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Laguerre Polynomials in the Forward and Backward Wave Profile Description for the Wave Equation on an Interval with the Robin Condition or the Attached Mass Condition

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Abstract—We obtain a formula describing the forward and backward wave profile for the solution of an initial–boundary value problem for the wave equation on an interval. The following combinations of boundary conditions are considered:
(i) The first-kind condition at the left endpoint of the interval and the third-kind condition at the right endpoint.
(ii) The second-kind condition at the left endpoint and the third-kind condition at the right endpoint.
(iii) The first-kind condition at the left endpoint and the attached mass condition at the right endpoint.
(iv) The second-kind condition at the left endpoint and the attached mass condition at the right endpoint.
The formula contains finitely many arithmetic operations, elementary functions, quadratures, and transformations of the independent argument of the initial data such as the multiplication by a number and taking the integer part of a number.

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1. INTRODUCTION

Consider the following initial–boundary value problem for a homogeneous one-dimensional wave equation (one boundary condition is of the first kind, and the other is of the third kind),

$$\begin{cases} u_{xx}(x,t) = u_{tt}(x,t) & (0 < x < l, t > 0), \\ u(0,t) = 0, \quad u_x(l,t) + ku(l,t) = 0 & (t > 0), \\ u(x,0) = \varphi(x), \quad u_t(x,0) = 0 & (0 \le x \le l), \end{cases} \tag{1}$$

in which l and k are fixed positive numbers, $\varphi \in C^2[0;l]$ with $\varphi(0) = 0$ and $\varphi'(l) + k\varphi(l) = 0$; the solution $u(x,t)$ is sought in the class of real-valued functions twice continuously differentiable on $(0;l) \times (0;+\infty)$ for which all second-order derivatives are definable by continuity on $[0;l] \times [0;+\infty)$. In addition, we assume that $\varphi''(0) = 0$, as otherwise a solution to problem (1) from the specified class of functions does not exist (see, for example, [1, Lecture IV, Sec. 2, p. 54] or [2]).

Writing the solution to (1) in d'Alembert's form, we arrive at the representation

$$u(x,t) = \frac{1}{2}(\tilde{\varphi}(x+t) + \tilde{\varphi}(x-t)), \tag{2}$$

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where $\tilde{\varphi}$ is a twice continuously differentiable continuation of φ from $[0;l]$ to the set of all real numbers $\mathbf{R}$, subject to the conditions $\tilde{\varphi}(-x) = -\tilde{\varphi}(x)$ and $(\tilde{\varphi}' + k\tilde{\varphi})\,(l+x) = -\,(\tilde{\varphi}' + k\tilde{\varphi})\,(l-x)$ $(x \in \mathbf{R})$ is finite if such a continuation $\tilde{\varphi}$ exists; the last relation can be replaced (taking into account the first one) by $(\tilde{\varphi}' + k\tilde{\varphi})\,(x) = -\,(\tilde{\varphi}' - k\tilde{\varphi})\,(x-2l)$ $(x \in \mathbf{R})$. In other words, $\tilde{\varphi}$ in (2) is an odd twice continuously differentiable function the restriction of which to $[-l;+\infty)$ is a solution to the initial value problem

$$\tilde{\varphi}'(x) + \tilde{\varphi}'(x-2l) + k\tilde{\varphi}(x) - k\tilde{\varphi}(x-2l) = 0 \quad (x \geq l), \tag{3}$$

$$\tilde{\varphi}(x) = \begin{cases} -\varphi(-x) & \text{if } -l \leq x \leq 0, \\ \varphi(x) & \text{if } 0 \leq x \leq l. \end{cases} \tag{4}$$

Our immediate goal is to obtain a formula expressing $\tilde{\varphi}$ in terms of φ using only finitely many arithmetic operations, elementary functions, quadratures, and such operations on the independent argument φ as multiplication by a number and taking the integer part. This kind of formula is useful in the numerical solution of problem (1), in the study of the corresponding controllability problem (see, for example, [3], [4]), and in the analysis of the dependence of the solution to the problem on the parameters included in it.

To achieve this goal, one can try to use various methods for solving problem (3), (4). Using the Laplace transform gives a representation of $\tilde{\varphi}$ as a contour integral, and calculating the latter by summing the residues leads to a representation of $\tilde{\varphi}$ as a series (see, for example, [5, Theorem 5.5]). The shift theorem would be effective if the initial condition in (4) had a special form (on this, see [5, Remark at the end of Sec. 4.7]), but in our case it is not this way. Finally, the representation of $\tilde{\varphi}$ in the form of a real definite integral (see, for example, [5, Theorem 5.4]) contains a fundamental solution, which, if it can be presented constructively, is only through the method of sequential integration (method of steps). It is for these reasons that in order to achieve this goal, we choose the step method despite its somewhat laborious nature.

To make further calculations less cumbersome, further in (3) and (4) we consider the case of $l = 1$, which results from the replacement $\hat{\varphi}(y) = \tilde{\varphi}(ly)$ with subsequent redesignation of lk again by k.

2. SOLVING PROBLEM (3), (4) BY SUCCESSIVE INTEGRATION

So let $l = 1$. First of all, we note that a solution to problem (3), (4) exists, is unique, and belongs to $C^2[-1;+\infty)$; to establish this it suffices to apply [5, Theorem 5.1] and take into account the fact that $\varphi \in C^2[0;1]$ and $\varphi(0) = 0, \varphi''(0) = 0$, and $\varphi'(1) + k\varphi(1) = 0$.

Let $n \in \mathbf{N}$, where $\mathbf{N}$ is the set of all positive integers. If we multiply Eq. (3) by e^{kx} and integrate from $2n-1$ to $x \in [2n-1;2n+1)$, then for the function $\psi(x) = e^{kx}\tilde{\varphi}(x)$ we obtain the relation

$$\psi(x) = \psi(2n-1) + \int_{2n-1}^{x} \Big[k\tilde{\varphi}(s-2)e^{ks} - \tilde{\varphi}'(s-2)e^{ks}\Big]\, ds.$$

Integrating the last term by parts here, we obtain (denoted by A as e^{2k})

$$\psi(x) = \psi(2n-1) - A\psi(x-2) + A\psi(2n-3) + 2kA\int_{2n-1}^{x} \psi(s-2)\, ds \tag{5}$$

for $x \in [2n-1;2n+1), n \in \mathbf{N}$.

The inclusion $x \in [2n-1;2n+1)$ is equivalent to the equality $n = [(x+1)/2]$ (here the square brackets mean taking the integer part of a number), so in what follows we assume that in the equality (5), $x \geq 1$ and $n = n(x) \overset{\text{def}}{=} [(x+1)/2]$.

Lemma 1. *There exist a sequence of finite tuples of polynomials*

$$\Big\{\{R_i^n(x)\}_{i=0}^{n-1}\Big\}_{n=1}^{\infty} \qquad (\deg R_i^n = i)$$

and a sequence of polynomials $\{Q^n(x)\}_{n=1}^{\infty}$ $(\deg Q^n = n-1)$ *such that*

$$\psi(x) = (-A)^n \psi(x-2n) + (2kA)^n \sum_{i=0}^{n-1} R_i^n(x) \int_{2n-1}^{x} t^{n-i-1} \psi(t-2n)\, dt + Q^n(x), \tag{6}$$

where $n = n(x)$ *is the integer part of the number* $(x+1)/2$.

Proof. We carry out the proof by induction on n.

The validity of (6) for $n(x) = 1$ follows from the validity of (5) for $n(x) = 1$ if we set $R_0^1(x) \equiv 1$, $Q^1(x) \equiv 0$.

Let us now suppose that the representation (6) is valid for $n(x) = m$ (i.e., for $x \in [2m-1; 2m+1)$), where $m \in \mathrm{N}$. Then relation (5) for $x \in [2m+1; 2m+3)$ will take the form (we substitute $\psi(s-2)$ for its representation in the form (6), integrate by parts, perform a suitable change of integration variable, and rearrange the terms)

$$\begin{aligned}
\psi(x) = (2kA)^{m+1}\Big[&-\frac{1}{2k}\sum_{i=0}^{m-1} R_i^m(x-2)\int_{2m+1}^{x}(t-2)^{m-i-1}\psi(t-2(m+1))\,dt \\
&+\sum_{i=0}^{m-1}\int R_i^m(x-2)\,dx\int_{2m+1}^{x}(t-2)^{m-i-1}\psi(t-2(m+1))\,dt \\
&-\sum_{i=0}^{m-1}\int_{2m+1}^{x}\left(\int R_i^m(t-2)\,dt\cdot(t-2)^{m-i-1}\psi(t-2(m+1))\right)dt \\
&+\left(-\frac{1}{2k}\right)^{m}\int_{2m+1}^{x}\psi(t-2(m+1))\,dt\Big] \\
&+(2kA)^m\sum_{i=0}^{m-1}R_i^m(2m+1)\int_{2m-1}^{2m+1}t^{m-i-1}\psi(t-2m)\,dt+Q^m(2m+1) \\
&-AQ^m(x-2)+AQ^m(2m-1)+2kA\int_{2m+1}^{x}Q^m(s-2)\,ds,
\end{aligned}$$

where $\int R_i^m(\tau-2)\,d\tau$ is some antiderivative of the polynomial $R_i^m(\tau-2)$, which we will further assume to be vanishing at the point $\tau = 2$. This implies the validity of (6) for $x \in [2m+1; 2m+3)$—it suffices to define the polynomials $R_i^{m+1}(x)$ by the relation

$$\begin{aligned}
\sum_{i=0}^{m} R_i^{m+1}(x)\int_{2m+1}^{x} t^{m-i}\psi(t-2(m+1))\,dt
&=\left(-\frac{1}{2k}\right)\sum_{i=0}^{m-1}R_i^m(x-2)\int_{2m+1}^{x}(t-2)^{m-i-1}\psi(t-2(m+1))\,dt \\
&\quad+\sum_{i=0}^{m-1}\int R_i^m(x-2)\,dx\int_{2m+1}^{x}(t-2)^{m-i-1}\psi(t-2(m+1))\,dt \\
&\quad-\sum_{i=0}^{m-1}\int_{2m+1}^{x}\left(\int R_i^m(t-2)\,dt\cdot(t-2)^{m-i-1}\psi(t-2(m+1))\right)dt \\
&\quad+\left(-\frac{1}{2k}\right)^{m}\int_{2m+1}^{x}\psi(t-2(m+1))\,dt
\end{aligned} \tag{7}$$

and set

$$Q^{m+1}(x) = (2kA)^m \sum_{i=0}^{m-1} R_i^m(2m+1)\int_{2m-1}^{2m+1} t^{m-i-1}\psi(t-2m)\,dt$$

$$+ Q^m(2m+1) - AQ^m(x-2) + AQ^m(2m-1) \tag{8}$$
$$+ (2kA)\int_{2m+1}^{x} Q^m(s-2)\,ds.$$

The fact that (7) allows one to define the polynomials $R_i^{m+1}(x)$ in terms of the polynomials $R_i^m(x)$ (we do not discuss the uniqueness of such a definition) can be seen by transforming the right-hand side of (7)—using the Newton binomial formula to expand the powers of $(t-2)$ and then grouping the resulting terms by the factors

$$\int_{2m+1}^{x} t^{m-i}\psi(t-2(m+1))\,dt.$$

The proof of the lemma is complete. □

Remark. In what follows, for definiteness, we assume that the polynomial $R_i^{m+1}(x)$ is determined from (7) via the polynomials $R_i^m(x)$ precisely in the way indicated at the end of the proof of Lemma 1.

Lemma 2. *The coefficients $b_r^{m,i}$ of the polynomials*

$$R_i^m(x) = b_i^{m,i}x^i + b_{i-1}^{m,i}x^{i-1} + \ldots + b_1^{m,i}x + b_0^{m,i}$$

are calculated using the formulas

$$b_{i-q}^{m,i} = \frac{(-1)^{m-i-1}\binom{m}{q}}{(m-i-1)!(i-q)!}\left(-\frac{1}{2k}\right)^q \qquad (q=0,\ldots,m-1, \quad i=q,\ldots,m-1), \tag{9}$$

where $\binom{m}{q}$ is the binomial coefficient.

Proof. Let us derive recurrence relations connecting the coefficients $b_p^{m+1,j}$ with the coefficients $b_r^{m,i}$. After expanding the powers of the binomial $t-2$ (including in the polynomial $\int R_i^m(t-2)\,dt$) and reduction of similar ones in $\int_{2m+1}^{x} t^p\psi(t-2(m+1))\,dt$, the expression on the right side of (7) takes the form

$$\left(-\frac{1}{2k}\right)\sum_{p=0}^{m-1}\sum_{i=0}^{m-1-p}(-2)^{m-p-i-1}\binom{m-i-1}{p}R_i^m(x-2)\int_{2m+1}^{x} t^p\psi(t-2(m+1))\,dt$$
$$+\sum_{p=0}^{m-1}\sum_{i=0}^{m-1-p}(-2)^{m-p-i-1}\binom{m-i-1}{p}\int R_i^m(x-2)\,dx\int_{2m+1}^{x} t^p\psi(t-2(m+1))\,dt$$
$$-\sum_{p=0}^{m-1}\sum_{i=0}^{m-p-1}\sum_{j=0}^{i}(-2)^{m-i+j-p}\binom{m-i+j}{p}\frac{b_j^{m,i}}{j+1}\int_{2m+1}^{x} t^p\psi(t-2(m+1))\,dt$$
$$-\sum_{p=1}^{m}\sum_{i=m-p}^{m-1}\sum_{j=p-(m-i)}^{i}(-2)^{m-i+j-p}\binom{m-i+j}{p}\frac{b_j^{m,i}}{j+1}\int_{2m+1}^{x} t^p\psi(t-2(m+1))\,dt$$
$$+\left(-\frac{1}{2k}\right)^m\int_{2m+1}^{x}\psi(t-2(m+1))\,dt.$$

Now it is already clear that (7) is fulfilled if, first,

$$R_0^{m+1}(x) = -\sum_{i=0}^{m-1}\frac{b_i^{m,i}}{i+1},$$

second, for $p = 1, \dots, m-1$

$$\begin{aligned} R_{m-p}^{m+1}(x) = \left(-\frac{1}{2k}\right) \sum_{i=0}^{(m-1)-p} (-2)^{m-i-1-p} \binom{m-i-1}{p} R_i^m(x-2) \\ + \sum_{i=0}^{(m-1)-p} \left((-2)^{m-i-1-p} \binom{m-i-1}{p} \int R_i^m(x-2)\,dx\right) \\ - \sum_{i=0}^{(m-1)-p} \sum_{j=0}^{i} (-2)^{m-i+j-p} \binom{m-i+j}{p} \frac{b_j^{m,i}}{j+1} \\ - \sum_{i=m-p}^{m-1} \sum_{j=p-(m-i)}^{i} (-2)^{m-i+j-p} \binom{m-i+j}{p} \frac{b_j^{m,i}}{j+1}, \end{aligned} \tag{10}$$

and third,

$$\begin{aligned} R_m^{m+1}(x) = \left(-\frac{1}{2k}\right) \sum_{i=0}^{m-1} (-2)^{m-i-1} R_i^m(x-2) \\ + \sum_{i=0}^{m-1} (-2)^{m-i-1} \int R_i^m(x-2)\,dx \\ - \sum_{i=0}^{m-1} \sum_{j=0}^{i} (-2)^{m-i+j} \frac{b_j^{m,i}}{j+1} + \left(-\frac{1}{2k}\right)^m. \end{aligned} \tag{11}$$

Expanding the powers of the binomial $x-2$ in $R_i^m(x-2)$ in these relations and grouping by the powers of x, we obtain the following recurrence relations connecting the coefficients of the polynomials $R_j^{m+1}(x)$ and $R_i^m(x)$:

$$b_0^{m+1,0} = -\sum_{i=0}^{m-1} \frac{b_i^{m,i}}{i+1},$$

$$\begin{aligned} b_0^{m+1,m-p} = \left(-\frac{1}{2k}\right) \sum_{i=0}^{m-p-1} \sum_{j=0}^{i} (-2)^{m-i+j-p-1} \binom{m-i-1}{p} b_j^{m,i} \\ + \sum_{i=0}^{m-p-1} \sum_{j=0}^{i} (-2)^{m-p-i+j} \binom{m-i-1}{p} \frac{b_j^{m,i}}{j+1} \\ - \sum_{i=0}^{m-p-1} \sum_{j=0}^{i} (-2)^{m-i+j-p} \binom{m-i+j}{p} \frac{b_j^{m,i}}{j+1} \\ - \sum_{i=m-p}^{m-1} \sum_{j=p-(m-i)}^{i} (-2)^{m-i+j-p} \binom{m-i+j}{p} \frac{b_j^{m,i}}{j+1} \quad (p = 1, \dots, m-1), \end{aligned}$$

$$\begin{aligned} b_q^{m+1,m-p} = \left(-\frac{1}{2k}\right) \sum_{i=q}^{m-p-1} \sum_{j=q}^{i} (-2)^{m-i+j-p-1} \binom{j}{q} \binom{m-i-1}{p} b_j^{m,i} \\ + \sum_{i=q-1}^{m-p-1} \sum_{j=q-1}^{i} (-2)^{m-p-i+j-q} \binom{j+1}{q} \binom{m-i-1}{p} \frac{b_j^{m,i}}{j+1}, \\ (p = 1, \dots, m-1,\ q = 1, \dots, m-p-1), \end{aligned}$$

$$b_{m-p}^{m+1,m-p} = \frac{b_{m-p-1}^{m,m-p-1}}{m-p} \quad (p=1,\dots,m-1),$$

$$b_0^{m+1,m} = \left(-\frac{1}{2k}\right)^m + \left(-\frac{1}{2k}\right)\sum_{p=1}^{m-1}(-2)^p \sum_{j=0}^{p} b_j^{m,m-1-p+j} + \left(-\frac{1}{2k}\right) b_0^{m,m-1},$$

$$b_q^{m+1,m} = \left(-\frac{1}{2k}\right)\sum_{i=q}^{m-1}\sum_{j=q}^{i}(-2)^{m-1-q-i+j}\binom{j}{q} b_j^{m,i}$$

$$+ \sum_{i=q-1}^{m-1}\sum_{j=q-1}^{i}(-2)^{m-q-i+j}\binom{j+1}{q}\frac{b_j^{m,i}}{j+1} \quad (q=1,\dots,m-1),$$

$$b_m^{m+1,m} = \frac{b_{m-1}^{m,m-1}}{m}.$$

It is directly verified that the substitution of (9) into the resulting recurrence relations turns them into true equalities.

The proof of the lemma is complete. □

Lemma 3. *The polynomial $Q^n(x)$ is representable in the form*

$$Q^n(x) = \sum_{j=1}^{n-1}(2kA)^j \sum_{i=0}^{j-1} R_i^j(x) \int_{2j-1}^{2j+1} t^{j-i-1}\psi(t-2j)\,dt. \tag{12}$$

Proof. We conduct the proof by induction on n. The validity of (12) for $n=1$ is obvious; for $n=2$ it follows from (8) (with $m=1$) taking into account the fact that $Q^1(x)\equiv 0$ and $R_0^1(x)\equiv 1$.

Now let the representation (12) be true for $n=m$. We substitute the representation of the polynomial $Q^m(x)$ in the form (12) into the right-hand side of (8) and group like terms on the right-hand side in the resulting representation of $Q^{m+1}(x)$: first, in powers of j factors $(2kA)^j$, $j=1,\dots,m$, and then, in each of the m sums with factors $(2kA)^j$, in factors of the form

$$\int_{2j-1}^{2j+1} t^{j-i-1}\psi(t-2j)\,dt, i=0,\dots,j-1.$$

As a result, we obtain

$$Q^{m+1}(x) = \sum_{j=1}^{m}(2kA)^j \sum_{i=0}^{j-1}\biggl[R_i^j(2m+1) + \sum_{p=0}^{i-1}(-2)^{i-p-1}\binom{j-p-2}{i-p-1} \times \biggl\{\int_{2m+1}^{x} R_p^{j-1}(s-2)\,ds - \frac{1}{2k}R_p^{j-1}(x-2) + \frac{1}{2k}R_p^{j-1}(2m-1)\biggr\}\biggr] \int_{2j-1}^{2j+1} t^{j-i-1}\psi(t-2j)\,dt.$$

Now it remains to show that

$$R_i^j(x) = R_i^j(2m+1) + \sum_{p=0}^{i-1}(-2)^{i-p-1}\binom{j-p-2}{i-p-1} \times \biggl\{\int_{2m+1}^{x} R_p^{j-1}(s-2)\,ds - \frac{1}{2k}R_p^{j-1}(x-2) + \frac{1}{2k}R_p^{j-1}(2m-1)\biggr\}. \tag{13}$$

The validity of (13) for $i=0$ follows from the fact that $R_0^j(x)\equiv R_0^j(2m+1)$, since $\deg R_0^j=0$. If $i=1,\dots,j-2$, then due to (10) (if we set $m=j-1$ and $m-p=i$ there), we have the equality

$$R_i^j(x)-R_i^j(2m+1)=\sum_{q=0}^{i-1}\biggl[(-2)^{i-q-1}\binom{j-q-2}{j-1-i}\biggl\{\biggl(-\frac{1}{2k}\biggr)\cdot\bigl(R_q^{j-1}(x-2)-R_q^{j-1}(2m-1)\bigr)+\int_{2m+1}^{x}R_q^{j-1}(s-2)\,ds\biggr\}\biggr],$$

which coincides with (13), since $\binom{j-q-2}{j-1-i}=\binom{j-q-2}{i-q-1}$. Finally, by virtue of (11), we have

$$R_{j-1}^j(x)-R_{j-1}^j(2m+1)=\sum_{q=0}^{j-2}(-2)^{j-q-2}\biggl[\biggl(-\frac{1}{2k}\biggr)\cdot\bigl(R_q^{j-1}(x-2)-R_q^{j-1}(2m-1)\bigr)+\int_{2m+1}^{x}R_q^{j-1}(s-2)\,ds\biggr],$$

which coincides with (13) for $i=j-1$.

The proof of the lemma is complete. □

Remark. From Lemma 2 we conclude that

$$R_i^j(x)=\frac{(-1)^{j-1}}{(j-i-1)!(2k)^i}L_i^{j-i}(2kx),$$

where

$$L_n^{\alpha}(y)=\sum_{q=0}^{n}\frac{\Gamma(\alpha+n+1)(-y)^q}{\Gamma(\alpha+q+1)q!(n-q)!}$$

is a generalized Laguerre polynomial, or a Laguerre–Sonin polynomial, see [6]–[9].

It should be noted that these polynomials L_n^{α} were originally introduced by E. Laguerre for $\alpha=0$, and in the general case they were introduced by N. Ya. Sonin, see [9], [10].

Now putting together the results of Lemmas 1–3 and returning to the case of arbitrary l (see the last paragraph in Sec. 1), we come to the following conclusion.

Theorem 1. *The function $\tilde\varphi$ in the representation* (2) *of the solution to problem* (1) *is a twice continuously differentiable odd function that coincides on the interval $[0;l]$ with φ and can be represented on any interval of the form $[(2n-1)l;(2n+1)l]$ ($n\in\mathbf{N}$) as a finite sum*

$$\begin{aligned}\tilde\varphi(x)&=(-1)^n\varphi_1(x-2nl)\\&\quad+(-1)^{n-1}\sum_{i=1}^{n}\frac{(2k)^i}{(i-1)!}L_i^{n-i}(2kx)\int_{(2n-1)l}^{x}t^{i-1}e^{k(t-x)}\varphi_1(t-2nl)\,dt\\&\quad+\sum_{j=1}^{n-1}(-1)^{j-1}\sum_{i=1}^{j}\frac{(2k)^i}{(i-1)!}L_i^{j-i}(2kx)\int_{(2j-1)l}^{(2j+1)l}t^{i-1}e^{k(t-x)}\varphi_1(t-2jl)\,dt,\end{aligned}\tag{14}$$

where φ_1 is the odd continuation of φ to $[-l;l]$.

3. SUMMATION OVER SINES WITH FREQUENCIES EQUAL TO THE EIGENFREQUENCIES OF OSCILLATIONS IN PROBLEM (1)

An interesting corollary can be obtained from the results in the previous section if, returning to problem (3), (4), we solve it using the Laplace transform. Namely, applying [5, Theorem 5.5], we conclude that the solution to problem (3), (4) can be represented as a series

$$\tilde\varphi(x)=\sum_{m=1}^{\infty}\operatorname{Res}\biggl[\frac{e^{xz}p(z)}{h(z)};iy_m\biggr]\qquad(x>-l),\tag{15}$$

where

$$p(z) = -2\varphi(l)\cosh(lz) + \int_{-l}^{l} \left(\tilde{\varphi}'(x) + k\tilde{\varphi}(x)\right) e^{-zx}\,dx, \quad h(z) = z + ze^{-2lz} + k - ke^{-2lz}$$

(here i is the imaginary unit) and y_m are the roots of the equation $y = -k \cdot \tan(ly)$ numbered in arbitrary order; moreover, the series (15) converges uniformly on any interval $[a; b] \subset (-l; +\infty)$. Expressing p only in terms of φ (i.e., by integrating by parts and getting rid of the presence of φ' in the expression for p) and calculating the residues in (15), we arrive (taking into account also the oddity of $\tilde{\varphi}$) at the representation of the function $\tilde{\varphi}$ in the form of a series the restriction of which to $[0; l]$ is the Fourier series of the function φ over the eigenfunctions of the spectral problem generated by problem (1),

$$\tilde{\varphi}(x) = \sum_{m=1}^{\infty} b_m \sin(\omega_m x) \quad (x \in \mathbf{R}),$$

where

$$b_m = \frac{2\left(k^2 + \omega_m^2\right)}{k + l\left(k^2 + \omega_m^2\right)} \cdot \int_0^l \varphi(s) \sin(\omega_m s)\, ds \quad (m = 1, \dots, \infty)$$

and $\{\omega_m\}_{m=1}^{\infty}$ is an increasing sequence composed of all positive roots of the equation $\omega = -k \cdot \tan(l\omega)$.

Thus, taking into account Theorem 1 we obtain the following assertion.

Theorem 2. *Let $f(x) \in C^2[0, l]$ with $f(0) = 0$ and $f''(0) = 0$. Let k be a positive number satisfying the equality $f'(l) + kf(l) = 0$, and let $\{b_m\}_{m=1}^{\infty}$ be the sequence of numbers defined by the relation*

$$b_m = \frac{2\left(k^2 + \omega_m^2\right)}{k + l\left(k^2 + \omega_m^2\right)} \cdot \int_0^l f(s) \sin(\omega_m s)\, ds \quad (m = 1, \dots, \infty),$$

in which $\{\omega_m\}_{m=1}^{\infty}$ is the increasing sequence of all positive roots of the equation $\omega = -k \cdot \tan(l\omega)$. Then the series $\sum_{m=1}^{\infty} b_m \sin(\omega_m x)$ converges uniformly on any segment of the real line, and its sum $s(x)$ is a twice continuously differentiable odd function that coincides on $[0;l]$ with $f(x)$ and for $x > l$ is defined by the relation

$$\begin{aligned} s(x) &= (-1)^n f_1(x - 2nl) \\ &\quad + (-1)^{n-1} \sum_{i=1}^{n} \frac{(2k)^i}{(i-1)!} L_i^{n-i}(2kx) \int_{(2n-1)l}^{x} t^{i-1} e^{k(t-x)} f_1(t - 2nl)\, dt \\ &\quad + \sum_{j=1}^{n-1} (-1)^{j-1} \sum_{i=1}^{j} \frac{(2k)^i}{(i-1)!} L_i^{j-i}(2kx) \int_{(2j-1)l}^{(2j+1)l} t^{i-1} e^{k(t-x)} f_1(t - 2jl)\, dt, \end{aligned}$$

where $n = n(x)$ is the integer part of the number $(x + l)/(2l)$, and f_1 is the odd continuation of f to $[-l; l]$.

4. THE CASE OF A BOUNDARY CONDITION OF THE THIRD KIND AT THE RIGHT ENDPOINT OF THE INTERVAL AND A BOUNDARY CONDITION OF THE SECOND KIND AT THE LEFT ENDPOINT

Theorem 1′. *Let $u(x, t)$ be a solution to the problem*

$$\begin{cases} u_{xx}(x,t) = u_{tt}(x,t) & (0 < x < l, t > 0) \\ u_x(0,t) = 0, \quad u_x(l,t) + ku(l,t) = 0 & (t > 0) \\ u(x,0) = \varphi(x), \quad u_t(x,0) = 0 & (0 \le x \le l) \end{cases} \qquad (1')$$

obtained from problem (1) by replacing the condition $u(0, t) = 0$ with the condition $u_x(0, t) = 0$ and correspondingly replacing the requirements $\varphi(0) = 0$ and $\varphi''(0) = 0$ (on the function φ) with the requirement: $\varphi'(0) = 0$. Then $u(x, t) = \frac{1}{2}(\hat{\varphi}(x + t) + \hat{\varphi}(x - t))$, where $\hat{\varphi}(x)$ is a twice *continuously differentiable even function coinciding with $\varphi(x)$ on $[0;l]$ and representable as a finite sum*

$$\begin{aligned}\hat{\varphi}(x) = (-1)^n\varphi_2(x-2nl) \\ -\sum_{i=1}^{n}\frac{(2k)^i}{(i-1)!}L_i^{n-i}(2kx)\int_{(2n-1)l}^{x}t^{i-1}e^{k(t-x)}\varphi_2(t-2nl)\,dt \\ -\sum_{j=1}^{n-1}\sum_{i=1}^{j}\frac{(2k)^i}{(i-1)!}L_i^{j-i}(2kx)\int_{(2j-1)l}^{(2j+1)l}t^{i-1}e^{k(t-x)}\varphi_2(t-2jl)\,dt\end{aligned} \tag{14'}$$

on any interval of the form $[(2n-1)l;(2n+1)l](n\in\mathbf{N})$, where φ_2 is an even continuation of φ to $[-l;l]$.

Proof. The proof of this theorem is practically no different from the proof of Theorem 1; this is why it is not given here. We only note that $\hat{\varphi}(x)$ is a solution to the equation

$$\hat{\varphi}'(x)-\hat{\varphi}'(x-2l)+k\hat{\varphi}(x)+k\hat{\varphi}(x-2l)=0. \quad \square \tag{3'}$$

At the same time, we note that the representation (14′) implies a statement similar to Theorem 2.

5. THE CASE OF THE ATTACHED MASS CONDITION AT THE RIGHT ENDPOINT OF THE INTERVAL AND A BOUNDARY CONDITION OF THE FIRST OR SECOND KIND AT THE LEFT ENDPOINT

Consider the problems

$$\begin{cases} u_{xx}(x,t)=u_{tt}(x,t) & (0<x<l, t>0), \\ u(0,t)=0, u_x(l,t)+mu_{tt}(l,t)=0 & (t>0), \\ u(x,0)=\alpha(x), u_t(x,0)=0 & (0\le x\le l) \end{cases} \tag{16}$$

and

$$\begin{cases} u_{xx}(x,t)=u_{tt}(x,t) & (0<x<l, t>0), \\ u_x(0,t)=0, u_x(l,t)+mu_{tt}(l,t)=0 & (t>0), \\ u(x,0)=\beta(x), u_t(x,0)=0 & (0\le x\le l), \end{cases} \tag{16'}$$

where l and m are fixed positive numbers, $\alpha,\beta\in C^2[0;l]$; the solution to each of these problems is sought in the same class of functions as the solution to problem (1). Regarding $\alpha(x)$, it is additionally assumed that $\alpha(0)=0$ and $\alpha''(0)=0$ and also that $\alpha'(l)+m\alpha''(l)=0$, and regarding $\beta(x)$, that $\beta'(0)=0$ and $\beta'(l)+m\beta''(l)=0$; these conditions are necessary and sufficient for the existence of solutions to problems (16) and (16′) in the indicated class.

Writing solutions to problems (16) and (16′) in d'Alembert's form

$$u(x,t)=\frac{1}{2}(\gamma(x+t)+\gamma(x-t)), \tag{17}$$

we arrive at the conclusion that in the case of problem (16), γ there is a twice continuously differentiable odd continuation of α to $[0;l]$ on $\mathbf{R}$ subject to the equation

$$\gamma''(x)-\gamma''(x-2l)+\frac{1}{m}\gamma'(x)+\frac{1}{m}\gamma'(x-2l)=0 \quad (x\ge l), \tag{18}$$

and in the case of problem (16′), γ is a twice continuously differentiable even continuation of β from $[0;l]$ to $\mathbf{R}$, subject to the equation

$$\gamma''(x)+\gamma''(x-2l)+\frac{1}{m}\gamma'(x)-\frac{1}{m}\gamma'(x-2l)=0 \quad (x\ge l). \tag{18'}$$

Comparing Eqs. (18) and (18′) with the equations, respectively, (3′) and (3), we arrive at the following two theorems.

Theorem 3. *Let $\hat{A}_k$ be an operator that associates any function $\varphi \in C^2[0;l]$ with an even function $\hat{\varphi}$ defined on $\mathbf{R}$ that coincides with $[0;l]$ with φ and is defined for $x \geq l$ by relation* (14′). *Then the solution to problem* (16) *can be represented in the form* (17)*, where*

$$\gamma(x) = \int_0^x \left(\hat{A}_{1/m}\alpha'\right)(s)\,ds.$$

Theorem 3′. *Let $\tilde{A}_k$ be the operator that associates any function $\varphi \in C^2[0;l]$ with an odd function $\tilde{\varphi}$ defined on $\mathbf{R}$ that coincides on $[0;l]$ with φ and is defined for $x \geq l$ by relation* (14). *Then the solution to problem* (16′) *can be represented in the form* (17)*, where*

$$\gamma(x) = \beta(0) + \int_0^x \left(\tilde{A}_{1/m}\beta'\right)(s)\,ds.$$

6. SOME NOTES AND COMMENTS

1. Above we limited ourselves to considering positive parameters k and m (based on the mechanical interpretation of problems (1), (1′), (16), and (16′)). However, it is easy to verify that all the proofs of theorems we have given are valid for arbitrary real k and m distinct from 0. Moreover, these proofs do not lose their validity in the case of complex-valued solutions to problems (1), (1′), (16), and (16′) and, accordingly, arbitrary nonzero complex parameters k and m.

2. We have so far limited ourselves to considering only zero initial velocity. If, for example, in problem (1) the condition $u_t(x,0) = 0$ is replaced by $u_t(x,0) = \Phi(x)$, where $\Phi \in C^1[0;l]$ with $\Phi(0) = 0$ and $\Phi'(l) + k\Phi(l) = 0$ (the last two conditions are necessary and sufficient for the existence of a solution in the class we are considering), then the solution can be represented in the form

$$u(x,t) = (C(t)\varphi)(x) + \int_0^t (C(\tau)\Phi)(x)\,d\tau,$$

   where the operator function $C(t)$ is defined by the relation

$$(C(t)\psi)(x) = \frac{1}{2}\left[\left(\tilde{A}_k\psi\right)(x+t) + \left(\tilde{A}_k\psi\right)(x-t)\right].$$

   For other problems ((1), (16), and (16′)) the situation is similar.

3. We have so far limited ourselves to considering only the homogeneous wave equation. However, as is well known, Duhamel's principle (see, for example, [2], [11]) allows one to express the solution to an inhomogeneous equation with zero initial data through a parametric family of solutions to a homogeneous equation in the form of an integral over a parameter.

4. It is clear a posteriori that the proofs of Theorems 1, 1′, 3, and 3′ can be constructed based on the known functional relations for Laguerre polynomials (however, there is only one necessary relation here: $(L_n^\alpha(x))' = -L_{n-1}^{\alpha+1}(x)$), and such a proof is less cumbersome. However, from a methodological point of view, such proofs—verification are inferior to the evidence—conclusions we have given.

5. It is of interest to generalize the results obtained in this paper to the case of B-hyperbolic equations in which the second derivatives with respect to time (or even simultaneously with respect to time and spatial variables) are replaced by the Bessel differential operator

$$B_v f(x) = \frac{d^2 f}{dx^2} + \frac{2v+1}{x}\frac{df}{dx}.$$

   For equations with Bessel operators, see, for example, [12], [13].

6. As far as the present authors know, the final formulas (14) and (14′) in terms of Laguerre polynomials were first obtained in the papers by the present authors [14], [15]. Moreover, in the paper [14] the formulas were obtained in terms of Laguerre polynomials expressed as sums in implicit form, without explicit indication of these polynomials, and in the paper [15], the expressions via Laguerre polynomials are given in explicit form. Subsequently, similar formulas in terms of Laguerre polynomials in similar problems were used by other authors, but in later works.

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CONFLICT OF INTEREST

The authors of this work declare that they have no conflicts of interest.

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