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ON DISCRETE BOUNDARY-VALUE PROBLEMS AND THEIR APPROXIMATION PROPERTIES

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Abstract. Discrete analogs of pseudo-differential operators and equations in discrete Sobolev–Slobodetsky spaces are considered. Using suitable discrete boundary conditions, we prove the unique solvability of the discrete boundary-value problem.

***Keywords and phrases*:** discrete pseudo-differential operator, discrete boundary-value problem, solvability, approximation, periodic factorization, error estimate.

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1. Introduction. The theory of approximate methods for solving boundary-value problems for elliptic differential equations is a fairly developed branch of mathematics. The most common methods are the grid method (see [16]), method of finite elements (see [17]), and various variational methods (see [14]). In recent years, the method of difference potentials (see [15]) has received some development; in a certain sense this method is a combination of the grid method and the method of boundary integral equations (see [6]). We also mention approximate methods for solving integral equations, in particular, the method of singular integral equations and the method of equations in convolutions (see [5, 9]), to which a large number of boundary-value problems are reduced. Unfortunately, the general theory of approximate methods of functional analysis (see [1, 4, 7, 8]) is difficult to apply to problems listed above; in fact, this led to the emergence of special developments.

In the overwhelming majority of cases, the technique of the discrete Fourier transform and spaces of periodic functions is not widespread; the exceptions are works [3, 10–13].

Let $\mathbb{Z}^m$ be a integer lattice in $\mathbb{R}^m$, $\mathbb{T}^m = [-\pi, \pi]^m$ be the m-dimensional cube, $h > 0$, and $\hbar = h^{-1}$. For function u_d of a discrete argument defined on $h\mathbb{Z}^m$, we introduce the discrete Fourier transform

$$\tilde{u}_d(\xi) \equiv (F_d u_d)(\xi) = \sum_{x \in h\mathbb{Z}^m} u_d(x) e^{ix\cdot\xi} h^m, \quad \xi \in \hbar\mathbb{T}^m.$$

Let $A_d(\xi)$ be a periodic function in $\mathbb{R}^m$ with the principal cube of periods $\hbar\mathbb{T}^m$. An operator of the form

$$(A_d u_d)(x) = \frac{1}{(2\pi)^m} \sum_{y \in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} A_d(\xi) e^{i(y-x)\cdot\xi} u_d(y) h^m d\xi, \quad x \in D_d = D \cap h\mathbb{Z}^m,$$

where D is a domain in $\mathbb{R}^m$, is called a *discrete pseudo-differential operator* in the discrete domain D_d. To this operator, we relate the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{1}$$

where v_d is a given function in D_d.

One of the main problems is the description of conditions for the unique solvability of Eq. (1) in suitable functional spaces over the discrete domain D_d and the application of the obtained results to the construction of suitable approximations of classical pseudo-differential equations (see [2]). The main difficulty is that if the domain is a part of $\mathbb{R}^m$ (for example, a half-space or a cone), the standard condition for the symbol $A(\xi)$ to be elliptic is no longer sufficient. Thus, for the case of a half-space, conditions for the solvability of Eq. (1) in discrete Sobolev–Slobodetsky spaces $H^s(D_d)$ were

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described in [23–25]. The theory of the periodic Riemann problem constructed earlier (see [21]) played a fundamental role.

Consider the boundary-value problem

$$(Au)(x) = 0, \quad x \in \mathbb{R}^m_+, \qquad \frac{\partial u}{\partial x_m}|_{x_m=0} = g(x'), \quad x' \in \mathbb{R}^{m-1}, \tag{2}$$

where A is an elliptic pseudo-differential operator whose symbol $A(\xi)$ satisfies the condition

$$c_1(1+|\xi|)^\alpha \le |A(\xi)| \le c_2(1+|\xi|)^\alpha, \quad \alpha \in \mathbb{R}; \tag{3}$$

here c_1 and c_2 are positive constants.

It is well known that the boundary-value problem (2) is uniquely solvable in Sobolev–Slobodetsky spaces $H^s(\mathbb{R}^m_+)$ for any right-hand side $g \in H^{s-3/2}(\mathbb{R}^{m-1})$ under some restrictions for the factorization index $\varkappa$ of the symbol $A(\xi)$, namely, $\varkappa - s = 1 + \delta$, $|\delta| < 1/2$ (see [2]).

To find an approximate solution of the problem (2), we construct a discrete pseudo-differential operator A_d acting in the spaces of functions of a discrete argument $H^s(h\mathbb{Z}^m_+)$ (see [23, 24]) and formulate the corresponding discrete boundary-value problem, whose unique solvability is studied by the methods proposed in [23–25] (see also [18, 22, 26–30]).

2. Discrete spaces and operators. For modeling the original operator A whose symbol satisfies the condition (3), we consider periodic symbols satisfying the following similar condition:

$$c_1\big(1+|\zeta^2|\big)^{\alpha/2} \le |A_d(\xi)| \le c_2\big(1+|\zeta^2|\big)^{\alpha/2}; \tag{4}$$

here c_1 and c_2 are positive constants independent of h and $\zeta^2 = h^{-2}\sum\limits_{k=1}^{m}\big(e^{-ih\cdot\xi_k}-1\big)^2$.

We use the discrete analog $S(h\mathbb{Z}^m)$ (see [23]) of the Schwartz space $S(\mathbb{R}^m)$ of infinitely differentiable functions rapidly decreasing at infinity.

To study simple pseudo-differential equations (equations of the convolution type), the authors previously used the restriction operator P_h to the lattice; this was expedient since integral operators are defined by their kernel functions. We introduce the restriction operator Q_h for functions $u \in S(\mathbb{R}^m)$. First, we calculate the Fourier transform $\tilde{u}(\xi)$, then we restrict it to $\hbar\mathbb{T}^m$ and continue periodically to $\mathbb{R}^m$. Further, applying the inverse discrete Fourier transform F_d^{-1}, we obtain the discrete function $(Q_h u)(\tilde{x})$, $\tilde{x} \in h\mathbb{Z}^m$. For many reasons, this projector Q_h is much more convenient than the projector P_h mentioned above. It turns out that, according to the following result, the projectors P_h and Q_h are almost the same.

Lemma 1. *For $u \in S(\mathbb{R}^m)$ and for all $\beta > 0$, the following estimate holds*:

$$\Big|(P_h u)(\tilde{x}) - (Q_h u)(\tilde{x})\Big| \le Ch^\beta \quad \forall \tilde{x} \in h\mathbb{Z}^m,$$

where the constant C depends only on u.

Proof. We must compare two Fourier transforms. Indeed, by definition,

$$(P_h u)(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\mathbb{R}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi, \quad (Q_h u)(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi;$$

therefore,

$$(P_h u)(\tilde{x}) - (Q_h u)(\tilde{x}) = \frac{1}{(2\pi)^m}\int\limits_{\mathbb{R}^m\setminus\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi}\tilde{u}(\xi)d\xi.$$

Now the required assertion follows from the invariance the Schwartz class $S(\mathbb{R}^m)$ under the Fourier transform and the simple estimate

$$\big|\tilde{u}(\xi)\big| \le C_u|\xi|^{-\gamma}, \quad \gamma > 0. \qquad \square$$

Further, we define the symbol $A_d(\xi)$ as follows: we periodically continue to $\mathbb{R}^m$ the restriction of $A(\xi)$ to the cube $\hbar\mathbb{T}^m$. We consider this h-operator as an approximating operator for A. Thus, ti find a discrete approximate solution of the equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{5}$$

in the case where $D = \mathbb{R}^m$ we can use the following discrete equation:

$$A_d u_d = Q_h v. \tag{6}$$

Definition 1. The space $H^s(h\mathbb{Z}^m)$ is the closure of the space $S(h\mathbb{Z}^m)$ with respect to the norm

$$\|u_d\|_s = \left(\int\limits_{\hbar\mathbb{T}^m} \left(1 + |\zeta^2|\right)^s \left|\tilde{u}_d(\xi)\right|^2 d\xi \right)^{1/2}.$$

Note that these discrete spaces were introduced and examined in [3].

For a domain $D \subset \mathbb{R}^m$, we introduce the discrete domain $D_d = D \cap h\mathbb{Z}^m$ and define the following functional spaces.

Definition 2. By definition, the space $H^s(D_d)$ consists of discrete functions of the space $H^s(h\mathbb{Z}^m)$ whose supports are contained in $\overline{D_d}$. The norm in $H^s(D_d)$ is induces by the initial space $H^s(h\mathbb{Z}^m)$. The right-hand side of Eq. (1) belongs to the space $H_0^s(D_d)$, which consists of discrete functions u_d with supports in D_d continuable to the whole space $H^s(h\mathbb{Z}^m)$. The norm in $H_0^s(D_d)$ is defined by the formula

$$\|u_d\|_s^+ = \inf \|\ell u_d\|_s,$$

infimum is taken over all continuations ℓ.

Of course, these norms are equivalent to the L_2-norm, but the equivalence constants depend on h. In our case, all constants involving in the estimates are independent of h (this is essential when comparing discrete and continuous solutions).

Definition 3. A *periodic factorization* of an elliptic symbol $A_d(\xi)$ is its representation in the form

$$A_d(\xi) = A_{d,+}(\xi) A_{d,-}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ admit analytical continuations with respect to the last variable ξ_m into the half-bands $\hbar\Pi_\pm$ for almost all fixed $\xi' \in \hbar\mathbb{T}^{m-1}$ and satisfy the estimates

$$\left|A_{d,+}^{\pm 1}(\xi)\right| \leq c_1 \left(1 + |\hat{\zeta}^2|\right)^{\pm \varkappa/2}, \quad \left|A_{d,-}^{\pm 1}(\xi)\right| \leq c_2 \left(1 + |\hat{\zeta}^2|\right)^{\pm \alpha - \varkappa/2}$$

with constants c_1 and c_2 independent of h,

$$\hat{\zeta}^2 \equiv \hbar^2 \left(\sum_{k=1}^{m-1} \left(e^{-ih\xi_k} - 1\right)^2 + \left(e^{-ih(\xi_m + i\tau)} - 1\right)^2 \right), \quad \xi_m + i\tau \in \hbar\Pi_\pm.$$

The number $\varkappa \in \mathbb{R}$ is called the *index of periodic factorization.*

3. Boundary conditions and solvability. Consider Eq. (1) under the following conditions. In the case $D = \mathbb{R}_+^m$, we search for a solution in the space $H^s(D_d)$. The symbol $A_d(\xi)$ admits a periodic factorization with index $\varkappa$, where $\varkappa - s = 1 + \delta$, $|\delta| < 1/2$. It was proved in [23, 24] that in this case, the structure of the general solution of Eq. (1) is as follows (for completeness, we present the corresponding result from [23]).

Theorem 1. *If $\varkappa - s = n + \delta$, where $n \in \mathbb{N}$ and $|\delta| < 1/2$, then the Fourier images of all solutions of Eq.* (1) *are described by the formula*

$$\tilde{u}_d(\xi) = \tilde{A}_{d,+}^{-1}(\xi) X_n(\xi) P_{\xi'}^{per}\Big(X_n^{-1}(\xi) \tilde{A}_{d,-}^{-1}(\xi) \widetilde{\ell v_d}(\xi)\Big) + \tilde{A}_{d,+}^{-1}(\xi) \sum_{k=0}^{n-1} c_k(\xi') \hat{\zeta}_m^k,$$

where $X_n(\xi)$ is an arbitrary polynomial of degree n that depends on the variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1)$, $k = 1, \ldots, m$, and satisfies the condition (3), *and $c_k(\xi')$, $j = 0, 1, \ldots, n-1$, are arbitrary function from $H^{s_k}(h\mathbb{T}^{m-1})$, $s_k = s - \varkappa + k - 1/2$. The following a priori estimate holds*:

$$\|u_d\|_s \leq a\left(\|f\|^+_{s-\alpha} + \sum_{k=0}^{n-1}[c_k]_{s_k}\right),$$

where $[\cdot]_{s_k}$ is the norm in the space $H^{s_k}(h\mathbb{T}^{m-1})$ and a is a constant independent of h.

Taking into account the conditions imposed above, we obtain the following formula for the general solution:

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)\tilde{c}_0(\xi'),$$

which involves only one arbitrary function. To define it uniquely and model the boundary condition of the problem (2), we impose a discrete boundary condition

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \hbar\big(e^{-ih\xi_m} - 1\big)\tilde{u}_d(\xi', \xi_m)d\xi_m = \tilde{g}_d(\xi'), \quad \xi' \in \hbar\mathbb{T}^{m-1}, \tag{7}$$

where $\tilde{g}_d(\xi')$ is a given function.

Thus, for numerical simulation of the problem (2), we consider the discrete problem (1), (7). It remains to prove the unique solvability of the problem (1), (7) and compare the discrete and continuous solutions.

Taking into account Theorem 1 and the condition (7), we have

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \hbar\big(e^{-ih\xi_m} - 1\big)\tilde{A}^{-1}_{d,+}(\xi', \xi_m)\tilde{c}_0(\xi')d\xi_m = \tilde{g}_d(\xi'), \quad \xi' \in \hbar\mathbb{T}^{m-1}.$$

Introduce the notation

$$\int\limits_{-\hbar\pi}^{\hbar\pi} \hbar\big(e^{-ih\xi_m} - 1\big)\tilde{A}^{-1}_{d,+}(\xi', \xi_m)d\xi_m = \tilde{t}_d(\xi'); \tag{8}$$

then we obtain the equality

$$\tilde{t}_d(\xi')\tilde{c}_0(\xi') = \tilde{g}_d(\xi').$$

Definition 4. The boundary-value problem (1), (7) is said to be *elliptic* if

$$\inf_{\xi' \in \hbar\mathbb{T}^{m-1}} \left|\tilde{t}_d(\xi')\right| > 0.$$

Remark 1. Note that as $h \to 0$, the condition (8) turns into the well-known condition

$$\int\limits_{-\infty}^{+\infty} (-i\xi_m)A^{-1}_+(\xi', \xi_m)d\xi_m = \tilde{t}(\xi'),$$

(see [2]), where $A_+(\xi', \xi_m)$ is an element of factorization in the Vishik–Eskin sense.

Theorem 2. *The elliptic boundary-value problem* (1), (7) *is uniquely solvable in the space $H^s(D_d)$ for any right-hand side $g \in H^{s-3/2}(h\mathbb{R}^{m-1})$. For $s \geq 1$, the following a priori estimate holds*:

$$\|u_d\|_s \leq C\|g_d\|_{s-3/2},$$

where C is a constant independent of h.

Proof. Indeed, according to the above calculations

$$\tilde{u}_d(\xi) = \frac{\tilde{g}_d(\xi')}{A_{d,+}(\xi', \xi_m)\tilde{t}_d(\xi')}.$$

Then

$$\|u_d\|_s^2 = \int\limits_{\hbar\mathbb{T}^m} \left| \frac{\tilde{g}_d(\xi')}{A_{d,+}(\xi', \xi_m)\tilde{t}_d(\xi')} \right|^2 \left(1 + |\zeta^2|\right)^s d\xi \leq \hbar^{2(s-\varkappa)} \|g_d\|_2^2.$$

Therefore,

$$\|u_d\|_s \leq C\|g_d\|_{s-\varkappa} \leq C\|g_d\|_{s-3/2},$$

since the inequality $s - \varkappa \leq s - 3/2$ is fulfilled due to the condition $\varkappa - s = 1 + \delta$ and the inequality $1 - \delta \leq s - 3/2$ is equivalent to the inequality $s \geq 1/2 - \delta$. Since $|\delta| < 1/2$, it suffices to require $s \geq 1$. □

4. Approximation estimates.

Lemma 2. *For $\xi_m \in [-\hbar\pi, \hbar\pi]$, the following estimate holds*:

$$\left|\hbar\left(e^{-ih\xi_m} - 1\right) + i\xi_m\right| \leq C\hbar,$$

where C is a constant independent of h.

Proof. Indeed, since

$$e^{-ih\xi_m} = \sum_{k=0}^{\infty} \frac{(-ih\xi_m)^k}{k!},$$

we have

$$\hbar\left(e^{-ih\xi_m} - 1\right) + i\xi_m = \sum_{k=1}^{\infty} \frac{h^k(-i\xi_m)^{k+1}}{(k+1)!},$$

and hence

$$\left|\hbar\left(e^{-ih\xi_m} - 1\right) + i\xi_m\right| \leq \hbar \sum_{k=1}^{\infty} \frac{1}{(k+1)!},$$

which implies the required estimate. □

It was proved earlier (see [2]) that the Fourier image of a solution of the problem (2) is defined by the formula

$$\tilde{u}(\xi) = \frac{\tilde{g}(\xi')}{A_+(\xi', \xi_m)\tilde{t}(\xi')}$$

if the ellipticity condition $\inf\limits_{\xi' \in \mathbb{R}^{m-1}} |t(\xi')| > 0$ is fulfilled. In the discrete case, we have the following similar formula

$$\tilde{u}_d(\xi) = \frac{\tilde{g}_d(\xi')}{A_{d,+}(\xi', \xi_m)\tilde{t}_d(\xi')}.$$

Below, we compare discrete and continuous solutions for a sufficiently smooth right-hand side of the boundary condition under some restrictions for the factorization index.

Lemma 3. *For $\varkappa > 2$, the following estimate holds*:

$$\left|\tilde{t}(\xi') - \tilde{t}_d(\xi')\right| \leq C\hbar^{-\varkappa+2}, \tag{9}$$

where C is a constant independent of h.

Proof. Due to the coincidence of the symbols $A_+(\xi',\xi_m)$ and $A_{d,+}(\xi',\xi_m)$ on $\hbar\mathbb{T}^m$, we must estimate the difference of the following two integrals:

$$\int\limits_{-\infty}^{+\infty}(-i\xi_m)A_+^{-1}(\xi',\xi_m)d\xi_m-\int\limits_{-\hbar\pi}^{\hbar\pi}\hbar\big(e^{-ih\xi_m}-1\big)\tilde{A}_{d,+}^{-1}(\xi',\xi_m)d\xi_m$$

$$=\left(\int\limits_{-\infty}^{-\hbar\pi}+\int\limits_{\hbar\pi}^{+\infty}\right)(-i\xi_m)A_+^{-1}(\xi',\xi_m)d\xi_m-\int\limits_{-\hbar\pi}^{\hbar\pi}\Big(\hbar\big(e^{-ih\xi_m}-1\big)+i\xi_m\Big)\tilde{A}_{d,+}^{-1}(\xi',\xi_m)d\xi_m.$$

We estimate the second integral by Lemma 2:

$$\left|\int\limits_{-\hbar\pi}^{\hbar\pi}\Big(\hbar\big(e^{-ih\xi_m}-1\big)+i\xi_m\Big)\tilde{A}_{d,+}^{-1}(\xi',\xi_m)d\xi_m\right|\le C_1h\int\limits_{-\hbar\pi}^{\hbar\pi}|\tilde{A}_{d,+}^{-1}(\xi',\xi_m)|d\xi_m$$

$$\le C_2h\int\limits_{0}^{\hbar\pi}\big(1+|\xi'|+\xi_m\big)^{-\varkappa}d\xi_m\le C_3\hbar^{-\varkappa+2}.$$

If $\varkappa>2$, then the last expression tends to zero as $h\to 0$. The first two terms are estimated similarly; consider one of them:

$$\left|\int\limits_{\hbar\pi}^{+\infty}(-i\xi_m)A_+^{-1}(\xi',\xi_m)d\xi_m\right|\le C_1\int\limits_{\hbar\pi}^{+\infty}|\xi_m|\big(1+|\xi'|+|\xi_m|\big)^{-\varkappa}d\xi_m\le C_2\hbar^{-\varkappa+2}.\qquad\square$$

Remark 2. Lemma 3 implies, in particular, that for $\varkappa>2$ we can assume that $\inf\limits_{\xi'\in\hbar\mathbb{T}^{m-1}}|\tilde{t}_d(\xi')|$ is independent of h.

Theorem 3. *If $g\in H^{s-3/2}(h\mathbb{Z}^{m-1})$, $\varkappa>2$, and $\beta\le 5/2+\delta$, then solutions of the problem* (2), (1), (7) *satisfy the estimate*

$$\|Q_hu-u_d\|_\beta\le Ch^{\varkappa-2}\|g_d\|_s,$$

where C is a constant independent of h.

Proof. Let $g\in S(\mathbb{R}^m)$. Owing to the above constructions, it remains to estimate the proximity of $\tilde{t}(\xi')$ and $\tilde{t}_d(\xi')$. Recall that the symbols $A_+(\xi',\xi_m)$ and $A_{d,+}(\xi',\xi_m)$ and the functions $\tilde{g}(\xi')$ and$\tilde{g}_d(\xi')$ coincide on $\hbar\mathbb{T}^m$.

We estimate the difference for $\xi\in\hbar\mathbb{T}^m$:

$$\tilde{u}(\xi)-\tilde{u}_d(\xi)=\frac{\tilde{g}(\xi')}{A_+(\xi',\xi_m)\tilde{t}(\xi')}-\frac{\tilde{g}_d(\xi')}{A_{d,+}(\xi',\xi_m)\tilde{t}_d(\xi')}=\frac{\tilde{g}(\xi')}{A_+(\xi',\xi_m)}\left(\frac{1}{\tilde{t}(\xi')}-\frac{1}{\tilde{t}_d(\xi')}\right)$$

$$=\frac{\tilde{g}(\xi')}{A_+(\xi',\xi_m)}\frac{\tilde{t}_d(\xi')-\tilde{t}(\xi')}{\tilde{t}(\xi')\tilde{t}_d(\xi')}.$$

Due to Lemma 3 and the ellipticity of the problem (2) (this means that $\inf\limits_{\xi'\in\mathbb{R}^{m-1}}|t(\xi')|>0$) we obtain

$$\big|\tilde{u}(\xi)-\tilde{u}_d(\xi)\big|\le C\hbar^{-\varkappa+2}\left|\frac{\tilde{g}_d(\xi')}{A_{d,+}(\xi',\xi_m)}\right|,\quad \xi\in\hbar\mathbb{T}^m.$$

The last inequality implies

$$\left|\widetilde{(Q_hu)}(\xi)-\tilde{u}_d(\xi)\right|\le C\hbar^{-\varkappa+2}\left|\frac{\tilde{g}_d(\xi')}{A_{d,+}(\xi',\xi_m)}\right|$$

due to the choice of the projector Q_h. Further, since $g_d \in H^{s-3/2}(h\mathbb{Z}^{m-1})$, we see that $\tilde{g}_d(\xi')A^{-1}_{d,+}(\xi',\xi_m) \in \tilde{H}^{\beta}(D_d)$. It remains to verify the accuracy of the exponent β:

$$\int\limits_{\hbar\mathbb{T}^m} \left|\tilde{g}_d(\xi')A^{-1}_{d,+}(\xi',\xi_m)\right|^2 \left(1+|\zeta^2|\right)^{\beta} d\xi \le C\hbar^{2(\beta-\varkappa)} \int\limits_{\hbar\mathbb{T}^m} \left|\tilde{g}_d(\xi')\right|^2 d\xi \le C\hbar^{2(\beta-\varkappa)}\hbar^{2(s-3/2)}\|g_d\|_s^2$$
$$= C\hbar^{2(\beta-\varkappa+s-3/2)}\|g_d\|_s^2 = C\hbar^{2(\beta-5/2-\delta)}\|g_d\|_s^2,$$

since $\varkappa - s = 1 + \delta$. The last value is finite as $h \to 0$ if $\beta \le 5/2 + \delta$. For this choice of β, we obtain the required estimate:

$$\|Q_h u - u_d\|_{\beta} \le Ch^{\varkappa-2}\|g_d\|_s. \qquad \square$$

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