex, and $\mathfrak{G}_B = \langle B, \Psi_B \rangle$ be unique block in this graph, which contains this vertex. Then the graph $\mathfrak{G}$ can be represented in the form*

$$\mathfrak{G} = \bigvee_{z \in B} \left[\mathfrak{G}_B \vee \left(\bigvee_{j=1}^{p(z)-1} \mathfrak{G}_j(z) \right) \right], \tag{2.3}$$

where $\mathfrak{G}_B$ is the block in the graph $\mathfrak{G}$ containing x, the numbers $p(z)$ are the degrees of each vertex $z \in B$ in the graph $\mathfrak{G}$, and the connected graphs $\mathfrak{G}_j(z)$, $j = 1, \dots, p(z)$, are the components of the cut set corresponding to the vertex z.

A *decomposition* $\mathcal{A}$ of a set $I_n = \{1, \dots, n\}$ is a disjunct set $\{\Gamma_1, \dots, \Gamma_s\}$ of subsets of I_n, called the components, such that $\bigcup_{j=1}^{s} \Gamma_j = I_n$ and $\Gamma_j \cap \Gamma_k = \varnothing$ for $j \neq k$. The number $s \equiv |\mathcal{A}|$ is called the *order* of the decomposition. We denote by $\mathfrak{S}_n^{(s)}$ the class of all decompositions of order s of the set I_n and by $\mathfrak{S} = \bigcup_{s=1}^{n} \mathfrak{S}_n^{(s)}$ the class of all decompositions. Note that the classes $\mathfrak{S}_n^{(1)}$ and $\mathfrak{S}_n^{(n)}$ contain one decomposition, respectively, $\mathfrak{S}_n^{(1)} = \{I_n\}$ and $\mathfrak{S}_n^{(n)} = \bigl\{\Gamma_j = \{j\},\ j = 1, \dots, n\bigr\}$.

Let $\bar{\mathcal{G}}[V; z]$ be the class of all connected graphs over the set of vertices $V \cup \{z\}$ with a marked vertex z. This class is a subclass of the class of all connected graphs over $V \cup \{z\}$. It is characterized by the invariance under renumbering of vertices of V. Namely, let P belongs to the group $\mathbb{P}_{|V|}$ of permutations of the set V. Then the numbers of the vertex z in the sets $\mathsf{P}V \cup \{z\}$ and $V \cup \{z\}$ coincide. Any renumbering P induces a transformation $\mathsf{P}\Psi \equiv \bigl\{\{\mathsf{P}x, \mathsf{P}y\} : \{x, y\} \in \Psi\bigr\}$ of the adjacency set Ψ of each graph $\mathfrak{G} = \langle V \cup \{z\}, \Psi \rangle$ and, therefore, a transformation $\mathsf{P}\mathfrak{G} = \langle \mathsf{P}V \cup \{z\}, \mathsf{P}\Psi \rangle$ of any graph $\mathfrak{G} \in \bar{\mathcal{G}}[V \cup \{z\}]$. Then the invariance of the class $\bar{\mathcal{G}}[V; z]$ with respect to P means that

$$\mathsf{P}\bar{\mathcal{G}}[V; z] \equiv \bigl\{\mathsf{P}\mathfrak{G};\ \mathfrak{G} \in \bar{\mathcal{G}}[V; z]\bigr\} = \bar{\mathcal{G}}[V; z].$$

Below, we need the following technical lemmas whose proofs are obvious.

Lemma 2.1. *The class $\bar{\mathcal{G}}[I_n; n+1]$, $n \in \mathbb{N}$, can be represented as the disjunct union*

$$\bar{\mathcal{G}}[I_n; n+1] = \bigcup_{s=1}^{n} \bar{\mathcal{G}}^{(s)}[I_n; n+1],$$

where $\bar{\mathcal{G}}^{(s)}[I_n; n+1]$ is the class of all connected graphs with the set of vertices I_{n+1} such that the marked vertex is a cut vertex of degree $s = 1, \dots, n$.

Moreover, it is obvious that $\mathsf{P}\bar{\mathcal{G}}^{(s)}[I_n; n+1] = \bar{\mathcal{G}}^{(s)}[I_n; n+1]$, $\mathsf{P} \in \mathbb{P}_n$.

Lemma 2.2. *If a marked vertex $n+1$ is a cut vertex of degree $s > 1$, then the class $\bar{\mathcal{G}}^{(s)}[I_n; n+1]$ can be represented as the disjunct union*

$$\bar{\mathcal{G}}^{(s)}[I_n; n+1] = \bigcup_{\mathcal{A} \in \mathfrak{S}_n^{(s)}} \bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]$$

with nonempty components, each of which is a class of all connected graphs over the set I_{n+1} with the marked vertex $n+1$ and the degree of the cut vertex s. In this case, the numbers of vertices (different from $n+1$) of the connected graphs $\mathfrak{G}_j$, $j = 1, \dots, s$, that are components of the cut vertex at the vertex $n+1$ form a decomposition $\mathcal{A} \in \mathfrak{S}_n^{(s)}$ with the number of components equal to s and the vertex $n+1$ is not a cut vertex in the graph $\mathfrak{G}_j$.

Each of the classes $\bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]$, $\mathcal{A} \in \mathfrak{S}_n^{(s)}$, is invariant under renumberings $\mathsf{P} \in \mathbb{P}_n$ of the set I_n that do not change the decomposition $\mathcal{A}$, i.e.,

$$\mathsf{P}\mathcal{A} = \bigl\langle \mathsf{P}A_j;\ j = 1, \dots, s \bigr\rangle = \bigl\langle A_j;\ j = 1, \dots, s \bigr\rangle \equiv \mathcal{A}.$$

Lemma 2.3. *For any disjunct decomposition* $\mathcal{A} = \{A_1, \dots, A_s\} \in \mathfrak{S}_n^{(s)}$*, the class of graphs* $\bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]$*,* $n \geq s > 1$*, is equivalent to the Cartesian product*

$$\bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1] = \bigotimes_{A \in \mathcal{A}} \bar{\mathcal{G}}^{(1)}[A; n+1],$$

where $\bar{\mathcal{G}}^{(1)}[A; n+1]$ *is the class of connected graphs with the set of vertices* $A \cup \{n+1\}$*,* $A \in \mathcal{A}$*, such that the vertex* $n+1$ *is not a cut vertex.*

Each of the classes $\bar{\mathcal{G}}^{(1)}[A; n+1]$ is invariant under renumberings $\mathsf{P} \in \mathbb{P}_n$ of vertices that transform the set $A \in \mathcal{A}$ into itself, $\mathsf{P}A = A$, $A \in \mathcal{A}$.

Based on the formula (2.3), one can prove the following assertion.

Lemma 2.4. *The class* $\bar{\mathcal{G}}^{(1)}[I_n; n+1]$*,* $n \geq 2$*, can be represented as the disjunct union*

$$\bar{\mathcal{G}}^{(1)}[I_n; n+1] = \bigcup_{B \subset I_n : |B| \geq 1} \; \bigcup_{C \subset B} \bar{\mathcal{G}}[I_n; B; C]$$

of nonempty classes $\bar{\mathcal{G}}[I_n; B; C]$ *of graphs with a marked vertex* $n+1$*, which is not a cut vertex. For each graph* $\mathfrak{G} \in \bar{\mathcal{G}}[I_n; B; C]$*, the nonempty set* B *consists of numbers of vertices of the block* $\mathfrak{G}_B$ *containing the vertex* $n+1$ *and* C *is the set of cut vertices of the graph with the degree greater than* 1*, which are contained in the block* $\mathfrak{G}_B$*.*

Each of the classes $\bar{\mathcal{G}}[I_n; B; C]$ is invariant under renumberings P of vertices that transform the sets B and C into themselves.

For each pair of sets $B \subset I_n$ and $C \subset B$, we denote by $\mathfrak{D}(B, C)$ the class of functions $\{B(z);\ z \in C\}$ on C, where the set of values forms the disjunct decomposition $\bigcup_{z \in C} B(z) = I_n \setminus B$, $B(z) \neq \varnothing$, $z \in C$, and $B(z_1) \cap B(z_2) = \varnothing$ for $z_1 \neq z_2$. The following assertion holds.

Lemma 2.5. *Each class* $\bar{\mathcal{G}}[I_n; B; C]$*,* $n \geq 2$*, can be represented as the disjunct union*

$$\bar{\mathcal{G}}[I_n; B; C] = \bigcup_{\{B(z);\ z \in C\} \in \mathfrak{D}(B,C)} \bar{\mathcal{G}}\big[I_n; B \mid \{B(z), z \in C\}\big]$$

of nonempty classes $\bar{\mathcal{G}}[I_n; B \mid \{B(z), z \in C\}]$*,* $\{B(z);\ z \in C\} \in \mathfrak{D}(B, C)$*, such that each graph* $\mathfrak{G} \in \bar{\mathcal{G}}\big[I_n; B \mid \{B(z),\ z \in C\}\big]$ *with the sets* B *and* C *defined in Lemma 2.4, for which the set of vertices of the graph glued to the block* $\mathfrak{G}_B$ *at the vertex* $z \in C$ *is* $B(z)$*.*

Here each of the classes $\bar{\mathcal{G}}[I_n; B \mid \{B(z), z \in C\}]$ is invariant under renumberings P of vertices that transform the sets B and C into themselves and do not change elements of the decomposition $\{B(z), z \in C\}$, $\mathsf{P}B(z) = B(z)$, $z \in C$.

Lemma 2.6. *Each class* $\bar{\mathcal{G}}\big[I_n; B \mid \{B(z),\ z \in C\}\big]$*,* $n \geq 2$*, can be represented as the Cartesian product*

$$\bar{\mathcal{G}}\big[I_n; B \mid \{B(z),\ z \in C\}\big] = \mathcal{F}[B; n+1] \otimes \left(\bigotimes_{z \in C} \bar{\mathcal{G}}[B(z); z] \right)$$

of the class $\mathcal{F}[B; n+1]$ *of graphs without cut vertices over the set of vertices* $B \cup \{n+1\}$ *and the set of nonempty classes* $\bar{\mathcal{G}}[B(z); z]$*,* $z \in C$*, where each class consists of all connected graphs over the set of vertices* $B(z) \cup \{z\}$ *with a marked vertex* z*.*

Here for a fixed vertex $z \in C$, each of the classes $\bar{\mathcal{G}}[B(z); z]$ is invariant under renumberings P of vertices that transform the set $B(z)$ into itself.

3. Algebras of symmetric functions. Let Ω be a set whose elements are denoted by $x, y, z, \dots$; we denote an ordered family $\langle x_1, x_2, \dots, x_n\rangle \in \Omega^n$ by X_n. A function $f_n(X_n)$, $n \geq 2$, on Ω^n with values in $\mathbb{C}$ is said to be *symmetric* if for any permutation P from the group $\mathbb{P}_n$ of permutations of the set I_n, $n \in \mathbb{N}$, the relation $f_n(\mathsf{P}X_n) = f_n(X_n)$ holds. The set of all symmetric functions on Ω^n is a linear variety $\mathbb{L}_n(\Omega)$. Consider the direct sum

$$\mathbb{L}_\infty(\Omega) = \bigoplus_{n=0}^{\infty} \mathbb{L}_n(\Omega)$$

of linear varieties $\mathbb{L}_0(\Omega) \equiv \mathbb{C}$, $\mathbb{L}_1(\Omega)$ is the linear variety of functions $f_1(x_1)$ on Ω and $\mathbb{L}_n(\Omega)$ are the linear varieties of symmetric functions $f_n(X_n)$ on Ω^n, $n \geq 2$. Thus, $\mathbb{L}_\infty(\Omega)$ consists of sequences $\mathsf{f} = \langle f_n(X_n);\ n \in \mathbb{N}_+\rangle$.

On the linear variety $\mathbb{L}_\infty(\Omega)$, we introduce the mapping $\mathbb{L}_\infty \times \mathbb{L}_\infty \mapsto \mathbb{L}_\infty$, which to any pair of sequences $\mathsf{f}^{(1)} = \langle f_n^{(1)};\ n \in \mathbb{N}_+\rangle$ and $\mathsf{f}^{(2)} = \langle f_n^{(2)};\ n \in \mathbb{N}_+\rangle$ assigns a sequence $\mathsf{f} = \langle f_n;\ n \in \mathbb{N}_+\rangle$ whose elements are defined by the formula

$$f_n(X_n) = \sum_{\Gamma \subset I_n} f^{(1)}_{|\Gamma|}\big(X(\Gamma)\big) f^{(2)}_{n-|\Gamma|}\big(X(I_n \setminus \Gamma)\big), \quad n \in \mathbb{N}_+,$$

where $X(\Gamma) = \langle x_{j_1}, \dots, x_{j_s}\rangle$ with $\Gamma = \{j_1, \dots, x_s\}$, $s = |\Gamma|$. We assume that f is the result of applying a binary operation denoted by $*$ to the ordered pair $\langle \mathsf{f}^{(1)}, \mathsf{f}^{(2)}\rangle$ from $\mathbb{L}_\infty(\Omega)$.

One can easily verify that the operation $*$ is commutative and associative. Moreover, it is distributive with respect to the addition of elements of $\mathbb{L}_\infty(\Omega)$ and bilinear with respect to the multiplication of elements $\mathsf{f} \in \mathbb{L}_\infty(\Omega)$ by numbers from $\mathbb{C}$. This allows one to call it *multiplication* on $\mathbb{L}_\infty(\Omega)$. The linear variety equipped with the multiplication operation $*$ is an algebra over the field $\mathbb{C}$, which is denoted by the same symbol $\mathbb{L}_\infty(\Omega)$. The neutral element of $\mathbb{L}_\infty(\Omega)$ is the sequence $\mathsf{e} = \langle \delta_{n,0};\ n \in \mathbb{N}_+\rangle$. Moreover, each element f of $\mathbb{L}_\infty(\Omega)$ with $f_0 \neq 0$ is invertible; we denote the inverse element by f_*^{-1}, so that $\mathsf{f} * \mathsf{f}_*^{-1} = \mathsf{e}$, i.e., the operation of division by elements $f_0 \neq 0$ is defined. For this reason, the set of elements $\mathbb{L}_\infty^{(0)}(\Omega) = \big\{\mathsf{f} \in \mathbb{L}_\infty(\Omega) : f_0 = 0\big\}$, which is a subalgebra in $\mathbb{L}_\infty(\Omega)$, is a maximal ideal in $\mathbb{L}_\infty(\Omega)$ (see [11]). The following assertion can be easily proved (we omit the proof).

Lemma 3.1. *For any element $\mathsf{f} \in \mathbb{L}_\infty^{(0)}(\Omega)$, the equality $(\mathsf{f}_*^l)_n(X_n) = 0$ is valid for $n < l$. For $n \geq l$, the following formula holds*:

$$(\mathsf{f}_*^l)_n(X_n) = l! \sum_{\mathcal{A} = \{\Gamma_1, \dots, \Gamma_l\} \in \mathfrak{S}_n^{(l)}} \prod_{j=1}^{l} \mathsf{f}_{|\Gamma_j|}(X(\Gamma_j)).$$

Since elements of the algebra $\mathbb{L}_\infty(\Omega)$ are $\mathbb{C}$-valued functions, one can consider power series in $\mathbb{L}_\infty(\Omega)$; in particular, we introduce the exponential function

$$\exp_* \mathsf{f} = \sum_{l=0}^{\infty} \frac{1}{l!}\, \mathsf{f}_*^l,$$

where $\mathsf{f}_*^0 \equiv \mathsf{e}$. Lemma 3.1 implies the following assertion.

Lemma 3.2. *For any element $\mathsf{f} \in \mathbb{L}_\infty^{(0)}(\Omega)$, the following formula holds*:

$$\Big(\exp_* \mathsf{f}\Big)_n(X_n) = \sum_{\mathcal{A} = \{\Gamma_1, \dots, \Gamma_{|\mathcal{A}|}\} \in \mathfrak{S}_n} \prod_{j=1}^{|\mathcal{A}|} f_{|\Gamma_j|}(X(\Gamma_j)).$$

On $\mathbb{L}_\infty(\Omega)$, we introduce the linear operators ∂_x, $x \in \Omega$, as follows:

$$(\partial_x \mathsf{f})_n(X_n) = f_{n+1}(x, X_n).$$

The following assertion is proved by a direct calculation.

Lemma 3.3. *Each operator ∂_x is a differentiation, i.e., for any pair of elements f and g from $\mathbb{L}_\infty(\Omega)$, the Leibnitz identity holds*:

$$\partial_x(\mathsf{f} * \mathsf{g}) = (\partial_x \mathsf{f}) * \mathsf{g} + \mathsf{f} * (\partial_x \mathsf{g}).$$

Corollary 3.1. *For any $x \in \Omega$ and any element $\mathsf{f} \in \mathbb{L}_\infty^{(0)}(\Omega)$, we have*

$$\partial_x \exp_* \mathsf{f} = (\partial_x \mathsf{f}) * \exp_* \mathsf{f}.$$

In the sequel, we assume that the set Ω is equipped with a measure structure on which a finite measure μ is defined. Then, introducing for each $n \in \mathbb{N}$ the product of measures $d\mu(x_1)d\mu(x_2)\dots d\mu(x_n)$ on Ω^n and considering only measurable and summable on Ω^n functions $f_n(X_n)$ in each of the functional spaces $\mathbb{L}_n(\Omega)$, for each measurable bounded function $\zeta(x)$ on Ω we define the linear functional on $\mathbb{L}_n(\Omega)$ by the rule

$$\mathsf{f}_n[\zeta; f_n] = \int\limits_{\Omega^n} \Big(\prod_{j=1}^{n} \zeta(x_j)\Big) f_n(X_n) d\mu(x_1)\dots d\mu(x_n). \tag{3.1}$$

We consider the restriction of the variety $\mathbb{L}_\infty(\Omega)$ containing elements $\mathsf{f} = \big\langle f_n \in \mathbb{L}_n(\Omega);\ n \in \mathbb{N}_+ \big\rangle \in \mathbb{L}_\infty(\Omega)$ with summable on Ω^n components f_n, $n \in \mathbb{N}$, such that the following series converges:

$$\sum_{n=0}^{\infty} \frac{1}{n!} M^n \int\limits_{\Omega^n} |f_n(X_n)| d\mu(x_1)\dots d\mu(x_n) < \infty, \quad M > 0. \tag{3.2}$$

We denote this restriction by the same symbol $\mathbb{L}_\infty(\Omega)$.

If a function $\zeta(x)$ is bounded by a constant $M > 0$, $|\zeta(x)| < M$, $x \in \Omega$, then for such elements f the following functional is defined:

$$\mathsf{f}[\zeta; \mathsf{f}] = \sum_{n=0}^{\infty} \frac{1}{n!} \mathsf{f}_n[\zeta; f_n].$$

This functional is *multiplicative*; namely, the following theorem holds.

Theorem 3.1. *If elements $\mathsf{f}^{(1)}$ and $\mathsf{f}^{(2)}$ possess the property* (3.2) *with a function $\zeta(x)$ satisfying the condition $|\zeta(x)| < M$, then their product $\mathsf{f}_1 * \mathsf{f}_2$ also possesses the property* (3.2) *and the following formula holds*:

$$\mathsf{f}[\zeta; \mathsf{f}^{(1)} * \mathsf{f}^{(2)}] = \mathsf{f}[\zeta; \mathsf{f}^{(1)}] \cdot \mathsf{f}[\zeta; \mathsf{f}^{(2)}].$$

Proof. By a direct calculation, we have

$$\begin{aligned}
\mathsf{f}_n[\zeta; \mathsf{f}^{(1)} * \mathsf{f}^{(2)}] &= \int\limits_{\Omega^n} \left(\prod_{j=1}^{n} \zeta(x_j)\right) \Big(\mathsf{f}^{(1)} * \mathsf{f}^{(2)}\Big)_n (X_n) d\mu(x_1)\dots d\mu(x_n) \\
&= \sum_{l=0}^{n} \binom{n}{l} \int\limits_{\Omega^n} \left(\prod_{j=1}^{n} \zeta(x_j)\right) f_l^{(1)}(X_l) f_{n-l}^{(2)}\big(X(I_n \setminus I_l)\big) d\mu(x_1)\dots d\mu(x_n) \\
&= \sum_{l=0}^{n} \binom{n}{l} \mathsf{f}_l[\zeta; f_l^{(1)}] \cdot \mathsf{f}_{n-l}[\zeta; f_{n-l}^{(2)}].
\end{aligned}$$

Substituting the expression obtained into $\mathsf{f}[\zeta; \mathsf{f}^{(1)} * \mathsf{f}^{(2)}]$, we obtain

$$\mathsf{f}[\zeta; \mathsf{f}^{(1)} * \mathsf{f}^{(2)}] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{l=0}^{n} \binom{n}{l} \mathsf{f}_l[\zeta; f_l^{(1)}] \cdot \mathsf{f}_{n-l}[\zeta; f_{n-l}^{(2)}] = \mathsf{f}[\zeta; \mathsf{f}^{(1)}] \cdot \mathsf{f}[\zeta; \mathsf{f}^{(2)}]. \qquad \square$$

Corollary 3.2. *The following formula is valid:*

$$\mathsf{f}[\zeta; \exp_* \mathsf{f}] = \exp \mathsf{f}[\zeta; \mathsf{f}]. \tag{3.3}$$

Corollary 3.3. *The following differentiation formula is valid:*

$$\mathsf{f}[\zeta; \partial_x \exp_* \mathsf{f}] = \mathsf{f}[\zeta; \partial_x \mathsf{f}] \cdot \exp \mathsf{f}[\zeta; \mathsf{f}].$$

4. Graphs and symmetric functions. Let $w(x, y)$ be an arbitrary symmetric function Ω^2; we call it the *generating function*. Fix $n \in \mathbb{N}$, $n \geq 2$, and assign to a pair $\langle w \in \mathbb{L}_2(\Omega), \mathfrak{G} = \langle I_n, \Psi \rangle \rangle$ the function on Ω^n defined by the formula

$$h_n(X_n; \mathfrak{G}) = \prod_{\{i,j\} \in \Psi} w(x_i, x_j).$$

Each such function is called the *function on* Ω^n *associated with the graph* $\mathfrak{G}$ by the generating function w.

Based on functions $h_n(\cdot; \mathfrak{G})$ associated with graphs, one can construct symmetric functions, which are elements of the space $\mathbb{L}_n(\Omega)$. Fix a class $\mathcal{H}$ of graphs over I_n, which is invariant under permutations, so that for $\mathsf{P} \in \mathbb{P}_n$ and $\mathfrak{G} \in \mathcal{H}$ we have $\mathsf{P}\mathfrak{G} = \langle I_n, \mathsf{P}\Psi \rangle \in \mathcal{H}$. Introduce the function

$$f_n(X_n) = \sum_{\mathfrak{G} \in \mathcal{H}} h_n(X_n; \mathfrak{G}) \tag{4.1}$$

on Ω^n, which is obviously symmetric. Functions $f_n(X_n)$ on Ω^n constructed by (4.1) are said to be *associated with the class* $\mathcal{H}$. As $\mathcal{H}$, we take the class $\mathcal{G}_n$ of all graphs over I_n.

Lemma 4.1. *Let $w(x, y)$ be a symmetric function on Ω^2. Then the function associated with the class $\mathcal{G}_n$ by the generating function $w(x, y)$ is equal to*

$$f_n(X_n) = \sum_{\mathfrak{G} = \langle I_n; \Psi \rangle \in \mathcal{G}_n} \prod_{\{i,j\} \in \Psi} w(x_i, x_j) = \prod_{\{i,j\} \in I_n^{(2)}} (1 + w(x_i, x_j)). \tag{4.2}$$

Proof. We apply induction by n based on the formula

$$\sum_{\mathfrak{G} = \langle I_{n+1}; \Psi \rangle \in \mathcal{G}_{n+1}} \prod_{\{i,j\} \in \Psi} w(x_i, x_j) = \sum_{\mathfrak{G} = \langle I_n; \Psi \rangle \in \mathcal{G}_n} \prod_{\{k,l\} \in \Psi} w(x_k, x_l)$$
$$+ \sum_{\mathfrak{G} = \langle I_n; \Psi \rangle \in \mathcal{G}_n} \sum_{\Gamma \subset I_n} \prod_{j \in \Gamma} w(x_{n+1}, x_j) \left[\prod_{\{k,l\} \in \Psi} w(x_k, x_l) \right],$$

which provides the step of induction. Here the first sum corresponds to graphs in which the vertex $n + 1$ is not connected with vertices of I_n and the second sum takes into account all graphs of the class $\mathcal{G}_{n+1}$ in which the vertex $n + 1$ is connected with vertices of graphs of the class $\mathcal{G}_n$ whose numbers form the set Γ. □

Note that the class $\mathcal{G}_n$ os all graphs over I_n is invariant under permutations $\mathsf{P} \in \mathbb{P}_n$; therefore, for any $n \in \mathbb{N}$, functions f_n associated with $\mathcal{G}_n$ are symmetric, i.e., belong to $\mathbb{L}_n(\Omega)$.

In addition to functions $f_n(X_n)$ associated with the classes $\mathcal{G}_n$, $n \in \mathbb{N}$, $n \geq 2$, introduce the sequence of functions $\bar{f}_n(X_n)$ with a generating function $w(x, y)$, each of which is associated with the class $\bar{\mathcal{G}}_n$ of all connected graphs over I_n, $n \geq 2$:

$$\bar{f}_n(X_n) = \sum_{\mathfrak{G} = \langle I_n; \Psi \rangle \in \bar{\mathcal{G}}_n} \prod_{\{i,j\} \in \Psi} w(x_i, x_j). \tag{4.3}$$

Since for any $n \in \mathbb{N}$ the class $\bar{\mathcal{G}}_n$ is invariant under permutations $\mathsf{P} \in \mathbb{P}_n$, the functions $\bar{f}_n(X_n)$ are symmetric for any $n \geq 2$. Thus, the functions f_n and $\bar{f}_n$ belong to $\mathbb{L}_n(\Omega)$ for any $n \geq 2$. We prove that the sequences of functions in $\mathbb{L}_\infty(\Omega)$ satisfy the following theorem.

Theorem 4.1. *Let elements* $\mathsf{f} = \langle f_n;\ n \in \mathbb{N}_+\rangle$ *and* $\bar{\mathsf{f}} = \langle \bar{f}_n;\ n \in \mathbb{N}_+\rangle$ *of the algebra* $\mathbb{L}_\infty(\Omega)$, *whose components for* $n \geq 2$ *are defined by the formulas* (4.2) *and* (4.3), *respectively, by the same generating function* $w(x, y) \in \mathbb{L}_2(\Omega)$ *and, moreover,* $f_0 = 1$, $\bar{f}_0 = 0$, *and* $\bar{f}_1 = f_1 = 1$, *then these elements are related as follows*:

$$\mathsf{f} = \exp_* \bar{\mathsf{f}}.$$

Proof. We prove that the functions f_n satisfy the relation

$$f_n(X_n) = \sum_{\mathcal{A}\in\mathfrak{S}_n} \prod_{\Gamma\in\mathcal{A}} \bar{f}_{|\Gamma|}(X(\Gamma)).$$

We divide all graphs of the class $\mathcal{G}_n$ into disjoint subclasses $\mathcal{G}_n(\mathcal{A})$, where $\mathcal{A} = \{\Gamma_1, \dots, \Gamma_s\} \in \mathfrak{S}_n$. Consider an arbitrary graph $\mathfrak{G} = \langle I_n, \Psi\rangle$ from $\mathcal{G}_n$. This graph is uniquely decomposed into the connected components $\mathfrak{G}_j = \langle \Gamma_j, \Psi_j\rangle$, $j = 1, \dots, s$, so that

$$\bigcup_{j=1}^{s} \Gamma_j = I_n, \qquad \bigcup_{j=1}^{n} \Psi_j = \Psi$$

and each of the graphs $\mathfrak{G}_j$ belongs to the class $\bar{\mathcal{G}}_{|\Gamma_j|}(\Gamma_j)$ of all connected graphs over the set of vertices Γ_j. In this case, we refer the graph $\mathfrak{G}$ to the class $\mathcal{G}_n(\mathcal{A})$, $\mathcal{A} = \{\Gamma_1, \dots, \Gamma_{|\mathcal{A}|}\}$. Clearly, the classes $\mathcal{G}_n(\mathcal{A}_1)$ and $\mathcal{G}_n(\mathcal{A}_2)$ are nonempty and disjoint if $\mathcal{A}_1 \neq \mathcal{A}_2$. Then the following representation of the sum over all graphs of the class $\mathcal{G}_n$ is valid:

$$\sum_{\mathfrak{G}\in\mathcal{G}_n} \cdots = \sum_{\mathcal{A}\in\mathfrak{S}_n} \left(\prod_{j=1}^{|\mathcal{A}|} \sum_{\mathfrak{G}_j\in\mathcal{G}_{|\Gamma_j|}(\Gamma_j)} \right) \cdots.$$

Moreover, for each term of the sum, we have the relation

$$\prod_{\{k,l\}\in\Psi} w(x_k, x_l) = \prod_{j=1}^{|\mathcal{A}|} \prod_{\{k,l\}\in\Psi_j} w(x_k, x_l),$$

due to the unconnectedness of the graphs $\mathfrak{G}_j$, $j = 1, \dots, |\mathfrak{D}|$. Substituting the product on the right-hand side into the sum defining $f_n(X_n)$, we have

$$\begin{aligned} f_n(X_n) &= \sum_{\mathfrak{G}\in\mathcal{G}_n} \prod_{\{k,l\}\in\Psi} w(x_k, x_l) \\ &= \sum_{\mathcal{A}\in\mathfrak{S}_n} \prod_{j=1}^{|\mathcal{A}|} \left(\sum_{\mathfrak{G}_j\in\mathcal{G}_{|\Gamma_j|}(\Gamma_j)} \prod_{\{k,l\}\in\Psi_j} w(x_k, x_l) \right) = \sum_{\mathcal{A}\in\mathfrak{S}_n} \left(\prod_{j=1}^{|\mathcal{A}|} \bar{f}_{|\Gamma_j|}(X(\Gamma_j)) \right). \quad \square \end{aligned}$$

Applying the formula (3.3), we obtain the following result.

Corollary 4.1. *Assume that a set* Ω *is equipped with a measure structure with a finite measure and a bounded measurable function* ζ *on* Ω. *Then the functional* $\mathsf{f}[\zeta; \cdot]$ *on summable elements* $\mathsf{f} \in \mathbb{L}_\infty(\Omega)$, $\bar{\mathsf{f}} \in \mathbb{L}_\infty^{(0)}(\Omega)$, *constructed by a fixed generating function* w, *is defined by the formula*

$$\mathsf{f}[\zeta, \mathsf{f}] = \exp \mathsf{f}[\zeta; \bar{\mathsf{f}}].$$

Note that this assertion remains valid in the case where the series defining this functional diverges.

5. Symmetric functions and graphs of the class $\mathcal{F}_n$. Assume that a group $\mathbb{T}$ of transformations T acts on a set Ω and $\Omega = \{\mathsf{T}x; \mathsf{T} \in \mathbb{T}\}$ for any $x \in \Omega$, i.e., Ω is invariant under transformations from this group. In this section, we obtain the main result of this paper: the equation that relates values of the functionals $\mathsf{f}[z;\bar{\mathsf{f}}]$ and $\mathsf{f}[z;\mathsf{g}]$ with $\zeta(x) \equiv z \in \mathbb{C}$, where the functions $\mathsf{g} = \langle g_n;\ n \in \mathbb{N}_+\rangle$ are associated with connected graphs without cut vertices and, together with the functions of the sequence $\bar{\mathsf{f}}$, are generated by a symmetric function $w(x,y)$, which is invariant under transformations $\mathsf{T} \in \mathbb{T}$.

Consider the subalgebra $\mathbb{L}_+(\Omega)$ of elements h of the algebra $\mathbb{L}_\infty(\zeta)$ whose components $h_n(X_n)$, $n \in \mathbb{N}_+$, are invariant under transformations of the group $\mathbb{T}$ (i.e., $\mathbb{T}$-invariant). In particular, for $n = 1$, the linear variety $\mathbb{L}_1$ in this algebra coincides with $\mathbb{C}$.

Moreover, we assume that all components of each sequence from $\mathbb{L}_+(\Omega)$ are summable with the $\mathbb{T}$-invariant measure μ in the following sense:

$$\int\limits_{\Omega^{n-1}} |h_n(X_n)| \prod_{j=1}^{n-1} d\mu(x_j) < \infty,$$

and the total collection of these components possesses the following property: for elements h of the maximal ideal $\mathbb{L}_+^{(0)}(\Omega) = \mathbb{L}_+(\Omega) \cap \mathbb{L}_\infty^{(0)}(\Omega)$ of the algebra $\mathbb{L}_+(\Omega)$, there exists a sufficiently small neighborhood of the point $z = 0$ on the plane $z \in \mathbb{C}$ in which the following power series converges:

$$\sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} \int\limits_{\Omega^{n-1}} |h_n(X_n)| \prod_{j=1}^{n-1} d\mu(x_j) < \infty.$$

We consider the functional

$$\mathsf{S}[z;\mathsf{h}] = \mathsf{f}[z;\partial_x \mathsf{h}] = \sum_{n=0}^{\infty} \frac{z^n}{n!} \mathsf{f}_n[h_{n+1}]$$

on elements $\mathsf{h} = \langle h_n(X_n);\ n \in \mathbb{N}_+\rangle \in \mathbb{L}_+^{(0)}(\Omega)$; here the functionals $\mathsf{f}_n[\cdot]$, $n \in \mathbb{N}$, are defined by the $\mathbb{T}$-invariant measure μ and the weight function $\zeta = 1$ according to (3.1),

$$\mathsf{f}_n[h_{n+1}] = \int\limits_{\Omega^n} h_{n+1}(X_{n+1}) \prod_{j=1}^{n} d\mu(x_j) \quad \text{and} \quad \mathsf{f}_0[h_1] = h_1.$$

To prove the main assertion, we will need the following simple combinatorial fact.

Lemma 5.1. *Let $\varphi(\xi_1, \dots, \xi_s)$ be an arbitrary function on $\mathbb{N}^s$. Then for $n \geq s$, the following formula holds:*

$$\sum_{\substack{\langle A_1,\dots,A_s\rangle:\\ A_j \neq \varnothing,\ A_j \subset I_n,\ j=1,\dots,s;\\ A_j \cap A_k = \varnothing,\ j \neq k;\ \bigcup_{j=1}^{s} A_j = I_n}} \varphi\big(|A_1|, \dots, |A_s|\big) = \sum_{\substack{\langle l_1,\dots,l_s\rangle:\ l_j \geq 1;\\ l_1+\dots+l_s=n}} \frac{n!}{l_1! \dots l_s!} \varphi(l_1, \dots, l_s). \tag{5.1}$$

Proof. We use induction in s. For $s = 1$, the sum in (5.1) consists of a single term and the formula takes the form

$$\sum_{A_1 = I_n} \varphi(|A_1|) = \sum_{l_1 = n} \frac{n!}{l_1!} \varphi(l_1).$$

The step of inductions is as follows:

$$\sum_{\substack{\langle A_1,\dots,A_{s+1}\rangle:\\ A_j\neq\varnothing,\ A_j\subset I_n,\ j=1,\dots,s+1;\\ A_j\cap A_k=\varnothing,\ j\neq k;\ \bigcup_{j=1}^{s+1}A_j=I_n}} \varphi\big(|A_1|,\dots,|A_{s+1}|\big)$$

$$= \sum_{\substack{\varnothing\neq A_{s+1}\subset I_n:\\ |A_{s+1}|=1,\dots,n-s}} \sum_{\substack{\langle A_1,\dots,A_s\rangle:\\ A_j\neq\varnothing,\ A_j\subset I_n\setminus A_{s+1},\ j=1,\dots,s;\\ A_j\cap A_k=\varnothing,\ j\neq k;\ \bigcup_{j=1}^{s}A_j=I_n\setminus A_{s+1}}} \varphi\big(|A_1|,\dots,|A_{s+1}|\big)$$

$$= \sum_{l_{s+1}=1}^{n-s}\binom{n}{l_{s+1}} \sum_{\substack{\langle A_1,\dots,A_s\rangle:\\ A_j\neq\varnothing,\ A_j\subset I_n\setminus A_{s+1},\ j=1,\dots,s;\\ A_j\cap A_k=\varnothing,\ j\neq k;\ \bigcup_{j=1}^{s}A_j=I_n\setminus A_{s+1}}} \varphi\big(|A_1|,\dots,|A_s|,l_{s+1}\big).$$

Using the induction hypothesis for the inner sum, we write

$$\sum_{l_{s+1}=1}^{n-s}\binom{n}{l_{s+1}} \sum_{\substack{\langle l_1,\dots,l_s\rangle:\ l_j\geq 1;\\ l_1+\dots+l_s=n-l_{s+1}}} \frac{(n-l_{s+1})!}{l_1!\dots l_s!}\varphi(l_1,\dots,l_s,l_{s+1})$$

$$= \sum_{\substack{\langle l_1,\dots,l_{s+1}\rangle:\ l_j\geq 1;\\ l_1+\dots+l_{s+1}=n}} \frac{n!}{l_1!\dots l_{s+1}!}\varphi(l_1,\dots,l_s,l_{s+1}). \quad \square$$

Corollary 5.1. *Let $\varphi(\xi_1,\dots,\xi_s)$ be an arbitrary function on $\mathbb{N}^s$. Then for summing over decompositions $\mathcal{A}=\{A_1,\dots,A_s\}\in\mathfrak{S}_n^{(s)}$ of the set I_n, $n\geq s$, the following formula holds*:

$$\sum_{\substack{\{A_1,\dots,A_s\}:\\ A_j\neq\varnothing,\ A_j\subset I_n,\ j=1,\dots,s;\\ A_j\cap A_k=\varnothing,\ j\neq k;\ \bigcup_{j=1}^{s}A_j=I_n}} \varphi\big(|A_1|,\dots,|A_s|\big) = \frac{1}{s!}\sum_{\substack{\langle l_1,\dots,l_s\rangle:\ l_j\geq 1;\\ l_1+\dots+l_s=n}} \frac{n!}{l_1!\dots l_s!}\varphi(l_1,\dots,l_s). \tag{5.2}$$

Proof. This assertion follows from the fact that each decomposition $\mathcal{A}=\{A_1,\dots,A_s\}\in\mathfrak{S}_n^{(s)}$ of the set I_n generates exactly $s!$ ordered collections $\langle A_1,\dots,A_s\rangle$. $\square$

Consider the functions

$$\bar f_{n+1}(X_{n+1})=\sum_{\mathfrak{G}\in\bar{\mathcal{G}}_n} h(X_{n+1};\mathfrak{G}),\quad h(X_{n+1};\mathfrak{G})=\prod_{\{j,k\}\in\Psi} w(x_j,x_k),$$

where $h(X_{n+1};\mathfrak{G})$ on Ω^{n+1} are associated with graphs $\mathfrak{G}=\langle I_{n+1},\Psi\rangle\in\bar{\mathcal{G}}_n$ by a $\mathbb{T}$-invariant generating function $w(x,y)$ on Ω^2. Moreover, $\bar{\mathsf{f}}=\langle\bar f_{n+1};\ n\in\mathbb{N}_+\rangle\in\mathbb{L}_+^{(0)}(\Omega)$. Similarly, we introduce the functions

$$g_{n+1}(X_{n+1})=\sum_{\mathfrak{G}\in\mathcal{F}[I_{n+1}]} h(X_{n+1};\mathfrak{G}),\quad n\in\mathbb{N}_+, \tag{5.3}$$

where $\mathcal{F}[I_{n+1}]$ is the class of graphs without cut vertices over I_{n+1}, which form the element $\mathsf{g}=\langle g_{n+1};\ n\in\mathbb{N}_+\rangle$ of the algebra $\mathbb{L}_+(\Omega)$. Now we formulate the main result of this paper.

Theorem 5.1. *The values $\mathsf{S}[z;\bar{\mathsf{f}}]$ and $\mathsf{S}[z;\mathsf{g}]$ of the functional $\mathsf{S}[z;\cdot]$ on the elements $\bar{\mathsf{f}}$ and g satisfy the following functional equation*:

$$\mathsf{S}[z;\bar{\mathsf{f}}]=\exp\Big(\mathsf{S}\big[z\mathsf{S}[z;\bar f];\mathsf{g}\big]-1\Big). \tag{5.4}$$

Proof.

1. Note that the classes $\bar{\mathcal{G}}_{n+1}[I_n; n+1]$, $n \in \mathbb{N}$, can be represented as disjunct unions

$$\bar{\mathcal{G}}[I_n, n+1] = \bigcup_{s=1}^{n} \bar{\mathcal{G}}^{(s)}[I_n; n+1],$$

where the marked cut vertex $n+1$ has degree s. Therefore, by Lemma 2.1, each of the functions $\bar{f}_{n+1}(X_{n+1})$ can be represented as the sum

$$\bar{f}_{n+1}(X_{n+1}) = \sum_{s=1}^{n} f_{n+1}^{(s)}(X_{n+1}), \quad f_{n+1}^{(s)}(X_{n+1}) = \sum_{\mathfrak{G} \in \bar{\mathcal{G}}^{(s)}[I_n, n+1]} h(X_{n+1}; \mathfrak{G}). \tag{5.5}$$

Due to the symmetry of the classes $\bar{\mathcal{G}}^{(s)}[I_n, n+1]$ under permutations $\mathsf{P} \in \mathbb{P}_n$ and the $\mathbb{T}$-invariance of $w(x, y)$, the functions $f_{n+1}^{(s)}(X_{n+1})$ are symmetric under such permutations P and $\mathbb{T}$-invariant. Moreover, the following equality holds:

$$\mathsf{f}_n[\bar{f}_{n+1}] = \sum_{s=1}^{n} \mathsf{f}_n\left[f_{n+1}^{(s)}\right].$$

2. By Lemma 2.2, if the vertex $n+1$ has degree $s > 1$, then the class $\bar{\mathcal{G}}^{(s)}[I_n; n+1]$ can be represented as the disjunct union

$$\bar{\mathcal{G}}^{(s)}[I_n; n+1] = \bigcup_{\mathcal{A} \in \mathfrak{S}_n^{(s)}} \bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1],$$

where the vertex $n+1$ of the connected graphs from $\bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]$ over I_{n+1} has degree s and the numbers of vertices of the connected graphs $\mathfrak{G}_j$, $j = 1, \dots, s$, that differ from the vertex $n+1$, form a decomposition $\mathcal{A} \in \mathfrak{S}_n^{(s)}$ with the number of components s, and $n+1$ is not a cut vertex. Then the functions $f_{n+1}^{(s)}$ can be represented as the sums

$$f_{n+1}^{(s)}(X_{n+1}) = \sum_{\mathcal{A} \in \mathfrak{S}_n^{(s)}} f_{n+1}^{(s)}\big(X_{n+1}; \mathcal{A}, n+1\big),$$

where the functions $f_{n+1}^{(s)}\big(X_{n+1}; \mathcal{A}, n+1\big)$ are defined by the formula

$$f_{n+1}^{(s)}\big(X_{n+1}; \mathcal{A}, n+1\big) = \sum_{\mathfrak{G} \in \bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]} h(X_{n+1}; \mathfrak{G}). \tag{5.6}$$

They are symmetric under all $\mathsf{P} \in \mathbb{P}_n$ that leave invariant the components of the decomposition $\mathcal{A}$. Moreover,

$$\mathsf{f}_n\left[f_{n+1}^{(s)}\right] = \sum_{\mathcal{A} \in \mathfrak{S}_n^{(s)}} \mathsf{f}_n\Big[f_{n+1}^{(s)}\big(X_{n+1}; \mathcal{A}, n+1\big)\Big].$$

3. By Lemma 2.3, for any decomposition $\mathcal{A} = \{A_1, \dots, A_s\} \in \mathfrak{S}_n^{(s)}$, the class $\bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]$, $n \geq s > 1$, is equivalent to the Cartesian product

$$\bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1] = \bigotimes_{l=1}^{s} \bar{\mathcal{G}}^{(1)}[A_l; n+1]$$

of the classes $\bar{\mathcal{G}}^{(1)}[A_l; n+1]$ of connected graphs over the set $A_l \cup \{n+1\}$, $l = 2, \dots, s$, with the marked vertex $n+1$, which is not a cut vertex. Then we have the following representation of this sum:

$$\sum_{\mathfrak{G} \in \bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]} h(X_{n+1}; \mathfrak{G}) = \left(\prod_{l=1}^{s} \sum_{\mathfrak{G} \in \bar{\mathcal{G}}^{(1)}[A_l; n+1]}\right) h(X_{n+1}; \mathfrak{G}).$$

Since for each graph $\mathfrak{G} \in \bar{\mathcal{G}}^{(s)}[I_n; \mathcal{A}, n+1]$, where $\mathcal{A} = \{A_j; j = 1, \dots, |\mathcal{A}|\}$, the associated symmetric function can be represented in the form

$$h(X_{n+1}; \mathfrak{G}) = \prod_{l=1}^{s} h\big(X(A_l \cup \{n+1\}); \mathfrak{G}_l\big), \quad \mathfrak{G} = \bigvee_{l=1}^{s} \mathfrak{G}_l, \tag{5.7}$$

we obtain

$$\begin{aligned}
\mathsf{f}_n[f(X_{n+1}; \mathcal{A})] &= \left(\prod_{l=1}^{s} \sum_{\mathfrak{G} \in \bar{\mathcal{G}}^{(1)}[A_l; n+1]}\right) \mathsf{f}_n\left[\prod_{l=1}^{s} h\big(X(A_l \cup \{n+1\}); \mathfrak{G}_l\big)\right] \\
&= \prod_{l=1}^{s} \mathsf{f}_{|A_l|}\left[\sum_{\mathfrak{G} \in \bar{\mathcal{G}}^{(1)}[A_l; n+1]} h\big(X(A_l \cup \{n+1\}); \mathfrak{G}_l\big)\right] = \prod_{l=1}^{s} \mathsf{f}_{|A_l|}\Big[f^{(1)}_{|A_l|+1}\big(X(A_l \cup \{n+1\})\big)\Big].
\end{aligned}$$

We used the following fact: for the graph $\mathfrak{G}_{l_1} \vee \mathfrak{G}_{l_2}$ with the sets of vertices A_{l_1} and A_{l_2}, respectively, glued from two graphs $\mathfrak{G}_{l_1}$ and $\mathfrak{G}_{l_2}$ at the vertex $n+1$, we have

$$\begin{aligned}
&\mathsf{f}_{|A_{l_1}|+|A_{l_2}|}\Big[h\Big(X(A_{l_1} \cup A_{l_2} \cup \{n+1\});\ \mathfrak{G}_{l_1} \vee \mathfrak{G}_{l_2}\Big)\Big] \\
&\qquad = \sum_{X(A_{l_1} \cup A_{l_2}) \in \Omega^{|A_{l_1}|+|A_{l_2}|}} h\Big(X\big(A_{l_1} \cup A_{l_2} \cup \{n+1\}\big);\ \mathfrak{G}_{l_1} \vee \mathfrak{G}_{l_2}\Big) \\
&= \left(\sum_{X(A_{l_1}) \in \Omega^{|A_{l_1}|}} h\Big(X\big(A_{l_1} \cup \{n+1\}\big);\ \mathfrak{G}_{l_1}\Big)\right)\left(\sum_{X(A_{l_2}) \in \Omega^{|A_{l_2}|}} h\Big(X\big(A_{l_2} \cup \{n+1\}\big);\ \mathfrak{G}_{l_2}\Big)\right) \\
&\qquad = \mathsf{f}_{|A_{l_1}|}\Big[h\Big(X\big(A_{l_1} \cup \{n+1\}\big);\ \mathfrak{G}_{l_1}\Big)\Big] \cdot \mathsf{f}_{|A_{l_2}|}\Big[h\Big(X\big(A_{l_2} \cup \{n+1\}\big);\ \mathfrak{G}_{l_2}\Big)\Big].
\end{aligned} \tag{5.8}$$

4. Using the formulas (5.5), (5.6), (5.7), and (5.2), we obtain the following expression for the functional f_n on the functions $\bar{f}_{n+1}(X_{n+1})$:

$$\begin{aligned}
\mathsf{f}_n\big[\bar{f}_{n+1}\big] &= \mathsf{f}_n\big[\bar{f}^{(1)}_{n+1}\big] + \sum_{s=2}^{n} \mathsf{f}_n\big[f^{(s)}_{n+1}\big] = \mathsf{f}_n\big[f^{(1)}_{n+1}\big] + \sum_{s=2}^{n} \sum_{\mathcal{A} \in \mathfrak{S}^{(s)}_n} \mathsf{f}_n\big[f^{(s)}_{n+1}(X_{n+1}; \mathcal{A})\big] \\
&= \mathsf{f}_n\big[\bar{f}^{(1)}_{n+1}\big] + \sum_{s=2}^{n} \sum_{\mathcal{A} \in \mathfrak{S}^{(s)}_n} \prod_{j=1}^{s} \mathsf{f}_{|A_j|}\Big[\bar{f}_{|A_j|+1}\big(X(A_j \cup \{n+1\})\big)\Big] \\
&= \mathsf{f}_n\big[\bar{f}^{(1)}_{n+1}\big] + \sum_{s=2}^{n} \frac{1}{s!} \sum_{\substack{l_j \geq 1,\ j=1,\dots,s: \\ l_1+\dots+l_s=n}} \frac{n!}{l_1! \dots l_s!} \prod_{j=1}^{s} \mathsf{f}_{l_j}\Big[\bar{f}^{(1)}_{l_j+1}\big(X_{l_j}, x_{n+1}\big)\Big] \\
&= \sum_{s=1}^{n} \frac{1}{s!} \sum_{\substack{l_j \geq 1,\ j=1,\dots,s: \\ l_1+\dots+l_s=n}} \frac{n!}{l_1! \dots l_s!} \prod_{j=1}^{s} \mathsf{f}_{l_j}\Big[\bar{f}^{(1)}_{l_j+1}\big(X_{l_j}, x_{n+1}\big)\Big].
\end{aligned} \tag{5.9}$$

5. Now we consider graphs of the class $\bar{\mathcal{G}}^{(1)}[I_n, n+1]$. By Lemma 2.4, the class $\bar{\mathcal{G}}^{(1)}[I_n, n+1]$, $n \geq 2$, can be represented as the disjunct union

$$\bar{\mathcal{G}}^{(1)}[I_n, n+1] = \bigcup_{B \subset I_n : |B| \geq 1} \ \bigcup_{C \subset B} \bar{\mathcal{G}}[I_n; B; C]$$

of classes of graphs such that the vertex $n+1$ is not a cut vertex and for each graph, the set B represents the numbers of vertices of the block containing the vertex $n+1$ and C is the set of cut vertices of the graph in this block. Then

$$f_{n+1}^{(1)}(X_{n+1}) = \sum_{B\subset I_n:|B|\geq 1}\ \sum_{C\subset B} f_{n+1}\big(X_{n+1};B;C\big),$$

$$f_{n+1}\big(X_{n+1};B;C\big) = \sum_{\mathfrak{G}\in\bar{\mathcal{G}}[I_n;B;C]} h(X_{n+1};\mathfrak{G}).$$

Therefore, the value $\mathsf{f}_n\big[f_{n+1}^{(1)}\big]$ of the functional $\mathsf{f}_n[\cdot]$ is equal to

$$\mathsf{f}_n\big[f_{n+1}^{(1)}\big] = \sum_{B\subset I_n\,:|B|\geq 1}\ \sum_{C\subset B} f_{n+1}\big(X_{n+1};B;C\big). \tag{5.10}$$

6. Since each class $\bar{\mathcal{G}}[I_n;B;C]$ of graphs, $n\geq 2$, can be represented as the disjunct union of nonempty classes

$$\bar{\mathcal{G}}\big[I_n;B;C\big] = \bigcup_{\{B(z);z\in C\}\in\mathfrak{D}(B,C)} \bar{\mathcal{G}}\big[I_n;B\mid\{B(z),z\in C\}\big]$$

(see Lemmas 2.4 and 2.5), the functions $f_{n+1}(X_{n+1};B;C)$ can be represented as the following sums:

$$f_{n+1}\big(X_{n+1};B;C\big) = \sum_{\{B(z);z\in C\}\in\mathfrak{D}(B,C)} f_{n+1}\big(X_{n+1};B\mid\{B(z),z\in C\}\big),$$

$$f_{n+1}\big(X_{n+1};B\mid\{B(z),z\in C\}\big) = \sum_{\mathfrak{G}\in\bar{\mathcal{G}}\left[I_n;\ B|\{B(z),z\in C\}\right]} h\big(X_{n+1};\mathfrak{G}\big).$$

Since the functional $\mathsf{f}_n[\cdot]$ is linear, we have

$$\mathsf{f}_n\big[f_{n+1}(X_{n+1};B;C)\big] = \sum_{\{B(z);\ z\in C\}\in\mathfrak{D}(B,C)} \mathsf{f}_n\Big[f_{n+1}\big(X_{n+1};B\mid\{B(z),z\in C\}\big)\Big]. \tag{5.11}$$

7. Finally, we recall (see Lemma 2.6) that each class $\bar{\mathcal{G}}\big[I_n;B\mid\{B(z),z\in C\}\big]$, $n\geq 2$, is equivalent to the Cartesian product

$$\bar{\mathcal{G}}\big[I_n;B\mid\{B(z),z\in C\}\big] = \mathcal{F}[B;n+1]\otimes\left(\bigotimes_{z\in C}\bar{\mathcal{G}}[B(z);z]\right),$$

where $\mathcal{F}_n[B;n+1]$ is the class of graphs without cut vertices over the set of vertices $B\cup\{n+1\}$, $\bar{\mathcal{G}}[B(z);z]$, $z\in C$ is a collection of nonempty classes such that $\bar{\mathcal{G}}_z[B(z);z]$ contains all connected graphs with the set of vertices $B(z)\cup\{z\}$ and a marked vertex $z\in C$. Then

$$\mathsf{f}_n\Big[f_{n+1}\big(X_{n+1};B\mid\{B(z),z\in C\}\big)\Big]$$
$$= \sum_{\mathfrak{G}_B\in\mathcal{F}_n[B;n+1]}\left(\prod_{z\in C}\ \sum_{\mathfrak{G}(z)\in\bar{\mathcal{G}}_z[B(z);z]}\right)\mathsf{f}_n\Big[h\Big(X_{n+1};\bigvee_{z\in C}\{\mathfrak{G}_B\vee\mathfrak{G}(z)\}\Big)\Big]. \tag{5.12}$$

Since

$$h\Big(X_{n+1};\bigvee_{z\in C}\{\mathfrak{G}_B\vee\mathfrak{G}(z)\}\Big) = h\big(X(B);\mathfrak{G}_B\big)\prod_{z\in C}h\big(X(B(z));\mathfrak{G}(z)\big)$$

(cf. (5.7)) and (5.8) is multiplicative, we obtain the following expression:

$$\mathsf{f}_n\Big[h\Big(X_{n+1};\bigvee_{z\in C}\{\mathfrak{G}_B\vee\mathfrak{G}(z)\}\Big)\Big] = \mathsf{f}_{|B|}\big[h(X(B);\mathfrak{G}_B)\big]\prod_{z\in C}\mathsf{f}_{|B(z)|}\big[h(X(B(z));\mathfrak{G}(z))\big].$$

Then, using (5.12), we have

$$\mathsf{f}_n\Big[f_{n+1}\big(X_{n+1};B\mid\{B(z),z\in C\}\big)\Big] = \mathsf{f}_{|B|}\big[g_{|B|+1}\big]\cdot\prod_{z\in C}\mathsf{f}_{|B(z)|}\big[\bar{f}_{|B(z)|+1}\big], \tag{5.13}$$

where we have used the functions $g_{n+1}(X_{n+1})$ introduced in (5.3) and the functions

$$\bar{f}_{B(z)|+1}\big(X_{|B|+1}\big) = \sum_{\mathfrak{G}(z)\in\bar{\mathcal{G}}_z[B(z);z]} h\big(X_{|B|+1};\mathfrak{G}(z)\big), \quad z\in C.$$

If $C=\varnothing$ and $\mathfrak{G}_B=\mathfrak{G}$, then the left-hand side of the formula (5.12) is equal to $\mathsf{f}_n\big[f_{n+1}(X_{n+1})\big]$.

8. From the formulas (5.10), (5.11), and (5.13) we obtain:

$$\begin{aligned}
\mathsf{f}_n\big[f^{(1)}_{n+1}(X_{n+1})\big] &= \sum_{B\subset I_n:|B|\ge 1}\ \sum_{C\subset B} f_{n+1}(X_{n+1};B;C)\\
&= \sum_{B\subset I_n:|B|\ge 1}\ \sum_{C\subset B}\ \sum_{\{B(z);z\in C\}\in\mathfrak{D}(B,C)} \mathsf{f}_n\Big[f_{n+1}\big(X_{n+1};B\mid\{B(z),z\in C\}\big)\Big]\\
&= \sum_{B\subset I_n:|B|\ge 1}\ \sum_{C\subset B}\ \sum_{\{B(z);z\in C\}\in\mathfrak{D}(B,C)} \mathsf{f}_{|B|}\big[g_{|B|+1}\big]\cdot\prod_{z\in C}\mathsf{f}_{|B(z)|}\big[\bar{f}_{|B(z)|+1}\big]\\
&= \sum_{m=1}^{n}\binom{n}{m}\sum_{C\subset I_m}\ \sum_{\{B(z);z\in C\}\in\mathfrak{D}(I_m,C)} \mathsf{f}_m[g_{m+1}]\cdot\prod_{z\in C}\mathsf{f}_{|B(z)|}\big[\bar{f}_{|B(z)|+1}\big]\\
&= \sum_{m=1}^{n}\binom{n}{m}\sum_{l=1}^{m}\binom{m}{l}\sum_{\{B(z_j);j\in I_l\}\in\mathfrak{D}(I_m,I_l)} \mathsf{f}_m[g_{m+1}]\cdot\prod_{j=1}^{l}\mathsf{f}_{|B(z_j)|}\big[\bar{f}_{|B(z_j)|+1}\big]\\
&= \sum_{m=1}^{n}\binom{n}{m}\mathsf{f}_m[g_{m+1}]\sum_{l=1}^{m}\binom{m}{l}\sum_{\substack{\langle k_1,\dots,k_l\rangle:\\ k_1+\dots+k_l=n-m,\\ k_j\ge 1,\ j=1,\dots,l}} \frac{(n-m)!}{k_1!\dots k_l!}\cdot\prod_{j=1}^{l}\mathsf{f}_{k_j}\big[\bar{f}_{k_j+1}\big]; \qquad (5.14)
\end{aligned}$$

here $\mathsf{f}_0\big[\bar{f}_{|B(z)|+1}\big]=1$ for $B(z)=\varnothing$.

9. Since the power series used below converge, due to (5.9) we obtain

$$\begin{aligned}
\mathsf{S}\big[z;\bar{\mathsf{f}}\big] &= \sum_{n=0}^{\infty}\frac{z^n}{n!}\mathsf{f}_n\big[\bar{f}_{n+1}\big] = 1+\sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{s=1}^{n}\frac{1}{s!}\sum_{\substack{l_j\ge 1,\ j=1,\dots,s:\\ l_1+\dots+l_s=n}}\frac{n!}{l_1!\dots l_s!}\prod_{j=1}^{s}\mathsf{f}_{l_j}\Big[\bar{f}^{(1)}_{l_j+1}\big(X_{l_j},x_{n+1}\big)\Big]\\
&= 1+\sum_{s=1}^{\infty}\frac{1}{s!}\sum_{s=n}^{\infty}\ \sum_{\substack{l_j\ge 1,\ j=1,\dots,s:\\ l_1+\dots+l_s=n}}\ \prod_{j=1}^{s}\frac{z^{l_j}}{l_j!}\mathsf{f}_{l_j}\Big[\bar{f}^{(1)}_{l_j+1}\big(X_{l_j},x_{n+1}\big)\Big]\\
&= 1+\sum_{s=1}^{\infty}\frac{1}{s!}\prod_{j=1}^{s}\sum_{l_j=1}^{\infty}\frac{z^{l_j}}{l_j!}\mathsf{f}_{l_j}\Big[\bar{f}^{(1)}_{l_j+1}\big(X_{l_j},x_{n+1}\big)\Big] = \exp\left[\sum_{n=1}^{\infty}\frac{z^n}{n!}\mathsf{f}_n\Big[\bar{f}^{(1)}_{n+1}(X_{n+1})\Big]\right].
\end{aligned}$$

Now we transform the sum in the exponent after substituting the value (5.14) of the functional $\mathsf{f}_n\big[\bar{f}^{(1)}_{n+1}(X_{n+1})\big]$:

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{z^n}{n!}\,\mathsf{f}_n\Big[\bar{f}^{(1)}_{n+1}(X_{n+1})\Big] &= \sum_{n=1}^{\infty}\frac{z^n}{n!}\sum_{m=1}^{n}\frac{n!}{m!}\mathsf{f}_m[g_{m+1}]\sum_{l=1}^{m}\binom{m}{l}\sum_{\substack{\langle k_1,\dots,k_l\rangle:\\ k_1+\dots+k_l=n-m,\\ k_j\ge 1,\ j=1,\dots,l}}\ \prod_{j=1}^{l}\frac{1}{k_j!}\mathsf{f}_{k_j}\big[\bar{f}_{k_j+1}\big]\\
&= \sum_{m=1}^{\infty}\frac{z^m}{m!}\mathsf{f}_m[g_{m+1}]\sum_{l=1}^{m}\binom{m}{l}\sum_{n=m}^{\infty}\ \sum_{\substack{\langle k_1,\dots,k_l\rangle:\\ k_1+\dots+k_l=n-m,\\ k_j\ge 1,\ j=1,\dots,l}}\ \prod_{j=1}^{l}\frac{z^{k_j}}{k_j!}\mathsf{f}_{k_j}\big[\bar{f}_{k_j+1}\big]
\end{aligned}$$

$$= \sum_{m=1}^{\infty} \frac{z^m}{m!} \mathsf{f}_m[g_{m+1}] \sum_{l=1}^{m} \binom{m}{l} \sum_{\substack{\langle k_1,\ldots,k_l\rangle: \\ k_j \geq 1,\ j=1,\ldots,l}} \prod_{j=1}^{l} \frac{z^{k_j}}{k_j!} \mathsf{f}_{k_j}\big[\bar{f}_{k_j+1}\big]$$

$$= \sum_{m=1}^{\infty} \frac{z^m}{m!} \mathsf{f}_m[g_{m+1}] \sum_{l=1}^{m} \binom{m}{l} \prod_{j=1}^{l} \sum_{k_j=1}^{\infty} \frac{z^{k_j}}{k_j!} \mathsf{f}_{k_j}\big[\bar{f}_{k_j+1}\big]$$

$$= \sum_{m=1}^{\infty} \frac{z^m}{m!} \mathsf{f}_m[g_{m+1}] \left(\sum_{k=0}^{\infty} \frac{z^k}{k!} \mathsf{f}_k\big[\bar{f}_{k+1}\big] \right)^m = \sum_{m=1}^{\infty} \frac{z^m}{m!} \mathsf{f}_m[g_{m+1}] \Big(\mathsf{S}\big[z; \bar{\mathsf{f}}\big]\Big)^m.$$

Then

$$\mathsf{S}\big[z; \bar{\mathsf{f}}\big] = \exp\Big(\mathsf{S}\big[z\mathsf{S}[z; \bar{\mathsf{f}}]; \mathsf{g}\big] - 1\Big).$$

The theorem is proved. □

6. Conclusion. In conclusion, we indicate some applications of the algebraic technique described in this paper. In equilibrium statistical mechanics of classical systems (see [10]), for expanding the equation of state $P(z,T)$ into a power series in the so-called activity z, the quantity $\ln \Xi$ arises, where Ξ is the partition function of the system defined by the formula

$$\Xi = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{\Omega^n} \exp\left(- \sum_{\{j,k\} \subset I_n} U(x_j - x_k)/T \right) \prod_{k=1}^{n} dx_k = \mathsf{f}[z; \mathsf{f}],$$

where dx is the Lebesgue measure in $\mathbb{R}^3$, which is invariant under the translation group, and $U(x)$ is the coupling potential at the point $x \in \mathbb{R}^3$. The components of the sequence f of symmetric functions are generated by the function $w(x,y) = \exp(-U(x-y)/T)$. The expression for $P(z,T)$ is given by the formula (2.2) in which the coefficients of the so-called *group* decomposition are defined by the functions of the sequence $\bar{\mathsf{f}}$. In these terms, the density ρ of the number of particles for systems considered is defined by the formula

$$\rho = \sum_{n=0}^{\infty} \frac{z^{n+1}}{n!} \int_{\Omega^n} \bar{f}_{n+1}(X_{n+1}) \prod_{k=1}^{n} dx_k = \mathsf{S}[z; \bar{\mathsf{f}}].$$

Then the formula (5.4) becomes the equation $\rho = \exp\big(\mathsf{S}\big[z\rho; \mathsf{g}\big] - 1\big)$ for ρ whose coefficients (called the irreducible integrals) are defined by the components of the sequence g.

Finally, we indicate a simple application of the formula (5.4) in the problem of enumerating graphs without cut vertices (see [12]).

Theorem 6.1. *Assume that in the sequences* $\langle N_n;\ n \in \mathbb{N}\rangle$ *and* $\langle M_n;\ n \in \mathbb{N}\rangle$, *the components* N_n *are the numbers of connected graphs with* $n \in \mathbb{N}$ *vertices* (*where* $N_1 = 1$) *and* M_m *are the numbers of connected graphs without cut vertices with* $m \in \mathbb{N}$ *vertices* ($M_1 \equiv 1$). *Then the numbers* M_{n+1}, N_m, $m = 1, \ldots, n+1$, *and* M_m, $m = 1, \ldots, n$, *satisfy the following recurrence formula*:

$$N_{n+1} = n! \sum_{s=1}^{n} \frac{1}{s!} \sum_{\substack{\langle k_1,\ldots,k_s\rangle,\ k_j \in \mathbb{N}: \\ k_1+\cdots+k_s = n}} \prod_{j=1}^{s} \left(\sum_{m=1}^{k_j} \frac{M_{m+1}}{m!} \sum_{\substack{\langle l_1,\ldots,l_m\rangle,\ l_i \in \mathbb{N}_+: \\ l_1+\cdots+l_m = k_j - m}} \prod_{i=1}^{m} \frac{N_{l_i+1}}{l_i!} \right). \tag{6.1}$$

Proof. It suffices to set $\mathsf{f}_m[g_{m+1}] = M_{m+1}$ and $\mathsf{f}_n[\bar{f}_{n+1}] = N_{n+1}$ in the formulas (5.9) and (5.14). □

Corollary 6.1 (see [9]). *The generating functions*

$$F(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} N_{n+1}, \quad G(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} M_{n+1} \tag{6.2}$$

are related by the following functional equation:

$$F(z) = \exp\Big[G\big(zF(z)\big) - 1\Big]. \tag{6.3}$$

Proof. Corollary 6.1 follows from (5.4) if we set $\mathsf{f}_m[g_{m+1}] = M_{m+1}$ and $\mathsf{f}_n[\bar{f}_{n+1}] = N_{n+1}$. □

Remark 6.1. The series (6.2) diverge at each nonzero point of the z-plane, i.e., they must be treated as asymptotic power series. Despite the divergence, they can be used for sequential calculating the numbers M_m, $m \in \mathbb{N}$, based on the generating function $F(z)$ of the numbers N_n and using the derivatives of order $n = 1, \dots, m$ (this possibility is justified in [12]). For example,

$$F(0;1) = 1,\ F'(0;1) = 1,\ F''(0;1) = N_3 = 4,\ F'''(0;1) = N_4 = 38,\ F^{IV}(0;1) = N_5 = 728,$$

and we obtain the following values:

$$G(0,1) = 1,\ G'(0,1) = M_2 = 1,\ G''(0;1) = M_3 = 1,\ G'''(0;1) = M_4 = 10,\ G^{IV}(0;1) = M_5 = 238.$$

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