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First Asymptotics of Solutions of Degenerate Second-Order Differential Equations

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Abstract—We propose a method for constructing asymptotic representations of solutions of linear degenerate second-order ordinary differential equations, which permits one to construct exact asymptotics of solutions in a neighborhood of the degeneracy point. An example where one finds a power-law asymptotics is given.

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1. INTRODUCTION

Linear second-order differential equations with constant sign of the coefficient of the highest derivative are studied in detail in classical courses of differential equations. However, studying the behavior of solutions near a point of degeneration of this coefficient requires some effort (see [1]–[3]). In the present paper, we are interested in the possibility of constructing exact asymptotics of solutions in a neighborhood of the degeneracy point as $t \to 0+$.

Consider the differential equation

$$\big(a(t)u'(t)\big)' + b(t)u'(t) + c(t)u(t) = f(t) \tag{1}$$

degenerating at $t = 0$ with real coefficients on the interval $[0, 1]$ such that $b(0) \neq 0$, $a(0) = 0$, and $a(t) > 0$ for $t \in (0, 1]$ under certain smoothness assumptions on the coefficients permitting one to make necessary transformations.

Asymptotic series expansions of solutions of Eq. (1) in specially chosen functions were constructed in the papers [3] and [4]. Here we present some of these results to be used in the subsequent research.

Let $d(t)$ be an arbitrary sufficiently smooth function such that $d(t) \neq 0$ for $t \in (0, 1]$. For any points $t_0, t \in (0, 1]$, we define two functions $v_k(t)$, $k = 1, 2$, by the formula

$$v_k\,(t, t_0) = \frac{1}{\sqrt{d(t)}} \exp\left(\int_t^{t_0} \frac{b(\tau) - (-1)^k d(\tau)}{2a(\tau)}\, d\tau\right) \tag{2}$$

and the functions

$$h(t) = \frac{1}{4d(t)}\left(a(t)\left(\frac{d'(t)}{d(t)}\right)^2 - 2\left(\frac{a(t)d'(t)}{d(t)}\right)' + \frac{d^2(t) - b^2(t) + 4a(t)c(t) - 2a(t)b'(t)}{a(t)}\right), \tag{3}$$

$$s(t) = \int_0^t h(\tau)\, d\tau, \qquad w\,(t, t_0) = \int_t^{t_0} \frac{d(\tau)}{a(\tau)}\, d\tau. \tag{4}$$

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For $f(t) \equiv 0$, linearly independent solutions of the homogeneous equation (1) can be represented in the form

$$u_1(t) = \Phi(t)v_1(t), \qquad u_2(t) = \Psi(t)v_2(t).$$

The function $\Phi(t)$ is a solution of the problem

$$\Phi(t) = 1 + K_1\Phi(t), \qquad \Phi(0) = 1, \tag{5}$$

where K_1 is the integral operator

$$K_1\varphi(t) = \int_0^{t_0} k_1(t,\tau)\varphi(\tau)\,d\tau$$

with kernel

$$k_1(t,\tau) = \begin{cases} h(\tau), & 0 \le \tau \le t \le t_0, \\ h(\tau)\exp\left(w\left(\tau,t_0\right) - w\left(t,t_0\right)\right), & t \le \tau \le t_0; \end{cases}$$

here $h(t)$ and $w\left(t,t_0\right)$ are the functions introduced in (3) and (4), respectively.

Likewise, the function $\Psi(t)$ is a solution of the problem

$$\Psi(t) = 1 + K_2\Psi(t), \qquad \Psi(0) = 1, \tag{6}$$

where K_2 is the integral operator

$$K_2\psi(t) = \int_0^{t} k_2(t,\tau)\psi(\tau)\,d\tau$$

with kernel $k_2(t,\tau) = -h(\tau)\left(1 - \exp\left(w\left(t,t_0\right) - w\left(\tau,t_0\right)\right)\right)$ for $0 \le \tau \le t$.

In what follows, we assume that $b(t) = b = \text{const} \ne 0$ and $a(t) = t^m a_0(t)$, $m \ge 2$, $a_0(t) > 0$, which means the case of strong degeneracy of Eq. (1).

Let $d(t) = \sqrt{b^2 - 4a(t)c(t)}$. We choose the point $t_0 > 0$ to satisfy $d(t) > 0$ for $t \in [0,t_0]$. For this choice of the function $d(t)$ and the point t_0, one can write the asymptotic representations

$$a(t) = t^m O(1), \quad h(t) = t^{2m-2}O(1), \quad d(t) = |b|\left(1 + t^m O(1)\right), \quad s(t) = \int_0^t h(\tau)\,d\tau = t^{2m-1}O(1) \tag{7}$$

as $t \to 0$, which permits one to use (7) when finding the asymptotics of solutions.

2. ASYMPTOTIC PROPERTIES OF THE INTEGRAL OPERATORS K_1 AND K_2

We assume in what follows that the functions $a(t)$ and $f(t)$ have power-law asymptotics of finite order at the point $t = 0$.

Lemma 1. *Assume that $a(t), f(t), d(t) \in C^2\left[0,t_0\right]$ for some $t_0 \in (0,1]$, $a(0) = a'(0) = 0$, $a(t) > 0$ and $f(t) \ne 0$ on $(0,t_0]$, and $d(t) > 0$ on $[0,t_0]$ Then one has the representations*

$$\int_t^{t_0} f(\xi)\exp\left(-\int_t^{\xi}\frac{d(\tau)}{a(\tau)}\,d\tau\right)d\xi = \frac{a(t)}{d(t)}\left(f(t) + \left(\frac{a(t)f(t)}{d(t)}\right)'\left(1 + a'(t)O(1)\right)\right), \tag{8}$$

$$\int_0^{t} f(\xi)\exp\left(-\int_\xi^{t}\frac{d(\tau)}{a(\tau)}\,d\tau\right)d\xi = \frac{a(t)}{d(t)}\left(f(t) - \left(\frac{a(t)f(t)}{d(t)}\right)'\left(1 + o(1)\right)\right). \tag{9}$$

Proof. Integrating by parts twice, we obtain the representation

$$\begin{aligned}
&\int_t^{t_0} f(\xi)\exp\left(-\int_t^{\xi}\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi \\
&\qquad=\frac{a(t)f(t)}{d(t)}+\int_t^{t_0}\left(\frac{a(\xi)f(\xi)}{d(\xi)}\right)'\exp\left(-\int_\xi^{t}\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi \\
&\qquad\qquad\qquad-\frac{a\,(t_0)\,f\,(t_0)}{d\,(t_0)}\exp\left(-\int_t^{t_0}\frac{d(\tau)}{a(\tau)}\,d\tau\right) \\
&\qquad=\frac{a(t)f(t)}{d(t)}+\int_t^{t_0}\left(\frac{a(\xi)f(\xi)}{d(\xi)}\right)'\exp\left(-\int_\xi^{t}\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi+o\left(t^{\infty}\right) \\
&\qquad=\frac{a(t)f(t)}{d(t)}+\frac{a(t)\tilde f(t)}{d(t)}+\int_t^{t_0}\left(\frac{a(\xi)\tilde f(\xi)}{d(\xi)}\right)'\exp\left(-\int_t^{\xi}\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi+o\left(t^{\infty}\right),
\end{aligned}\tag{10}$$

where

$$\tilde f(t)=\left(\frac{a(t)f(t)}{d(t)}\right)'.$$

Next, consider the ratio

$$\frac{\int_t^{t_0}(a(\xi)\tilde f(\xi)/d(\xi))'\exp\left(-\int_t^{\xi}d(\tau)/a(\tau)\,d\tau\right)\,d\xi}{\tilde f(t)a(t)a'(t)/d(t)}=\frac{\int_t^{t_0}(a(\xi)\tilde f(\xi)/d(\xi))'\exp\left(\int_\xi^{t_0}d(\tau)/a(\tau)\,d\tau\right)\,d\xi}{\tilde f(t)a(t)a'(t)/d(t)\exp\left(\int_t^{t_0}d(\tau)/a(\tau)\,d\tau\right)\,d\xi}.$$

Since the functions $a(t)$ and $f(t)$ have power-law asymptotics of finite order at the point $t=0$, we can apply l'Hôpital's rule to this ratio and obtain

$$\lim_{t\to0+}\frac{-(a(t)\tilde f(t)/d(t))'}{\left(a(t)a'(t)\tilde f(t)/d(t)\right)'-a'(t)\tilde f(t)}=\mathrm{const}=O(1).\tag{11}$$

Thus, by (11),

$$\int_t^{t_0}(a(\xi)\tilde f(\xi)/d(\xi))'\exp\left(-\int_t^{\xi}d(\tau)/a(\tau)\,d\tau\right)\,d\xi=\frac{\tilde f(t)a(t)a'(t)}{d(t)}\,O(1),$$

which, together with (10), establishes the asymptotic representation (8).

Now let us prove the representation (9). As in the proof of (8), integrating by parts twice, we obtain

$$\begin{aligned}
&\int_0^t f(\xi)\exp\left(-\int_\xi^t\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi \\
&\qquad=\int_0^t\frac{a(\xi)f(\xi)}{d(\xi)}\frac{d(\xi)}{a(\xi)}\exp\left(-\int_\xi^t\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi \\
&\qquad=\frac{a(t)f(t)}{d(t)}-\int_0^t\frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)f(\xi)}{d(\xi)}\right)'\frac{d(\xi)}{a(\xi)}\exp\left(-\int_\xi^t\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi \\
&\qquad=\frac{a(t)f(t)}{d(t)}-\frac{a(t)}{d(t)}\left(\frac{a(t)f(t)}{d(t)}\right)' \\
&\qquad\qquad+\int_0^t\left(\frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)f(\xi)}{d(\xi)}\right)'\right)'\exp\left(-\int_\xi^t\frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi.
\end{aligned}\tag{12}$$

Since

$$\lim_{t\to0+}\frac{\int_0^t\tilde f(\xi)\exp\left(-\int_\xi^t d(\tau)/a(\tau)\,d\tau\right)\,d\xi}{\tilde f(t)a(t)a'(t)/d(t)}$$

$$= \lim_{t\to 0+} \frac{\int_0^t \tilde f(\xi)\exp\left(-\int_\xi^{t_0} d(\tau)/a(\tau)\,d\tau\right)\,d\xi}{\tilde f(t)a(t)a'(t)/d(t)\exp\left(-\int_t^{t_0} d(\tau)/a(\tau)\,d\tau\right)\,d\xi}$$

$$= \lim_{t\to 0+} \frac{\tilde f(t)\exp\left(-\int_t^{t_0} d(\tau)/a(\tau)\,d\tau\right)}{\tilde f(t)\exp\left(-\int_t^{t_0} d(\tau)/a(\tau)\,d\tau\right) + (a(t)\tilde f(t)/d(t))'\exp\left(-\int_t^{t_0} d(\tau)/a(\tau)\,d\tau\right)}$$

$$= \lim_{t\to 0+} \frac{\tilde f(t)}{\tilde f(t) + (a(t)\tilde f(t)/d(t))'} = 1,$$

we have

$$\int_0^t f(\xi)\exp\left(-\int_\xi^t \frac{d(\tau)}{a(\tau)}\,d\tau\right)\,d\xi = \frac{a(t)}{d(t)}\left(f(t) - \frac{a(t)f(t)}{d(t)}\right)'(1+o(1)),$$

and the representation (9) follows from (12). The proof of the lemma is complete. □

We introduce the following notation:

$$K_{10}\varphi(t) = \int_0^t h(\tau)\varphi(\tau)\,d\tau,$$

$$K_{11}\varphi(t) = \int_t^{t_0} h(\tau)\varphi(\tau)\exp\left(w(\tau,t_0) - w(t,t_0)\right)\,d\tau, s(t) = \int_0^t h(\tau)\,d\tau,$$

where the functions $h(t)$ and $w(t,t_0)$ are defined in (3) and (4), respectively.

Integrating by parts, we have

$$K_{10}\varphi(t) = \int_0^t h(\tau)\varphi(\tau)\,d\tau = s(t)\varphi(t) - \int_0^t s(\tau)\varphi'(\tau)\,d\tau,$$

and applying Lemma 1 for $\varphi(t) \in C^2[0,t_0]$, we obtain

$$\begin{aligned} K_{11}\varphi(t) &= \int_t^{t_0} h(\tau)\varphi(\tau)\exp\left(w(\tau,t_0) - w(t,t_0)\right)\,d\tau \\ &= \frac{a(t)}{d(t)}\left(h(t)\varphi(t) + \left(\frac{a(t)}{d(t)}h(t)\varphi(t)\right)'\left(1 + a'(t)O(1)\right)\right). \end{aligned} \tag{13}$$

Therefore, $K_1\varphi(t)$ has the asymptotic representation

$$\begin{aligned} K_1\varphi(t) &= K_{10}\varphi(t) + K_{11}\varphi(t) \\ &= s(t)\varphi(t) - \int_0^t s(\tau)\varphi'(\tau)\,d\tau + \frac{a(t)}{d(t)}\left(h(t)\varphi(t) + \left(\frac{a(t)}{d(t)}h(t)\varphi(t)\right)'\left(1 + a'(t)O(1)\right)\right). \end{aligned} \tag{14}$$

By differentiating (5) and by applying (13), we obtain an asymptotic representation for the derivative

$$\begin{aligned} (K_1\varphi(t))' &= \frac{d(t)}{a(t)}\exp\left(-w(t,t_0)\right)\int_t^{t_0} h(\tau)\varphi(\tau)\exp\left(w(\tau,t_0)\right)\,d\tau = \frac{d(t)}{a(t)}K_{11}\varphi(t) \\ &= h(t)\varphi(t) + \left(\frac{a(t)}{d(t)}h(t)\varphi(t)\right)'\left(1 + a'(t)O(1)\right) = h(t)\varphi(t)(1+o(1)). \end{aligned} \tag{15}$$

In a similar way, by applying Lemma 1 for $\psi(t) \in C^2[0,t_0]$, we obtain

$$\begin{aligned} K_2\psi(t) &= -\int_0^t h(\tau)\left(1 - \exp\left(w(t,t_0) - w(\tau,t_0)\right)\right)\psi(\tau)\,d\tau = K_{20}\psi(t) + K_{21}\psi(t) \\ &= -s(t)\psi(t) + \int_0^t s(\tau)\psi'(\tau)\,d\tau + \frac{a(t)}{d(t)}\left(h(t)\psi(t) - \left(\frac{a(t)}{d(t)}h(t)\psi(t)\right)'(1+O(1))\right). \end{aligned} \tag{16}$$

By differentiating (6) and by applying Lemma 1 with regard to the asymptotic representations (7), we have

$$\begin{aligned}(K_2\psi(t))' &= k_2(t,t)\psi(t) - \frac{d(t)}{a(t)}\int_0^t h(\tau)\exp\left(w(t,t_0) - w(\tau,t_0)\right)\psi(\tau)\,d\tau \\ &= -\frac{d(t)}{a(t)}\int_0^t h(\tau)\exp\left(\int_\tau^t \frac{d(\xi)}{a(\xi)}\,d\xi\right)\psi(\tau)\,d\tau \\ &= -h(t)\psi(t) + \left(\frac{a(t)}{d(t)}h(t)\psi(t)\right)'(1+o(1)) \\ &= -h(t)\psi(t)(1+o(1)).\end{aligned} \tag{17}$$

3. FIRST ASYMPTOTICS OF THE FUNCTIONS $\Phi(t)$ AND $\Psi(t)$

We use the results of the preceding section to study the asymptotics of the functions $\Phi(t)$ and $\Psi(t)$. Note that the smoothness requirements imposed on the functions in question are due to the method and are not sharp.

Lemma 2. *Assume that*

$$a(t), c(t) \in C^4[0,t_0], \qquad a(t) = t^m a_0(t), \quad m \ge 2, \qquad a_0(t) > 0, \quad t \in [0,t_0],$$
$$b(t) = b = \text{const}, \qquad d(t) = \sqrt{b^2 - 4a(t)c(t)} > 0.$$

Then the functions $\Phi(t)$ and $\Psi(t)$ defined by Eqs. (5) *and* (6) *admit the asymptotic representations*

$$\Phi(t) = 1 + s(t) + \frac{a(t)}{d(t)}\left(h(t) + \left(\frac{a(t)}{d(t)}h(t)\right)'\right) + s^2(t)O(1), \tag{18}$$

$$\Psi(t) = 1 - s(t) + \frac{a(t)}{d(t)}\left(h(t) - \left(\frac{a(t)}{d(t)}h(t)\right)'\right) + s^2(t)O(1) \tag{19}$$

as $t \to 0+$, where the functions $h(t)$ and $s(t)$ are defined in (3) *and* (4)*, respectively.*

Proof. By differentiating (5) and by using (15), we obtain

$$\Phi'(t) = (K_1\Phi(t))' = h(t)\Phi(t) + \left(\frac{a(t)}{d(t)}h(t)\Phi(t)\right)'\left(1 + a'(t)O(1)\right) = h(t)\Phi(t)(1+o(1)). \tag{20}$$

Taking into account Eqs. (5), (14), and (20), we have

$$\begin{aligned}\Phi(t) &= 1 + K_1\Phi(t) \\ &= 1 + s(t)\Phi(t) - \int_0^t s(\tau)\Phi'(\tau)\,d\tau \\ &\quad + \frac{a(t)}{d(t)}\left(h(t)\Phi(t) + \left(\frac{a(t)}{d(t)}h(t)\Phi(t)\right)'\right)\left(1 + a'(t)O(1)\right) \\ &= 1 + s(t)\Phi(t) - \frac{1}{2}s^2(t)\Phi(t)(1+o(1)) \\ &\quad + \frac{a(t)}{d(t)}h(t)\Phi(t) + \left(\frac{a(t)\Phi(t)}{d(t)}\left(\frac{a(t)h(t)}{d(t)}\right)' + h(t)\Phi'(t)\left(\frac{a(t)}{d(t)}\right)^2\right)\left(1 + a'(t)O(1)\right),\end{aligned}$$

because

$$\int_0^t s(\tau)\Phi'(\tau)\,d\tau = \int_0^t s(\tau)h(\tau)\Phi(\tau)(1+o(1))\,d\tau = \frac{1}{2}s^2(t)\Phi(t)(1+o(1)).$$

For small $t > 0$, one has $\Phi'(t) = h(t)\Phi(t)(1 + o(1))$, and therefore, the relation

$$\Phi(t) = 1 + s(t)\Phi(t) + \frac{a(t)}{d(t)}h(t)\Phi(t) + \frac{a(t)\Phi(t)}{d(t)}\left(\frac{a(t)h(t)}{d(t)}\right)' + s^2(t)\Phi(t)O(1)$$

is valid; solving this relation for $\Phi(t)$, we finally obtain

$$\Phi(t) = 1 + s(t) + \frac{a(t)}{d(t)}\left(h(t) + \left(\frac{a(t)}{d(t)}h(t)\right)'\right) + s^2(t)O(1),$$

which proves the asymptotic representation (18).

Likewise, for $\Psi(t)$ from (6), (16), and (17) we derive

$$\Psi'(t) = (K_2\Psi(t))' = -h(t)\Psi(t)(1 + o(1)),$$

$$\Psi(t) = 1 + K_2\Psi(t) = 1 - s(t)\Psi(t) - \int_0^t s(\tau)h(\tau)\Psi(\tau)(1 + o(1))\,d\tau + \frac{a(t)}{d(t)}\left(h(t)\Psi(t) - \left(\frac{a(t)}{d(t)}h(t)\Psi(t)\right)'\right)(1 + o(1)),$$

$$\Psi(t) = 1 - s(t) + \frac{a(t)}{d(t)}\left(h(t) - \left(\frac{a(t)}{d(t)}h(t)\right)'\right) + s^2(t)O(1).$$

Thus, the asymptotic representation (19) is also established. The proof of the lemma is complete. □

4. FIRST ASYMPTOTICS OF SOLUTIONS OF THE HOMOGENEOUS EQUATION

The behavior of solutions of the homogeneous equation

$$\big(a(t)u'(t)\big)' + b(t)u'(t) + c(t)u(t) = 0 \tag{21}$$

near the degeneracy point $t = 0$ is mainly determined by the functions $v_1(t, t_0)$ and $v_2(t, t_0)$ given by Eqs. (2) (for more detail, see [3]). These functions are represented via specific functions, and their asymptotics can be obtained by standard methods or by using well-known mathematical computation software, such as *Wolfram Mathematica*. The asymptotic representations of the solutions

$$u_1(t) = \Phi(t)v_1(t, t_0), \quad u_2(t) = \Psi(t)v_2(t, t_0), \tag{22}$$

indicated in the introduction, of the homogeneous equation (21) are established in the following theorem, where excessive smoothness conditions are imposed on the coefficients to simplify the formulation.

Theorem 1. *Assume that* $a(t), c(t) \in C^\infty[0, t_0]$, $a(t) = t^m a_0(t)$, $m \ge 2$, $a_0(t) > 0$ *for* $t \in [0, t_0]$, $b(t) = b = \text{const}$, *and* $d(t) = \sqrt{b^2 - 4a(t)c(t)} > 0$ *in Eq.* (21). *Then Eq.* (21) *has linearly independent solutions* $u_1(t)$ *and* $u_2(t)$ *admitting the following asymptotic expansions as* $t \to 0+$:

$$u_1(t) = v_1(t, t_0)\left(1 + s(t) + \frac{a(t)}{d(t)}\left(h(t) + \left(\frac{a(t)}{d(t)}h(t)\right)'\right) + s^2(t)O(1)\right), \tag{23}$$

$$u_2(t) = v_2(t, t_0)\left(1 - s(t) + \frac{a(t)}{d(t)}\left(h(t) - \left(\frac{a(t)}{d(t)}h(t)\right)'\right) + s^2(t)O(1)\right), \tag{24}$$

where the functions $h(t)$ *and* $s(t)$ *are defined in* (3) *and* (4), *respectively. Moreover,*

$$\lim_{t\to 0+} u_1(t) = v_1(0, t_0) \ne 0, \qquad \lim_{t\to 0+} u_2^{(n)}(t) = 0 \tag{25}$$

for all $n \ge 0$ *and* $b < 0$, *and*

$$\lim_{t\to 0+} u_1^{(n)}(t) = +\infty, \qquad \lim_{t\to 0+} u_2(t) = v_2(0, t_0) \ne 0 \tag{26}$$

for $n \ge 0$ *and* $b > 0$.

Proof. Applying Lemma 2 to the representations (22), we obtain the asymptotic expansions (23) and (24).

The properties of the functions $v_1(t,t_0)$ and $v_2(t,t_0)$ are described in detail in the paper [3], where it is established that

$$\lim_{t\to 0+} v_1(t,t_0) = v_1(0,t_0) > 0, \qquad \lim_{t\to 0+} v_2^{(n)}(t,t_0) = 0$$

for $b<0$ and $n\ge 0$ and

$$\lim_{t\to 0+} v_1^{(n)}(t,t_0) = +\infty, \qquad \lim_{t\to 0+} v_2(t,t_0) = v_2(0,t_0) \neq 0$$

for $b>0$, whence Eqs. (25) and (26) follow. The proof of the theorem is complete. □

5. FIRST ASYMPTOTICS OF SOLUTIONS OF THE INHOMOGENEOUS EQUATION

The existence of a twice continuously differentiable solution $u_*(t)$ of the inhomogeneous differential equation (1) with sufficiently smooth coefficients and right-hand side was established in the papers [3] and [4]. This solution can be expressed via the functions $\Phi(t)$ and $\Psi(t)$ defined in Eqs. (5) and (6) as follows:

$$u_*(t) = A(t)\Phi(t) + B(t)\Psi(t), \tag{27}$$

where

$$A(t) = -\int_0^t \frac{\Psi(\xi)f(\xi)}{\sqrt{d(t)d(\xi)}} \exp\left(-\int_\xi^t \frac{b+d(\tau)}{2a(\tau)}\,d\tau\right) d\xi,$$

$$B(t) = -\int_t^{t_0} \frac{\Phi(\xi)f(\xi)}{\sqrt{d(t)d(\xi)}} \exp\left(\int_t^\xi \frac{b-d(\tau)}{2a(\tau)}\,d\tau\right) d\xi.$$

Moreover, $\lim_{t\to 0+} u_*(t) = 0$ for $b<0$ and $\lim_{t\to 0+} u_*(t) = u_*(0) \neq 0$ for $b>0$.

The asymptotic obtained in (18) and (19) for the functions $\Phi(t)$ and $\Psi(t)$ permit one to state the following theorem.

Theorem 2. *Let the coefficients of Eq.* (1) *satisfy the assumptions of Theorem* 1. *If $b<0$, then, for any function $f(t)\in C^\infty[0,t_0]$, the inhomogeneous equation* (1) *has a solution $u_*(t)\in C^\infty[0,t_0]$ that is infinitesimal in a neighborhood of zero and can be written in the form*

$$\begin{aligned} u_*(t) &= A_0(t)\left(1+s(t)+\frac{a(t)h(t)}{d(t)}\right) + \frac{a(t)}{d(t)}\left(\frac{a(t)h(t)}{d(t)}\right)' \\ &\quad + B_0(t)\left(1-s(t)+\frac{a(t)h(t)}{d(t)}\right) - \frac{a(t)}{d(t)}\left(\frac{a(t)h(t)}{d(t)}\right)' + s^2(t)O(1), \end{aligned} \tag{28}$$

where

$$A_0(t) = -\int_0^t \left(1-s(\xi)+\frac{a(\xi)h(\xi)}{d(\xi)} - \frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)'\right) \exp\left(\int_\xi^t \frac{c(\tau)\,d\tau}{d_1(\tau)}\right) \frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}} = o(1),$$

$$B_0(t) = -\int_t^{t_0} \left(1+s(\xi)+\frac{a(\xi)h(\xi)}{d(\xi)} + \frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)'\right) \exp\left(\int_\xi^t \frac{d_1(\tau)\,d\tau}{a(\tau)}\right) \frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}} = o(1),$$

$$d_1(t) = \frac{1}{2}(|b| + d(t)).$$

If $b>0$, then the infinitesimal solution $\tilde{u}_*(t)\in C^\infty[0,t_0]$ of the inhomogeneous equation (1) *in a neighborhood of zero can be written in the form*

$$\begin{aligned} \tilde{u}_*(t) &= \tilde{A}_0(t)\left(1+s(t)+\frac{a(t)h(t)}{d(t)}\right) + \frac{a(t)}{d(t)}\left(\frac{a(t)h(t)}{d(t)}\right)' \\ &\quad + \tilde{B}_0(t)\left(1-s(t)+\frac{a(t)h(t)}{d(t)}\right) - \frac{a(t)}{d(t)}\left(\frac{a(t)h(t)}{d(t)}\right)' + s^2(t)O(1), \end{aligned} \tag{29}$$

where

$$\tilde{A}_0(t) = -\int_0^t \left(1 - s(\xi) + \frac{a(\xi)h(\xi)}{d(\xi)} - \frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)'\right) \exp\left(\int_t^\xi \frac{d_1(\tau)\,d\tau}{a(\tau)}\right) \frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}} = o(1),$$

$$\tilde{B}_0(t) = -\int_t^{t_0} \left(1 + s(\xi) + \frac{a(\xi)h(\xi)}{d(\xi)} + \frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)'\right) \exp\left(\int_\xi^t \frac{c(\tau)\,d\tau}{d_1(\tau)}\right) \frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}} = o(1).$$

Proof. First, note that the functions $A_0(t)$, $\tilde{A}_0(t)$, $B_0(t)$, and $\tilde{B}_0(t)$ can be expressed via given functions, and their asymptotics can be obtained by standard methods, possibly with the use of Lemma 1. In accordance with the assumptions of the theorem and formulas (28) and (29), the accuracy of these asymptotics should be of the order of t^{4m-2}. The smoothness conditions can be weakened; they are only determined by the method used for constructing the asymptotics and by the accuracy of the asymptotics.

As was noted earlier, the solution of the inhomogeneous equation (1) can be written in the form (27). For $b < 0$, we transform $A(t)$ and $B(t)$ using the asymptotics (18) and (19) obtained in Lemma 2. Then we obtain

$$\begin{aligned} A(t) &= -\int_0^t \frac{\Psi(\xi)f(\xi)}{\sqrt{d(t)d(\xi)}} \exp\left(\int_\xi^t \frac{b-d(\tau)}{2a(\tau)}\,d\tau\right) d\xi \\ &= \int_0^t \left(-1 + s(\xi) - \frac{a(\xi)h(\xi)}{d(\xi)} + \frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)' - s^2(\xi)O(1)\right) \\ &\qquad\qquad \times \exp\left(\int_\xi^t \frac{c(\tau)\,d\tau}{d_1(\tau)}\right) \frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}} \\ &= A_0(t) + s^2(t)o(1), \end{aligned} \tag{30}$$

$$\begin{aligned} B(t) &= -\int_t^{t_0} \frac{\Phi(\xi)f(\xi)}{\sqrt{d(t)d(\xi)}} \exp\left(\int_t^\xi \frac{b-d(\tau)}{2a(\tau)}\,d\tau\right) d\xi \\ &= -\int_t^{t_0} \left(1 + s(\xi) + \frac{a(\xi)h(\xi)}{d(\xi)} + \frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)' + s^2(\xi)O(1)\right) \\ &\qquad\qquad \times \exp\left(\int_\xi^t \frac{d_1(\tau)\,d\tau}{a(\tau)}\right) \frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}} \\ &= B_0(t) + s^2(t)O(1). \end{aligned} \tag{31}$$

By substituting the representations (30) and (31) into Eq. (27) and by again using (18) and (19), we arrive at the desired asymptotics (28).

For $b > 0$, we consider another solution $\tilde{u}_*(t)$ of the inhomogeneous equation (1) such that $\tilde{u}_*(0) = 0$. Take this solution in the form

$$\tilde{u}_*(t) = u_*(t) - C\,(t_0)\,u_2(t), \tag{32}$$

where

$$C(t) = -\int_0^t \frac{f(\xi)\Phi(\xi)}{\sqrt{d(\xi)}} \exp\left(-\int_\xi^{t_0} \frac{c(\tau)\,d\tau}{d_1(\tau)}\right) d\xi.$$

Since

$$\begin{aligned} B(t)\Psi(t) &= -\Psi(t)\int_0^t \frac{f(\xi)\Phi(\xi)}{\sqrt{d(t)d(\xi)}} \exp\left(\int_t^\xi \frac{b-d(\tau)\,d\tau}{2a(\tau)}\right) d\xi \\ &= -v_2\,(t,t_0)\,\Psi(t)\int_0^t \frac{f(\xi)\Phi(\xi)}{\sqrt{d(\xi)}} \exp\left(-\int_{t_0}^\xi \frac{c(\tau)\,d\tau}{d_1(\tau)}\right) d\xi \end{aligned}$$

$$= v_2(t,t_0)\,\Psi(t)\left(\int_0^t \frac{f(\xi)\Phi(\xi)}{\sqrt{d(\xi)}}\exp\left(\int_{t_0}^{t_0}\frac{c(\tau)\,d\tau}{d_1(\tau)}\right)d\xi - \int_0^{t_0}\frac{f(\xi)\Phi(\xi)}{\sqrt{d(\xi)}}\exp\left(\int_{t_0}^{\xi}\frac{c(\tau)\,d\tau}{d_1(\tau)}\right)d\xi\right)$$
$$= v_2(t,t_0)\,\Psi(t)\,(C(t_0)-C(t)) = u_2(t)\,(C(t_0)-C(t)),$$

it follows from the last equality that

$$\tilde{u}_*(t) = u_*(t) - C(t_0)\,u_2(t) = A(t)\Phi(t) - C(t)u_2(t), \qquad \tilde{u}_*(0)=0.$$

We write the solution $\tilde{u}_*(t)$ defined by Eq. (32) in the form

$$\tilde{u}_*(t) = A(t)\Phi(t) + B_*(t)\Psi(t), \tag{33}$$

where $B_*(t) = -v_2(t,t_0)\,C(t)$, and we transform $B_*(t)$ using (18) and (19) by analogy with the preceding case. Then we obtain

$$B_*(t) = \frac{1}{\sqrt{d(t)}}\exp\left(\int_t^{t_0}\frac{c(\tau)\,d\tau}{d_1(\tau)}\right)\int_0^t \frac{f(\xi)\Phi(\xi)}{\sqrt{d(\xi)}}\exp\left(-\int_\xi^{t_0}\frac{c(\tau)\,d\tau}{d_1(\tau)}\right)d\xi$$
$$= \int_0^t\left(1+s(\xi)+\frac{a(\xi)h(\xi)}{d(\xi)}+\frac{a(\xi)}{d(\xi)}\left(\frac{a(\xi)h(\xi)}{d(\xi)}\right)'+s^2(\xi)O(1)\right)\exp\left(\int_t^\xi\frac{c(\tau)\,d\tau}{d_1(\tau)}\right)\frac{f(\xi)\,d\xi}{\sqrt{d(t)d(\xi)}}$$
$$= B_{*,0}(t) + s^2(t)o(1). \tag{34}$$

Substituting (30) and (34) into (33) and again applying (18) and (19), we establish the theorem for $b>0$. The proof of the theorem is complete. □

6. EXAMPLE OF CONSTRUCTING POWER-LAW ASYMPTOTICS

Let us use an example to show the possibility of obtaining power-law asymptotics. Assume that

$$a(t)=t^m, \quad m\ge 2, \quad b=\text{const}\ne 0, \quad c=\text{const}, \quad t_0=1, \quad f(t)\in C^\infty[0,1] \tag{35}$$

in Eq. (1), so that we deal with the equation

$$\left(t^m u'(t)\right)' + bu'(t) + cu(t) = f(t) \tag{36}$$

Taking into account the specific form (35) of the coefficients of Eq. (36), we perform the calculations necessary to obtain the asymptotics. We have

$$h(t) = \frac{1}{4d(t)}\left(a(t)\left(\frac{d'(t)}{d(t)}\right)^2 - 2\left(\frac{a(t)d'(t)}{d(t)}\right)'\right) = \frac{1}{d^5(t)}\left(c_{2m-2}t^{2m-2}+c_{3m-2}t^{3m-2}\right), \tag{37}$$

where

$$c_{2m-2} = m(2m-1)cb^2, \quad c_{3m-2} = m(4-3m)c^2, \quad d(t)=\sqrt{b^2-4a(t)c(t)} = |b|\sqrt{1-\frac{4ct^2}{b^2}}.$$

Using the well-known formula

$$(1-x)^p = 1+\sum_{i=1}^n \frac{(-1)^i p(p-1)\cdots(p-i+1)}{i!}x^i + O\left(x^{n+1}\right),$$

which holds for $|x|<1$, we also find

$$\frac{1}{\sqrt{d(t)}} = \frac{1}{\sqrt{|b|}}\left(1+\frac{1}{4}\frac{4c}{b^2}t^m + \frac{5}{32}\left(\frac{4c}{b^2}\right)^2 t^{2m} + \frac{5\cdot 9}{4^3\cdot 6}\left(\frac{4c}{b^2}\right)^3 t^{3m}\right)$$
$$+ \frac{1}{\sqrt{|b|}}\left(\frac{5\cdot 9\cdot 13}{4^5\cdot 6}\left(\frac{4c}{b^2}\right)^4 t^{4m} + O\left(\left(\frac{4c}{b^2}\right)^5 t^{5m}\right)\right), \tag{38}$$

$$\frac{1}{d(t)} = \frac{1+d_{11}t^m}{|b|} + t^{2m}O(1), \quad d_{11} = \frac{2c}{b^2}, \quad \frac{1}{d^5(t)} = \frac{1+d_{51}t^m}{|b|^5} + t^{2m}O(1), \quad d_{51} = \frac{10c}{b^2}. \tag{39}$$

By substituting (39) into (37), we obtain

$$\begin{aligned} h(t) &= \frac{1}{|b|^5}\left(1+d_{51}t^m\right)\left(c_{2m-2}t^{2m-2} + c_{3m-2}t^{3m-2}\right) + t^{4m-2}O(1) \\ &= h_1 t^{2m-2} + h_2 t^{3m-2} + t^{4m-2}O(1), \end{aligned} \tag{40}$$

where

$$h_1 = \frac{m(2m-1)c}{|b|^3}, \quad h_2 = \frac{m(4-3m)b^2c^2 + 10c}{|b|^7},$$

and moreover,

$$s(t) = \int_0^t h(\tau)\,d\tau = s_1 t^{2m-1} + s_2 t^{3m-1} + t^{4m-2}O(1), \quad s_1 = \frac{h_1}{2m-1}, \quad s_2 = \frac{h_2}{3m-1}. \tag{41}$$

Using the asymptotics (38)–(41) in the earlier-obtained expansions (18), (19) and (23), (24), we have

$$\Phi(t) = 1 + s_1 t^{2m-1} + \frac{h_1}{|b|}t^{3m-2} + s_2 t^{3m-1} + \frac{(3m-2)h_1}{|b|^2}t^{4m-3} + t^{4m-2}O(1),$$

$$\Psi(t) = 1 - s_1 t^{2m-1} + \frac{h_1}{|b|}t^{3m-2} - s_2 t^{3m-1} + \frac{(3m-2)h_1}{|b|^2}t^{4m-3} + t^{4m-2}O(1),$$

$$\begin{aligned} u_1(t) = v_1(t,t_0)\,\Phi(t) = v_1(t,t_0)\Bigg(1 &+ s_1 t^{2m-1} + \frac{h_1}{|b|}t^{3m-2} + s_2 t^{3m-1} \\ &+ \frac{(3m-2)h_1}{|b|^2}t^{4m-3} + t^{4m-2}O(1)\Bigg), \end{aligned} \tag{42}$$

$$\begin{aligned} u_2(t) = v_2(t,t_0)\,\Psi(t) = v_2(t,t_0)\Bigg(1 &- s_1 t^{2m-1} + \frac{h_1}{|b|}t^{3m-2} - s_2 t^{3m-1} \\ &+ \frac{(3m-2)h_1}{|b|^2}t^{4m-3} + t^{4m-2}O(1)\Bigg). \end{aligned} \tag{43}$$

As was already noted in the proof of Theorem 1, for $b < 0$ all derivatives of the function $v_2(t,t_0)$ at zero vanish, and so finding the power-law asymptotics is only meaningful for $u_1(t)$, and if

$$v_1(t,t_0) = \sum_{j=0}^{4m-3} \frac{v_1^{(j)}(0,t_0)}{j!}t^j + t^{4m-2}O(1),$$

then from (42) we derive

$$\begin{aligned} u_1(t) = &\left(1 + s_1 t^{2m-1} + \frac{h_1}{|b|}t^{3m-2} + s_2 t^{3m-1} + \frac{(3m-2)h_1}{|b|^2}t^{4m-3}\right) \\ &\times \sum_{j=0}^{4m-3} \frac{v_1^{(j)}(0,t_0)}{j!}t^j + t^{4m-2}O(1). \end{aligned}$$

For $b > 0$, the function $v_1(t,t_0)$ is unbounded at zero, and so finding the power-law asymptotics only makes sense for $u_2(t)$, and if

$$v_2(t,t_0) = \sum_{j=0}^{4m-3} \frac{v_2^{(j)}(0,t_0)}{j!}t^j + t^{4m-2}O(1),$$

then it follows from (43) that

$$u_2(t) = \left(1 - s_1 t^{2m-1} + \frac{h_1}{|b|} t^{3m-2} - s_2 t^{3m-1} + \frac{(3m-2)h_1}{|b|^2} t^{4m-3}\right) \times \sum_{j=0}^{4m-3} \frac{v_2^{(j)}(0, t_0)}{j!} t^j + t^{4m-2} O(1).$$

To construct the power-law asymptotics of the solutions $u_*(t)$ for $b < 0$ and $\tilde{u}_*(t)$ for $b > 0$ of the inhomogeneous equation (36), in (28) and (29) one can also use formulas (37)–(41) and the expansion

$$f(t) = \sum_{j=0}^{4m-3} \frac{f^{(j)}(0)}{j!} t^j + t^{4m-2} O(1),$$

make the necessary transformations, and integrate the result.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

REFERENCES

1. V. P. Glushko, *Linear Degenerating Differential Equations* (Voronezh. Gos. Univ., Voronezh, 1972) [in Russian].
2. N. Kh. Rozov, V. G. Sushko, and D. I. Chudova, “Differential equations with degenerate coefficient of the highest-order derivative,” Fundam. Prikl. Mat. **4** (3), 1063–1095 (1998).
3. V. P. Arkhipov, “Linear second-order differential equations with degenerating coefficient of the second derivative,” Differ. Equ. **47** (10), 1397–1407 (2011).
4. V. P. Arkhipov and A. V. Glushak, “Degenerate differential equations of the second order. Asymptotic representations of solutions,” Nauchn. Vedom. Belgorod. Gosud. Univ. Mat. Fiz. **44** (20 (241)), 5–22 (2016).