ISSN 1995-0802, Lobachevskii Journal of Mathematics, 2023, Vol. 44, No. 8, pp. 3191–3196. © Pleiades Publishing, Ltd., 2023.

On a Discrete Boundary Value Problem in a Quarter-Plane

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Received May 31, 2023; revised June 29, 2023; accepted July 10, 2023

Abstract—We study a special discrete boundary value problem for a digital elliptic pseudo-differential operator in a discrete quadrant. Using a special periodic wave factorization for a symbol of the pseudo-differential operator we can construct a general solution of the pseudo-differential equation and then to choose appropriate boundary conditions for its unique solvability in corresponding Sobolev–Slobodetskii spaces. We also give a comparison between discrete and continuous solutions for boundary value problems under consideration.

DOI: 10.1134/S1995080223080024

Keywords and phrases: *discrete operator, discrete boundary value problem, solvability, error estimate.*

1. INTRODUCTION

We interested in studying discrete pseudo-differential equations and their solvability in appropriate discrete functional spaces. There are certain approaches to studying discrete boundary value problems for partial differential equations [1, 2]. But these approaches are not applicable to studying discrete boundary value problems for elliptic pseudo-differential equations. According to this statement one of authors with colleagues has started to develop discrete theory for elliptic pseudo-differential equations [3]. First considerations were related to discrete m-dimensional space and half-space, and here we consider discrete quadrant.

Let $\mathbb{Z}^2$ be an integer lattice in a plane. Let $K = \{x \in \mathbb{R}^2 : x = (x_1, x_2), x_1 > 0, x_2 > 0\}$ be a quadrant, $K_d = h\mathbb{Z}^2 \cap K, h > 0$. We consider functions of discrete variable $u_d(\tilde{x}), \tilde{x} = (\tilde{x}_1, \tilde{x}_2) \in h\mathbb{Z}^2$.

Let us denote $\mathbb{T}^2 = [-\pi, \pi]^2, \hbar = h^{-1}$. We consider functions defined in $\hbar\mathbb{T}^2$ as periodic functions defined in $\mathbb{R}^2$ with basic square of periods $\hbar\mathbb{T}^2$.

One can define the discrete Fourier transform for the function u_d

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in h\mathbb{Z}^2} e^{-i\tilde{x}\cdot\xi} u_d(\tilde{x}) h^2, \quad \xi \in \hbar\mathbb{T}^2,$$

under the condition of convergence of the above series, the function of the form $\tilde{u}_d(\xi)$ will be periodic in $\mathbb{R}^2$ while the base square of the periods will be $\hbar\mathbb{T}^2$. The Fourier transform of this form will have the same properties as the integral Fourier transform. The inverse discrete Fourier transform will look like this

$$(F_d^{-1}\tilde{u}_d)(\tilde{x}) = \frac{1}{(2\pi)^2} \int\limits_{\hbar\mathbb{T}^2} e^{i\tilde{x}\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in h\mathbb{Z}^2.$$

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Working with the discrete Fourier transform will allow one to obtain a one-to-one correspondence between the spaces $L_2(h\mathbb{Z}^2)$ and $L_2(\hbar\mathbb{T}^2)$ with norms

$$||u_d||_2 = \left(\sum_{\tilde{x}\in h\mathbb{Z}^2} |u_d(\tilde{x})|^2 h^2\right)^{1/2}, \quad ||\tilde{u}_d||_2 = \left(\int_{\xi\in\hbar\mathbb{T}^2} |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}.$$

We need more general discrete functional spaces and we introduce such spaces using divided differences [1] and their Fourier transforms.

We introduce discrete analogue of the Schwartz space $S(h\mathbb{Z}^2)$ and the notation $\zeta^2 = h^{-2}((e^{-ih\cdot\xi_1} - 1)^2 + (e^{-ih\cdot\xi_2} - 1)^2)$.

Definition 1. The space $H^s(h\mathbb{Z}^2)$ consists of discrete distributions and it is a closure of the space $S(h\mathbb{Z}^2)$ with respect to the norm

$$||u_d||_s = \left(\int_{\hbar\mathbb{T}^2} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 d\xi\right)^{1/2}. \tag{1}$$

We have the space $H^s(K_d)$. This space consists of discrete distributions from $H^s(h\mathbb{Z}^2)$. Distribution data carriers belong to the set $\overline{K_d}$. A norm in the space $H^s(K_d)$ is induced by the norm of the space $H^s(h\mathbb{Z}^2)$. The space $H^s_0(K_d)$ will consist of discrete distributions represented as functions $f_d \in S'(h\mathbb{R}^2)$ with supports inside of K_d. These discrete distributions will allow continuation into the space $H^s(h\mathbb{Z}^2)$. A norm in the space $H^s_0(K_d)$ will be determined by the following formula

$$||f_d||^+_s = \inf ||\ell f_d||_s,$$

where infimum is taken for all continuations ℓ.

Denote the Fourier image of the space $H^s(K_d)$ as $\widetilde{H}^s(K_d)$. Denote a measurable periodic function in $\mathbb{R}^2$ as $\widetilde{A}_d(\xi)$ with the basic square of periods $\hbar\mathbb{T}^2$. Such functions we call symbols.

Definition 2. A digital pseudo-differential operator A_d with the symbol $A_d(\xi)$ in the discrete quadrant K_d is called an operator of the following type

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^2} h^2 \int_{\hbar\mathbb{T}^2} \widetilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in K_d, \tag{2}$$

We say that the operator A_d is elliptic one if

$$ess \inf_{\xi\in\hbar\mathbb{T}^2} |A_d(\xi)| > 0.$$

A more general digital pseudo-differential operator with the symbol $\widetilde{A}_d(\tilde{x}, \xi)$ depending on a spatial variable $\tilde{x}$

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y}\in h\mathbb{Z}^2} h^2 \int_{\hbar\mathbb{T}^2} A_d(\tilde{x}, \xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) d\xi, \quad \tilde{x} \in \text{пЫ}_d,$$

can be defined in the same way, but here we consider only operators of type (2).

We consider symbols satisfying the condition

$$c_1(1 + |\zeta^2|)^{\alpha/2} \le |A_d(\xi)| \le c_2(1 + |\zeta^2|)^{\alpha/2} \tag{3}$$

with constants c_1, c_2 non-depending on h. The number $\alpha \in \mathbb{R}$ is called an order of digital pseudo-differential operator A_d.

It is well known that a digital pseudo-differential operator A_d with the symbol $\widetilde{A}_d(\xi)$ is a linear bounded operator $H^s(h\mathbb{Z}^2) \to H^{s-\alpha}(h\mathbb{Z}^2)$ with a norm non-depending on h.

2. BASIC METHODS

We study a solvability of the discrete equation

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in K_d, \tag{4}$$

in the space $H^s(K_d)$ assuming that $v_d \in H_0^{s-\alpha}(K_d)$.

We will use certain special domain in two-dimensional complex space $\mathbb{C}^2$. A domain of the type $\mathcal{T}_h(K) = \hbar\mathbb{T}^2 + iK$ is called a tube domain over the quadrant K, and we will consider analytical functions $f(x+i\tau)$ in the domain $\mathcal{T}_h(K) = \hbar\mathbb{T}^2 + iK$.

Let us introduce the periodic Bochner kernel similar [4]

$$B_h(z) = \sum_{\tilde{x}\in K_d} e^{i\tilde{x}\cdot(\xi+i\tau)}h^2, \quad \xi \in \hbar\mathbb{T}^2, \quad \tau \in K,$$

and corresponding integral operator

$$(B_h\tilde{u}_d)(\xi) = \lim_{\tau\to 0, \tau\in K} \frac{1}{4\pi^2} \int\limits_{\hbar\mathbb{T}^2} B_h(\xi + i\tau - \eta)\tilde{u}_d(\eta)d\eta.$$

Using calculations from [5] one can verify that for the quadrant K the operator B_h has the following form

$$\begin{aligned}(B_h\tilde{u}_d)(\xi) = \frac{h^2}{8\pi^2}\int\limits_{\mathbb{T}^2}\tilde{u}_d(\eta)d\eta + \lim_{\tau\to 0+}\frac{ih}{8\pi^2}\int\limits_{\mathbb{T}^2}\cot\frac{h(\xi_1-\eta_1+i\tau_1)}{2}\tilde{u}_d(\eta)d\eta \\ + \lim_{\tau\to 0+}\frac{ih}{8\pi^2}\int\limits_{\mathbb{T}^2}\cot\frac{h(\xi_2-\eta_2+i\tau_2)}{2}\tilde{u}_d(\eta)d\eta \\ - \lim_{\tau\to 0+}\frac{h^2}{8\pi^2}\int\limits_{\mathbb{T}^2}\cot\frac{h(\xi_1-\eta_1+i\tau_1)}{2}\cot\frac{h(\xi_2-\eta_2+i\tau_2)}{2}\tilde{u}_d(\eta)d\eta,\end{aligned}$$

and B_h is a linear bounded operator $H^s(\hbar\mathbb{T}^2) \to H^s(\hbar\mathbb{T}^2)$ for $|s| < 1/2$. Moreover, the operator B_h is a projector $\widetilde{H}^s(h\mathbb{Z}^2) \to \widetilde{H}^s(K_d)$.

Such operator B_h is so called periodic bi-singular operator. Using classical results for Cauchy type integral [6, 7] one can evaluate the boundary value, but it is not important this time. Since these formulas are very huge we can do some simplifications without lost of generality. For example, we can consider the space $S_1(h\mathbb{Z}^2) \subset S_1(h\mathbb{Z}^2)$ with zeroes in coordinate axes and consider the space $H^s(h\mathbb{Z}^2)$ as closure of the set $S_1(h\mathbb{Z}^2)$ assuming that all functions of discrete variable vanish on coordinate axes. For this case the first three summands in B_h will be zero. One more property of the operator B_h is that for $|s| < 1/2$ the space $\widetilde{H}^s(h\mathbb{Z}^2)$ is uniquely represented as the direct sum

$$\widetilde{H}^s(h\mathbb{Z}^2) = \widetilde{H}^s(K_d) \oplus \widetilde{H}^s(h\mathbb{Z}^2 \setminus K_d).$$

To describe a solvability picture for the discrete equation (4) we need some additional elements of multidimensional complex analysis.

This concept is a periodic analogue of the wave factorization [8]. Some first preliminary considerations and results were obtained earlier for a half-space case.

Definition 3. A periodic wave factorization for the elliptic symbol $A_d(\xi) \in E_\alpha$ is called its representation in the form $A_d(\xi) = A_{d,\neq}(\xi)A_{d,=}(\xi)$, where the factors $A_{d,\neq}(\xi), A_{d,=}(\xi)$ admit analytical continuation into tube domains $\mathcal{T}_h(K), \mathcal{T}_h(-K)$ respectively with estimates

$$c_1(1+|\hat{\zeta}^2|)^{\frac{æ}{2}} \le |A_{d,\neq}(\xi+i\tau)| \le c_1'(1+|\hat{\zeta}^2|)^{\frac{æ}{2}},$$

$$c_2(1+|\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}} \le |A_{d,=}(\xi-i\tau)| \le c_2'(1+|\hat{\zeta}^2|)^{\frac{\alpha-æ}{2}},$$

and constants c_1, c_1', c_2, c_2' non-depending on h, where

$$\hat{\zeta}^2 \equiv \hbar^2 \left((e^{-ih(\xi_1+i\tau_1)} - 1)^2 + (e^{-ih(\xi_2+i\tau_2)} - 1)^2 \right), \quad \xi = (\xi_1, \xi_2) \in \hbar\mathbb{T}^2, \quad \tau - (\tau_1, \tau_2) \in K.$$

The number æ $\in \mathbb{R}$ is called an index of periodic wave factorization.

Unfortunately, we have no an algorithm to construct the factorization. But there are certain examples of periodic symbols which admit such factorization. We give one of them.

If f is an arbitrary function of a discrete variable, $f \in S(h\mathbb{Z}^2)$, $supp\, f \subset K_d \cup (-K_d)$, then we have $f = \chi_+ f + \chi_- f$, where $\chi_\pm$ are indicators of $\pm K_d$. Applying the discrete Fourier transform, we obtain the representation $\tilde{f} = \tilde{f}_+ + \tilde{f}_-$, and $\tilde{f}_\pm$ admit an analytical continuation into $\mathcal{T}_h(\pm K)$. Thus, we can write $\exp \tilde{f} = \exp \tilde{f}_+ \cdot \exp \tilde{f}_-$, therefore, we obtain periodic wave factorization with index zero for the function $\exp \tilde{f}$.

Everywhere below we assume existence of such periodic wave factorization for the symbol $A_d(\xi)$ with index æ.

3. MAIN RESULTS

For some cases a solution of the equation (4) exists and it is unique.

Theorem 1. *Let* $|æ - s| < 1/2$. *Then, the equation (4) has a unique solution for arbitrary right hand side* $v_d \in H_0^{s-\alpha}(K_d)$, *and it is given by the formula*

$$\tilde{u}_d(\xi) - A_{d,\neq}^{-1}(\xi) B_h (A_{d,=}^{-1}(\xi) \widetilde{(\ell v_d)}(\xi)),$$

where ℓv_d *is an arbitrary continuation of* v_d *into* $H^{s-\alpha}(h\mathbb{Z}^2)$.

Other cases are more interesting when the equation (4) have a lot of solutions.

We use here some results from [3] concerning a form of a discrete distribution supported at the origin.

Theorem 2. *Let* $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. *Then, a general solution of the equation (4) has the following form*

$$\tilde{u}_d(\xi) = A_{d,\neq}^{-1}(\xi) Q_n(\xi) B_h (Q_n^{-1}(\xi) A_{d,=}^{-1}(\xi) \widetilde{(\ell v_d)}(\xi)) + A_{d,\neq}^{-1}(\xi) \left(\sum_{k=0}^{n-1} \tilde{c}_k(\xi_1) \hat{\zeta}_2^k + \tilde{d}_k(\xi_2) \hat{\zeta}_1^k \right), \tag{5}$$

in this case $Q_n(\xi)$ *will be an arbitrary polynomial of order* n *with variables* $\zeta_k = \hbar(e^{-ih\xi_k} - 1)$, $k = 1, 2$, *which will satisfy condition (3) with* $\alpha = n$, $\tilde{c}_k(\xi_1), \tilde{d}_k(\xi_2)$, $k = 0, 1, \cdots, n-1$, *are arbitrary functions from* $H^{s_k}(h\mathbb{T}), s_k = s - æ + k - 1/2$.

The a priori estimate

$$||u_d||_s \leq const \left(||f||_{s-\alpha}^+ + \sum_{k=0}^{n-1} ([c_k]_{s_k} + [d_k]_{s_k}) \right),$$

holds, where $[\cdot]_{s_k}$ *denotes a norm in* $H^{s_k}(h\mathbb{T})$, *and const does not depend on* h.

Now we consider here the case $æ - s = 1 + \delta, |\delta| < 1/2$ and the homogeneous equation

$$(A_d u_d)(\tilde{x}) = 0, \quad \tilde{x} \in K_d, \tag{6}$$

with the following boundary conditions

$$\sum_{\tilde{x}_1 \in h\mathbb{Z}_+} u_d(\tilde{x}_1, \tilde{x}_2) h = f_d(\tilde{x}_2), \quad \sum_{\tilde{x}_2 \in h\mathbb{Z}_+} u_d(\tilde{x}_1, \tilde{x}_2) h = g_d(\tilde{x}_1), \quad \sum_{\tilde{x} \in h\mathbb{Z}_{++}} u_d(\tilde{x}_1, \tilde{x}_2) h^2 = 0. \tag{7}$$

These additional conditions will help us to determine uniquely the unknown functions c_0, d_0 in the solution (5).

Indeed, using the discrete Fourier transform we rewrite the conditions (7) as follows

$$\tilde{u}_d(0, \xi_2) = \tilde{f}_d(\xi_2), \quad \tilde{u}_d(\xi_1, 0) = \tilde{g}_d(\xi_1), \quad \tilde{u}_d(0, 0) = 0. \tag{8}$$

Now we substitute the formulas (8) into (5). The first two equality are

$$\tilde{u}_d(0,\xi_2) = A^{-1}_{d,\neq}(0,\xi_2)(\tilde{c}_0(0) + \tilde{d}_0(\xi_2)) = \tilde{f}_d(\xi_2),$$

$$\tilde{u}_d(\xi_1,0) = A^{-1}_{d,\neq}(\xi_1,0)(\tilde{c}_0(\xi_1) + \tilde{d}_0(0)) = \tilde{g}_d(\xi_1).$$

It implies the following relations according to the third condition

$$\tilde{f}_d(0) = \tilde{g}_d(0), \quad \text{and from which} \quad \tilde{c}_0(0) + \tilde{d}_0(0) = 0, \quad \text{and} \quad \tilde{c}_0(0) = \tilde{d}_0(0) = 0.$$

Then, we have at least formally

$$\tilde{u}_d(\xi) = A^{-1}_{d,\neq}(\xi)\left(A_{d,\neq}(\xi_1,0)\tilde{g}_d(\xi_1) + A_{d,\neq}(0,\xi_2)\tilde{f}_d(\xi_2)\right). \tag{9}$$

Theorem 3. *Let $f_d, g_d \in H^{s+1/2}(h\mathbb{Z})$. Then the discrete problem (6), (7) has unique solution which is given by the formula (9).*

The a priori estimate

$$||u_d||_s \leq const(||f_d||_{s+1/2} + ||g_d||_{s+1/2})$$

holds with a const non-depending on h.

Now we will describe the continuous analogue of the discrete boundary value problem is the following [9].

Let A be a pseudo-differential operator with the symbol $A(\xi), \xi = (\xi_1, \xi_2)$ satisfying the condition

$$c_1(1+|\xi|)^{\alpha} \leq |A(\xi)| \leq c_2(1+|\xi|)^{\alpha}$$

and admitting the wave factorization with respect to the quadrant K with index æ.

We consider the equation

$$(Au)(x) = 0, \quad x \in K, \tag{10}$$

with the following additional conditions

$$\int\limits_0^{+\infty} u(x_1,x_2)dx_1 = f(x_2), \quad \int\limits_0^{+\infty} u(x_1,x_2)dx_2 = g(x_1), \quad \int\limits_{-K} u(x)dx = 0. \tag{11}$$

A solution of the problem (10) and (11) is sought in the space $H^s(K)$ [8] and boundary functions are taken from the space $H^{s+1/2}(\mathbb{R}_+)$. Such problem was considered in [9] and it has the solution

$$\tilde{u}(\xi) = A^{-1}_{\neq}(\xi)\left(A_{\neq}(\xi_1,0)\tilde{g}(\xi_1) + A_{\neq}(0,\xi_2)\tilde{f}(\xi_2)\right) \tag{12}$$

under condition that the symbol $A(\xi)$ admits the wave factorization with respect to the quadrant K

$$A(\xi) = A_{\neq}(\xi)A_{=}(\xi)$$

with index æ such that $æ - s = 1 + \delta, |\delta| < 1/2$.

To construct a discrete boundary value problem which is good approximation to (10) and (11) we need to choose $A_d(\xi)$ and f_d, g_d in a special way. First, we introduce the operator l_h which acts as follows. For a function u defined in $\mathbb{R}$ we take its Fourier transform $\tilde{f}$ then we take its restriction on $\hbar T$ and periodically extend it to $\mathbb{R}$. Finally, we take its inverse discrete Fourier transform and obtain the function of discrete variable $(l_h u)(\tilde{x}), \tilde{x} \in h\mathbb{R}$. Thus, we put

$$f_d = l_h f, \quad g_d = l_h g.$$

Second, the symbol of digital operator A_d we construct in the same way. If we have the wave factorization for the symbol $A(\xi)$ then we take restrictions of factors on $\hbar\mathbb{T}^2$ and the periodic symbol $A_d(\xi)$ is a product of these restrictions. For such f_d, g_d and the symbol $A_d(\xi)$ we obtain the following result.

Theorem 4. *Let $f, g \in S(\mathbb{R})$, æ > 1. Then, we have the following estimate for solutions u andu_d of the continuous problem (10) and (11) and the discrete one (6) and (7)*

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq C(f,g)h^{\beta},$$

where the const $C(f,g)$ depends on functions f, g, $\beta > 0$ can be an arbitrary number.

4. CONCLUSIONS

In this paper, we have considered two-dimensional cone only, but the authors continue to work in multidimensional situations and we hope to obtain results similar to a discrete half-space. Also it has the sense to consider different boundary conditions.

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