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PARTIAL DIFFERENTIAL EQUATIONS

On the Solvability of Initial and Boundary Value Problems for Abstract Functional-Differential Euler–Poisson–Darboux Equations

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Abstract—In a Banach space, we consider the Cauchy problem and the Dirichlet and Neumann boundary value problems for a functional-differential equation generalizing the Euler–Poisson–Darboux equation. A sufficient condition for the solvability of the Cauchy problem is proved, and an explicit form of the resolving operator is indicated, which is written using the Bessel and Struve operator functions introduced by the author. For boundary value problems in the hyperbolic case, we establish conditions imposed on the operator coefficient of the equation and the boundary elements that are sufficient for the unique solvability of these problems.

Keywords: functional-differential equation, Cauchy problem, Dirichlet problem, Neumann problem, unique solvability, Bessel operator function, Struve operator function

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INTRODUCTION

Let A be a closed operator with dense domain $D(A)$ in a Banach space E. For $k > 0$, consider the Euler–Poisson–Darboux equation

$$u''(t) + \frac{k}{t}u'(t) = Au(t), \quad t > 0. \tag{1}$$

It follows from the results in the papers [1, 2] that the well-posed statement of initial conditions for Eq. (1) consists in setting the initial conditions at the point $t = 0$,

$$u(0) = u_0, \quad u'(0) = 0; \tag{2}$$

moreover, if $k \geq 1$, then the initial condition $u'(0) = 0$ is removed; this is typical for a number of equations with a singularity in the coefficients at $t = 0$.

The well-posed statement of the initial conditions depending on the parameter $k \in \mathbb{R}$, as well as the solution of the corresponding initial value problems in the case where A is the Laplace operator in the spatial variables, is given in [3, Ch. 1]. Further research on the theory of singular partial differential equations can be found in [4–8]. As to the abstract Euler–Poisson–Darboux equation (1), it was considered earlier in [9; 10, Ch. 1; 11] under various assumptions about the operator A.

The papers [1, 2] provide conditions on the operator A that ensure the well-posed solvability of problem (1), (2). In [2], these conditions are stated in terms of an estimate for the norm of the resolvent $R(\lambda, A)$ of the operator A and its weighted derivatives, and in [1], in terms of the fractional power of the resolvent and its ordinary derivatives. The set of operators A for which problem (1), (2) is uniformly well posed for $k \geq 0$ will be denoted by G_k, and the resolving operator of this problem will be denoted by $Y_k(t)$ and called the *Bessel operator function*. In what follows, the assumption $A \in G_k$ for some $k \geq 0$ means, in particular, that the Cauchy problem (1), (2) with the operator A is uniformly well posed and $Y_k(t)$ is the resolving operator of this problem, with $Y_0(t) = C(t)$ being the cosine operator function (for more details on the cosine operator function, e.g., see [12; 13, p. 175; 14; 15]).

The Bessel operator function $Y_k(t)$ ($Y_k(0) = I$, $Y_k'(0) = 0$) was introduced in [1, 2] as the resolving operator of the Cauchy problem for the Euler–Poisson–Darboux equation. However, just as in the theory of semigroups and cosine operator functions, the family of Bessel operator functions can be introduced (see [16]) independently of the Euler–Poisson–Darboux differential equation with which it is ultimately connected.

The paper [17] studied the Cauchy problem for the Bessel–Struve equation

$$u''(t) + \frac{k}{t}\big(u'(t) - u'(0)\big) = Au(t), \quad t > 0, \tag{3}$$

$$u(0) = 0, \quad u'(0) = u_1, \tag{4}$$

indicated the explicit form of the resolving operator of this problem, which was called the Struve operator function and denoted by $L_k(t)$ ($L_k(0) = 0$, $L_k'(0) = I$), and also provided formulas that connect the Bessel operator function $Y_k(t)$ and the Struve operator function $L_k(t)$ with the integrated cosine operator function.

The concept of integrated cosine operator function was inspired (see [18–21]) by the researchers' desire to relax the requirements on the operator coefficient of the Cauchy problem for abstract second-order differential equations. Let us further recall the definition of integrated cosine operator function.

Definition. Let $\alpha > 0$. A one-parameter family of linear bounded operators $C_\alpha(t)$, $t \geq 0$, is called an *α times integrated cosine operator function* if

1. $2\Gamma(\alpha)C_\alpha(t)C_\alpha(s) = \int_t^{t+s}(t+s-r)^{\alpha-1}C_\alpha(r)\,dr - \int_0^s(t+s-r)^{\alpha-1}C_\alpha(r)\,dr + \int_{t-s}^t(r-t+s)^{\alpha-1} \times C_\alpha(r)\,dr + \int_0^s(r+t-s)^{\alpha-1}C_\alpha(r)\,dr$, $t > s > 0$, where $\Gamma(\cdot)$ is the Euler gamma function.
2. $C_\alpha(0) = 0$.
3. For each $x \in E$, the function $C_\alpha(t)x$ is continuous in $t \geq 0$.
4. There exist constants $M > 0$ and $\omega \geq 0$ such that
$$\big\|C_\alpha(t)\big\| \leq Me^{\omega t}, \quad t \geq 0.$$

The generator A of the integrated cosine operator function $C_\alpha(t)$ is defined as follows: $D(A)$ the set of elements $x \in E$ such that there exists an element $y \in E$ satisfying the equality

$$C_\alpha(t)x - \frac{t^\alpha}{\Gamma(\alpha+1)}x = \int_0^t (t-r)C_\alpha(r)y\,dr, \quad t \geq 0, \tag{5}$$

and in this case we set $Ax = y$.

A criterion for the operator A to be the generator of an integrated cosine operator function $C_\alpha(t)$ is that its resolvent $R(\lambda^2, A)$ has the estimate (e.g., see Theorem 2.2.5 in [21])

$$\left\|\frac{d^n}{d\lambda^n}\big(\lambda^{1-\alpha}R(\lambda^2, A)\big)\right\| \leq \frac{Mn!}{(\lambda-\omega)^{n+1}}, \quad \lambda > \omega, \quad n \in \mathbb{N}.$$

Let $P_\nu(t)$ be the spherical Legendre function (see [22, p. 205]). The formulas connecting the Bessel operator function $Y_k(t)$ and the Struve operator function $L_k(t)$ with the integrated cosine operator function $C_\alpha(t)$ are contained in the following two theorems.

Theorem 1 [17]. *Let $k = 2\alpha > 0$, let an operator A be the generator of an α times integrated cosine operator function $C_\alpha(t)$, and let $u_0 \in D(A)$. Then problem (1), (2) is uniformly well posed; i.e., $A \in G_k$, and the corresponding Bessel operator function is representable in the form*

$$Y_k(t)u_0 = \frac{2^\alpha \Gamma(\alpha+1/2)}{\sqrt{\pi}t^\alpha}\left(C_\alpha(t)u_0 - \int_0^1 P'_{\alpha-1}(\tau)C_\alpha(t\tau)u_0\,d\tau\right). \tag{6}$$

It follows from Theorem 1 that the function $u(t) = Y_k(t)u_0$ defined by relation (6) is the only solution of the Cauchy problem (1), (2).

Theorem 2 [17]. *Let $u_1 \in D(A)$, let $k = 2\alpha > 0$, and let an operator A be the generator of an α times integrated cosine operator function $C_\alpha(t)$. Then the function $u(t) = L_k(t)u_1$, where*

$$L_k(t)u_1 = \frac{2^\alpha\ \Gamma(\alpha+1)}{t^{\alpha-1}}\int_0^1 P_{\alpha-1}(\tau)C_\alpha(t\tau)u_1\,d\tau, \tag{7}$$

is a solution of problem (3), (4).

Note also that the condition for the existence of the operator functions $Y_k(t)$ and $L_k(t)$ in relations (6) and (7) is the condition $A \in G_k$ (see [17, 23]).

Example 1. If the operator A is the operator of multiplication by a number, then

$$Y_k(t) = \Gamma(k/2+1/2)\sum_{j=0}^{\infty}\frac{(t^2A/4)^j}{j!\,\Gamma(j+k/2+1/2)} = \Gamma(k/2+1/2)\left(t\sqrt{A}/2\right)^{1/2-k/2} I_{k/2-1/2}\left(t\sqrt{A}\right),$$

$$L_k(t) = \frac{\sqrt{\pi}}{2}\Gamma(k/2+1)\sum_{j=0}^{\infty}\frac{t(t^2A/4)^j}{\Gamma(j+3/2)\Gamma(j+k/2+1)} = \frac{2^{k/2-1/2}\sqrt{\pi}\Gamma(k/2+1)}{A^{k/4+1/4}t^{k/2-1/2}}\mathbf{L}_{k/2-1/2}\left(t\sqrt{A}\right),$$

where $I_\nu(z)$ is the modified Bessel function and $\mathbf{L}_\nu(z)$ is the modified Struve function [24, p. 655]. That is why the operator function $Y_k(t)$ was called the Bessel operator function and the operator function $L_k(t)$ was called the Struve operator function.

In the present paper, we consider an Euler–Poisson–Darboux functional-differential equation of the form

$$u''(t) + \frac{2(\mu+\nu)+1}{t}\big(u'(t)-u'(0)\big) + \frac{4\mu\nu}{t^2}\big(u(t)-u(0)-tu'(0)\big) = Au(t), \quad t>0. \tag{8}$$

Following [21, 22], the functional-differential equation (8) generalizing the Euler–Poisson–Darboux and Bessel–Struve equations can also be called a *weakly loaded Euler–Poisson–Darboux equation*. The interest in the study of loaded differential equations is explained by the scope of their applications and the fact that loaded equations constitute a special class of functional-differential equations with their own specific problems. A survey of publications on loaded differential equations can be found in the monographs [25, 26].

1. CAUCHY PROBLEM

Next, we consider the initial value problem and find a solution of the singular functional-differential equation (8) with the initial conditions

$$u(0) = u_0, \quad u'(0) = u_1. \tag{9}$$

If $\nu = 0$, $\mu \geq -1/2$, and $A \in G_{2\mu+1}$, then Eq. (8) turns into the Bessel–Struve equation, and by virtue of Theorems 1 and 2, the only solution of problem (8), (9) is the function

$$u_{\mu,0}(t) = Y_{2\mu+1}(t)u_0 + L_{2\mu+1}(t)u_1. \tag{10}$$

Now let $\nu > 0$ and $\mu \geq -1/2$. In this case, we will seek a solution of problem (8), (9) in the form of the Erdelyi–Kober type integral

$$u(t) = \int_0^1 s(1-s^2)^{\nu-1} U(ts)u_2\, ds, \tag{11}$$

where $U(ts)$ and u_2 are the twice differentiable operator function and the initial element to be determined.

Let us calculate the derivatives of the function $u(t)$ defined by relation (11) to obtain, after integrating by parts,

$$\begin{aligned} u'(t) &= \int_0^1 s^2(1-s^2)^{\nu-1} U'(ts)u_2\, ds = -\frac{1}{2\nu}\int_0^1 \frac{d}{ds}(1-s^2)^{\nu} sU'(ts)u_2\, ds \\ &= \frac{1}{2\nu}\int_0^1 (1-s^2)^{\nu}\big(tsU''(ts)u_2 + U'(ts)u_2\big)\, ds, \end{aligned} \tag{12}$$

$$u''(t) = \int_0^1 s^3(1-s^2)^{\nu-1} U''(ts)u_2\, ds. \tag{13}$$

Assume that the function $U(t)u_2$ satisfies the relation

$$AU(t)u_2 = U''(t)u_2 + \frac{2\mu+1}{t}\big(U'(t) - U'(0)\big)u_2; \tag{14}$$

then, taking into account (12)–(14) and the elementary integral (2.2.4.8) in [27], after integration by parts we have

$$\begin{aligned} &u''(t) + \frac{2(\mu+\nu)+1}{t}\big(u'(t)-u'(0)\big) + \frac{4\mu\nu}{t^2}\big(u(t)-u(0)-tu'(0)\big) - Au(t) \\ &= \int_0^1 s(1-s^2)^{\nu-1}\left(\frac{2(\mu+\nu)+1}{2\nu}(1-s^2) + s^2 - 1\right)U''(ts)u_2\, ds \\ &\quad + \frac{1}{t}\int_0^1 (1-s^2)^{\nu-1}\left(\frac{2(\mu+\nu)+1}{2\nu}(1-s^2) - 2\mu - 1\right)U'(ts)u_2\, ds \\ &\quad + \frac{2\mu+1}{t}\int_0^1 (1-s^2)^{\nu-1}U'(0)u_2\, ds - \frac{2(\mu+\nu)+1}{t}\int_0^1 s^2(1-s^2)^{\nu-1}U'(0)u_2\, ds \\ &\quad - \frac{4\mu\nu}{t}\int_0^1 s^2(1-s^2)^{\nu-1}U'(0)u_2\, ds + \frac{4\mu\nu}{t^2}\int_0^1 s(1-s^2)^{\nu-1}U(ts)u_2\, ds \end{aligned}$$

$$- \frac{4\mu\nu}{t^2} \int_0^1 s(1-s^2)^{\nu-1} U(0) u_2 \, ds$$

$$= \frac{2\mu+1}{2\nu t} \int_0^1 s(1-s^2)^{\nu} \frac{d}{ds} U'(ts) u_2 \, ds$$

$$+ \frac{1}{t} \int_0^1 (1-s^2)^{\nu-1} \left(\frac{2(\mu+\nu)+1}{2\nu} (1-s^2) - 2\mu - 1 \right) U'(ts) u_2 \, ds$$

$$+ \frac{4\mu\nu}{t^2} \int_0^1 s(1-s^2)^{\nu-1} U(ts) u_2 \, ds - \frac{2\mu}{t^2} U(0) u_2$$

$$= -\frac{2\mu}{t} \int_0^1 (1-s^2)^{\nu} U'(ts) u_2 \, ds + \frac{4\mu\nu}{t^2} \int_0^1 s(1-s^2)^{\nu-1} U(ts) u_2 \, ds - \frac{2\mu}{t^2} U(0) u_2$$

$$= \frac{2\mu}{t^2} U(0) u_2 - \frac{4\mu\nu}{t^2} \int_0^1 s(1-s^2)^{\nu-1} U(ts) u_2 \, ds$$

$$+ \frac{4\mu\nu}{t^2} \int_0^1 s(1-s^2)^{\nu-1} U(ts) u_2 \, ds - \frac{2\mu}{t^2} U(0) u_2 = 0.$$

Therefore, if relation (14) is satisfied, then the function $u(t)$ defined by relation (11) is a solution of Eq. (8).

As follows from Theorem 1, the function $Y_{2\mu+1}(t)u_2$ satisfies relation (14), and if we take $U(t) = Y_{2\mu+1}(t)$ and $u_2 = 2\nu u_0$, then the function $u(t) = Y_{2\mu+1}(t)u_2$ will obviously satisfy the conditions $u(0) = u_0$ and $u'(0) = 0$.

By Theorem 2, the function $L_{2\mu+1}(t)u_2$ also satisfies relation (14), and if we take

$$U(t) = L_{2\mu+1}(t), \quad u_2 = \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} u_1,$$

then the function $u(t) = L_{2\mu+1}(t)u_2$ will satisfy the conditions $u(0) = 0$ and $u'(0) = u_1$.

Thus, if $\nu > 0$ and $\mu \geq -1/2$, then the solution of problem (8), (9) is the function

$$u_{\mu,\nu}(t) = 2\nu \int_0^1 s(1-s^2)^{\nu-1} Y_{2\mu+1}(ts) u_0 \, ds + \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\,\Gamma(\nu)} \int_0^1 s(1-s^2)^{\nu-1} L_{2\mu+1}(ts) u_1 \, ds. \qquad (15)$$

Example 2. If the operator A is the operator of multiplication by a number, then, taking into account the results in Example 1, the integrals in the expression (15) are calculated (see, respectively, the integrals (2.15.2.5) in [28] and (2.7. 4.1) in [24]) and the solution $u_{\mu,\nu}(t; u_0, u_1)$ of problem (8), (9) takes the form

$$u_{\mu,\nu}(t) = {}_1F_2\left(1; \mu+1, \nu+1; \frac{t^2 A}{4}\right) u_0 + t \, {}_1F_2\left(1; \mu+\frac{3}{2}, \nu+\frac{3}{2}; \frac{t^2 A}{4}\right) u_1, \qquad (16)$$

where ${}_1F_2(\cdot)$ is the hypergeometric function

$${}_1F_2\left(1; \alpha, \beta; \frac{t^2 A}{4}\right) = \sum_{j=0}^{\infty} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(j+\alpha)\Gamma(j+\beta)} \left(\frac{t^2 A}{4}\right)^j.$$

Naturally, the definition of a solution in the form of hypergeometric series according to formula (16) also holds for any bounded operator A acting in E. We also point out that earlier results on the solvability of some integro-differential equations for other values of the parameters of hypergeometric series were found in the paper [29].

The above reasoning leads to the following assertion.

Theorem 3. *Let $\nu > 0$ and $\mu \geq -1/2$, let $u_0, u_1 \in D(A)$, and let $A \in G_{2\mu+1}$. Then the function $u_{\mu,\nu}(t)$ defined by relation (15) is the only solution of problem (8), (9).*

Proof. To prove the theorem, all that remains is to prove the uniqueness of the solution of problem (8), (9), which we will establish by contradiction. Let $u_1(t)$ and $u_2(t)$ be two solutions of the problem. Consider the function of two variables

$$w(t,s) = f\Big(Y_{2\mu+1}(s)\big(u_1(t) - u_2(t)\big)\Big),$$

where $f \in E^*$ (E^* is the dual space) and $t, s \geq 0$. Obviously, this function satisfies the equation

$$\frac{\partial^2 w(t,s)}{\partial t^2} + \frac{2(\mu+\nu)+1}{t}\frac{\partial w(t,s)}{\partial t} + \frac{4\mu\nu}{t^2} w(t,s) = \frac{\partial^2 w(t,s)}{\partial s^2} + \frac{2\mu+1}{s}\frac{\partial w(t,s)}{\partial s}, \quad t, s > 0, \tag{17}$$

and the conditions

$$\lim_{t\to 0} w(t,s) = \lim_{t\to 0}\frac{\partial w(t,s)}{\partial t} = \lim_{s\to 0}\frac{\partial w(t,s)}{\partial s} = 0. \tag{18}$$

Just as was done in [30], by $w(t,s)$ we understand a distribution of tempered growth and apply the Fourier–Bessel transform over the variable s,

$$\hat{w}(t,\lambda) = \int_0^\infty s^{2\mu+1}\, j_\mu(\lambda s) w(t,s)\, ds,$$

$$w(t,s) = \gamma_\mu \int_0^\infty \lambda^{2\mu+1} j_\mu(\lambda s)\hat{w}(t,\lambda)\, d\lambda,$$

$$\gamma_\mu = \frac{1}{2^{2\mu}\Gamma^2(\mu+1)}, \quad j_\mu(s) = \frac{2^\mu \Gamma(\mu+1)}{s^\mu} J_\mu(s),$$

where $J_\mu(\cdot)$ is the Bessel function.

From (17) and (18), for the transform $\hat{w}(t,\lambda)$ we obtain the problem

$$\frac{\partial^2 \hat{w}(t,\lambda)}{\partial t^2} + \frac{2(\mu+\nu)+1}{t}\frac{\partial \hat{w}(t,\lambda)}{\partial t} + \frac{4\mu\nu}{t^2}\hat{w}(t,\lambda) = -\lambda^2\, \hat{w}(t,\lambda), \quad t > 0, \tag{19}$$

$$\lim_{t\to 0}\hat{w}(t,\lambda) = \lim_{t\to 0}\frac{\partial \hat{w}(t,\lambda)}{\partial t} = 0. \tag{20}$$

By virtue of Example 2, the general solution of the differential equation (19) has the form

$$\hat{w}(t,\lambda) = c_1(\lambda)\, {}_1F_2\left(1; \mu+1, \nu+1; -\frac{t^2\lambda^2}{4}\right) + c_2(\lambda) t\, {}_1F_2\left(1; \mu+\frac{3}{2}, \nu+\frac{3}{2}; -\frac{t^2\lambda^2}{4}\right),$$

and the initial conditions (20) obviously imply the equalities $c_1(\lambda) = c_2(\lambda) = 0$. Consequently, $\hat{w}(t,\lambda) = w(t,s) = 0$ for each $s \geq 0$. In view of the arbitrariness of the functional $f \in E^*$ for $s = 0$, we obtain the relation $u_1(t) \equiv u_2(t)$, therewith establishing the uniqueness of the solution of the problem under consideration. The proof of the theorem is complete.

The Struve operator function $L_{2\mu+1}(t)$ defined by relation (7) is expressed via the Bessel operator function $Y_{2\mu+2}(t)$ according to the formula (see [17])

$$L_{2\mu+1}(t)u_1 = \int_0^t \frac{\xi}{\sqrt{t^2-\xi^2}} Y_{2\mu+2}(\xi)u_1\, d\xi;$$

therefore, the second term in the representation (15) can be brought to the form

$$\begin{aligned}
&\frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \int_0^1 s(1-s^2)^{\nu-1} L_{2\mu+1}(ts)u_1\, ds \\
&\quad = \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \int_0^1 s(1-s^2)^{\nu-1} \int_0^{ts} \frac{\xi}{\sqrt{t^2s^2-\xi^2}} Y_{2\mu+2}(\xi)u_1\, d\xi\, ds \\
&\quad = \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \int_0^t \xi Y_{2\mu+2}(\xi)u_1 \int_{\xi/t}^1 \frac{s(1-s^2)^{\nu-1}\, ds}{\sqrt{t^2s^2-\xi^2}}\, d\xi \\
&\quad = \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)t^{2\nu}} \int_0^t \xi Y_{2\mu+2}(\xi)u_1 \int_{\xi^2}^{t^2} \frac{(t^2-\eta)^{\nu-1}\, d\eta}{\sqrt{\eta-\xi^2}}\, d\xi \\
&\quad = \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)t^{2\nu}} \frac{\sqrt{\pi}\Gamma(\nu)}{\Gamma(\nu+1/2)} \int_0^t \xi(t^2-\xi^2)^{\nu-1/2} Y_{2\mu+2}(\xi)u_1\, d\xi \\
&\quad = (2\nu+1)t \int_0^1 s(1-s^2)^{\nu-1/2} Y_{2\mu+2}(ts)u_1\, ds,
\end{aligned}$$

by virtue of which we have the following assertion.

Corollary 1. *Under the assumptions of Theorem 3, the solution $u_{\mu,\nu}(t)$ defined by relation (15) can be written in the form*

$$u_{\mu,\nu}(t) = 2\nu \int_0^1 s(1-s^2)^{\nu-1} Y_{2\mu+1}(ts)u_0\, ds + (2\nu+1)t \int_0^1 s(1-s^2)^{\nu-1/2} Y_{2\mu+2}(ts)u_1\, ds. \tag{21}$$

Since the parameters μ and ν occur in Eq. (8) symmetrically, naturally, the following two statements are true as well.

Theorem 4. *Let $\mu > 0$ and $\nu \geq -1/2$, let $u_0, u_1 \in D(A)$, and let $A \in G_{2\nu+1}$. Then the function $u_{\mu,\nu}(t)$ defined by the relation*

$$u_{\mu,\nu}(t) = 2\mu \int_0^1 s(1-s^2)^{\mu-1} Y_{2\nu+1}(ts)u_0\, ds + \frac{4\Gamma(\mu+3/2)}{\sqrt{\pi}\Gamma(\mu)} \int_0^1 s(1-s^2)^{\mu-1} L_{2\nu+1}(ts)u_1\, ds$$

is the unique solution of problem (8), (9).

Corollary 2. *Under the assumptions of Theorem 4, the solution $u_{\mu,\nu}(t)$ can be written in the form*

$$u_{\mu,\nu}(t) = 2\mu \int_0^1 s(1-s^2)^{\mu-1} Y_{2\nu+1}(ts)u_0\, ds + (2\mu+1)t \int_0^1 s(1-s^2)^{\mu-1/2} Y_{2\nu+2}(ts)u_1\, ds.$$

If the assumptions of Theorems 3 and 4 are simultaneously satisfied and it is necessary to relax the requirements on the operator A, then one should choose a theorem that uses operator functions with a larger index, since $G_k \subset G_m$ for $k < m$ (see [1, 16]).

Let us give examples of equations for which integrals in the representation (15) can be calculated.

Example 3. In Eq. (8), let $\mu > 0$ and $\nu = 1/2$, and let A be the operator of multiplication by a number $A \neq 0$. Then, according to formula (16), the only solution of the problem

$$u''(t) + \frac{2\mu+2}{t}\big(u'(t) - u'(0)\big) + \frac{2\mu}{t^2}\big(u(t) - u(0) - tu'(0)\big) = Au(t), \quad t > 0, \tag{22}$$

$$u(0) = u_0, \quad u'(0) = u_1, \tag{23}$$

by virtue of relations (7.14.1.11) and (7.14.1.12) in [24], is the function

$$\begin{aligned} u_{\mu,1/2}(t) &= {}_1F_2\left(1; \frac{3}{2}, \mu+1; \frac{t^2 A}{4}\right) u_0 + t\,{}_1F_2\left(1; 2, \mu + \frac{3}{2}; \frac{t^2 A}{4}\right) u_1 \\ &= \frac{\sqrt{\pi}}{2}\Gamma(\mu+1)\left(\frac{t\sqrt{A}}{2}\right)^{-\mu-1/2} \mathbf{L}_{\mu-1/2}\left(t\sqrt{A}\right) u_0 \\ &\qquad + \frac{4\mu+2}{tA}\left(\Gamma\left(\mu+\frac{1}{2}\right)\left(\frac{t\sqrt{A}}{2}\right)^{1/2-\mu} I_{\mu-1/2}\left(t\sqrt{A}\right) - 1\right) u_1 \\ &= \frac{1}{t} L_{2\mu}(t) u_0 + \frac{4\mu+2}{tA}\big(Y_{2\mu}(t) - 1\big) u_1. \end{aligned}$$

If an unbounded operator $A \in G_{2\mu}$ has an inverse, then the representation of the solution of problem (22), (23) in the form

$$u_{\mu,1/2}(t) = \frac{1}{t} L_{2\mu}(t) u_0 + \frac{4\mu+2}{t}\big(Y_{2\mu}(t) - I\big) A^{-1} u_1$$

is retained, as easily shown by a straightforward verification.

In particular, if the operator A is the generator of a cosine operator function $C(t)$, then, taking into account the examples of operator functions given in [17], for $\mu = 1$ we obtain

$$u_{1,1/2}(t) = \frac{1}{t^2}\big(C(t) - I\big) A^{-1} u_0 + \frac{6}{t^2}\big(S(t) - tI\big) A^{-1} u_1,$$

where $S(t) = C_1(t)$ is the sine operator function.

Theorem 5. *Let the conditions in Theorem 3 be satisfied. Then, uniformly in $t \in [0, T]$, $T > 0$, one has the limit relation*

$$\lim_{\nu \to +0} u_{\mu,\nu}(t) = u_{\mu,0}(t), \tag{24}$$

where $u_{\mu,0}(t)$ is defined by relation (10).

Proof. Taking into account the representations (15) and (10), for each $\delta > 0$ we have

$$\begin{aligned} u_{\mu,\nu}(t) - u_{\mu,0}(t) &= 2\nu \left(\int\limits_0^{1-\delta/T} + \int\limits_{1-\delta/T}^1 \right) s(1-s^2)^{\nu-1}\big(Y_{2\mu+1}(ts) - Y_{2\mu+1}(t)\big) u_0\, ds \\ &\quad + \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \left(\int\limits_0^{1-\delta/T} + \int\limits_{1-\delta/T}^1 \right) s(1-s^2)^{\nu-1}\big(L_{2\mu+1}(ts) - L_{2\mu+1}(t)\big) u_1\, ds. \end{aligned}$$

Owing to the strong continuity of the operator functions $Y_{2\mu+1}(t)$ and $L_{2\mu+1}(t)$, for any $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$\Big\|\big(Y_{2\mu+1}(ts) - Y_{2\mu+1}(t)\big)u_0\Big\| + \Big\|\big(L_{2\mu+1}(ts) - L_{2\mu+1}(t)\big)u_1\Big\| < \varepsilon$$

as long as $|s-1| < \delta/T$. Let us fix such a $\delta > 0$. In addition, we also suppose that

$$M(T) = \sup_{[0,T]}\Big(\big\|Y_{2\mu+1}(t)u_0\big\| + \big\|L_{2\mu+1}(t)u_1\big\|\Big).$$

Then after obvious estimates we obtain

$$\begin{aligned}
&\|u_{\mu,\nu}(t) - u_{\mu,0}(t)\| \\
&\quad\le 2\nu\left(1 + \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu+1)}\right)\left(2M(T)\int\limits_0^{1-\delta/T} s(1-s^2)^{\nu-1}\,ds + 2\varepsilon\int\limits_{1-\delta/T}^{1} s(1-s^2)^{\nu-1}\,ds\right) \\
&\quad= \left(1 + \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu+1)}\right)\left(2M(T)\left(1-\left(1-\frac{\delta^2}{T^2}\right)^{\nu}\right) + 2\varepsilon\right).
\end{aligned}$$

Since the term

$$2M(T)\left(1 + \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu+1)}\right)\left(1-\left(1-\frac{\delta^2}{T^2}\right)^{\nu}\right)$$

tends to zero as $\nu \to +0$, while the term

$$2\varepsilon\left(1 + \frac{2\Gamma(\nu+3/2)}{\sqrt{\pi}\,\Gamma(\nu+1)}\right)$$

can be made smaller than an arbitrary number $\varepsilon_1 > 0$, this implies the validity of the limit relation (24). In particular, if the operator A is bounded, then, taking into account Example 2, it takes the form

$$\begin{aligned}
\lim_{\nu\to+0} u_{\mu,\nu}(t) &= \lim_{\nu\to+0}\left({}_1F_2\left(1;\mu+1,\nu+1;\frac{t^2A}{4}\right)u_0 + t\,{}_1F_2\left(1;\mu+\frac{3}{2},\nu+\frac{3}{2};\frac{t^2A}{4}\right)u_1\right) \\
&= {}_1F_2\left(1;\mu+1,1;\frac{t^2A}{4}\right)u_0 + t\,{}_1F_2\left(1;\mu+\frac{3}{2},\frac{3}{2};\frac{t^2A}{4}\right)u_1 \\
&= \Gamma(\mu+1)\left(t\sqrt{A}/2\right)^{-\mu} I_\mu\left(t\sqrt{A}\right)u_0 + \frac{2^\mu\sqrt{\pi}\Gamma(\mu+3/2)}{A^{\mu/2+1/2}t^\mu}\mathbf{L}_\mu\left(t\sqrt{A}\right) \\
&= Y_{2\mu+1}(t)u_0 + L_{2\mu+1}(t)u_1 = u_{\mu,0}(t).
\end{aligned}$$

The proof of the theorem is complete.

Consider the case of $\nu < 0$, which is not covered by Theorem 3. If $\mu \ge \nu - 1/2$, then a straightforward verification shows that the function

$$u_{\mu,\nu}(t) = t^{-2\nu}Y_{2\mu-2\nu+1}(t)u_0 + \frac{1}{2}t^{-2\nu}L_{2\mu-2\nu+1}(t)u_1 + \frac{1}{2}t^{1-2\nu}u_1 \tag{25}$$

satisfies Eq. (8) and the conditions

$$\begin{aligned}
\lim_{t\to0} t^{2\nu}u(t) &= u_0, \\
\lim_{t\to0}\big(t^{2\nu}u(t)\big)' &= u_1.
\end{aligned} \tag{26}$$

Indeed, having calculated the derivatives of the function $u_{\mu,\nu}(t)$ defined by relation (25), we obtain

$$\begin{aligned} u'_{\mu,\nu}(t) &= -2\nu t^{-2\nu-1} Y_{2\mu-2\nu+1}(t)u_0 + t^{-2\nu} Y'_{2\mu-2\nu+1}(t)u_0 - \\ &\qquad - \nu t^{-2\nu-1} L_{2\mu-2\nu+1}(t)u_1 + \frac{1}{2} t^{-2\nu} L'_{2\mu-2\nu+1}(t)u_1 + \frac{1}{2}(1-2\nu)t^{-2\nu}, \\ u''_{\mu,\nu}(t) &= -2\nu(-2\nu-1)t^{-2\nu-2} Y_{2\mu-2\nu+1}(t)u_0 - 4\nu t^{-2\nu-1} Y'_{2\mu-2\nu+1}(t)u_1 \\ &\qquad + t^{-2\nu} Y''_{2\mu-2\nu+1}(t)u_0 - \nu(-2\nu-1)t^{-2\nu-2} L_{2\mu-2\nu+1}(t)u_1 \\ &\qquad\quad - 2\nu t^{-2\nu-1} L'_{2\mu-2\nu+1}(t)u_1 + \frac{1}{2} t^{-2\nu} L''_{2\mu-2\nu+1}(t)u_1 - \nu(1-2\nu)t^{-2\nu-1}, \end{aligned}$$

$$\begin{aligned} u''_{\mu,\nu}(t) &+ \frac{2(\mu+\nu)+1}{t} u'_{\mu,\nu}(t) + \frac{4\mu\nu}{t^2} u_{\mu,\nu}(t) \\ &= t^{-2\nu}\left(Y''_{2\mu-2\nu+1}(t)u_0 + \frac{2(\mu-\nu)+1}{t} Y'_{2\mu-2\nu+1}(t)u_0 \right) \\ &\qquad + \frac{1}{2} t^{-2\nu}\left(L''_{2\mu-2\nu+1}(t)u_1 + \frac{2(\mu-\nu)+1}{t} L'_{2\mu-2\nu+1}(t)u_1 \right) \\ &= t^{-2\nu} A Y_{2\mu-2\nu+1}(t)u_0 + \frac{1}{2} t^{-2\nu}\left(A L_{2\mu-2\nu+1}(t)u_1 + \frac{2(\mu-\nu)+1}{t} u_1 \right) \\ &\qquad - \frac{2(\mu-\nu)+1}{2t^{2\nu+1}} u_1 = A u_{\mu,\nu}(t), \quad A \in G_{2\mu-2\nu+1}. \end{aligned}$$

Here it was taken into account that the function $v(t) = t^{1-2\nu}$ satisfies the inhomogeneous differential equation

$$v''(t) + \frac{2(\mu+\nu)+1}{t} v'(t) + \frac{4\mu\nu}{t^2} v(t) = (2\mu-2\nu+1)t^{-2\nu-1} u_1.$$

Note also that the representation of the solution in the form (25) was found by substituting the operator function $U(t) = t^{-2\mu} Y_{1-2\mu}(t)$, $\mu < 0$, into relation (11).

The validity of the initial conditions (26) obviously follows from the properties of the operator functions $Y_{2\mu-2\nu+1}(t)$ and $L_{2\mu-2\nu+1}(t)$; thereby, this completes the verification of the assertion.

Thus, the following two theorems are valid, in which the statement about the uniqueness of the solution is established by contradiction, just as in Theorem 3.

Theorem 6. *Let $\nu < 0$, $\mu \ge \nu - 1/2$ and $u_0, u_1 \in D(A)$, and let $A \in G_{2\mu-2\nu+1}$. Then the function $u_{\mu,\nu}(t)$ defined by relation (25) is the unique solution of the differential equation*

$$u''(t) + \frac{2(\mu+\nu)+1}{t} u'(t) + \frac{4\mu\nu}{t^2} u(t) = Au(t), \quad t > 0, \tag{27}$$

with conditions (26).

Theorem 7. *Let $\mu < 0$, $\nu \ge \mu - 1/2$, $u_0, u_1 \in D(A)$, and $A \in G_{2\nu-2\mu+1}$. Then the function*

$$u_{\mu,\nu}(t) = t^{-2\mu} Y_{2\nu-2\mu+1}(t)u_0 + \frac{1}{2} t^{-2\mu} L_{2\nu-2\mu+1}(t)u_1 + \frac{1}{2} t^{1-2\mu} u_1$$

is the unique solution of the differential equation (27) with the conditions

$$\lim_{t\to 0} t^{2\mu} u(t) = u_0,$$
$$\lim_{t\to 0} \left(t^{2\mu} u(t)\right)' = u_1.$$

Corollary 3. *If, in addition,* $\nu \le -1/2$ *in the assumptions of Theorem 6 and* $\mu \le -1/2$ *in the assumptions of Theorem 7, then the functions* $u_{\mu,\nu}(t)$ *have the property* $u_{\mu,\nu}(0) = u'_{\mu,\nu}(0) = 0$ *and satisfy not only the differential equation (27) but also the functional-differential equation (8).*

Note that in Theorems 6 and 7, the classical ("nonweighted") formulation of the initial conditions is not suitable for identifying a unique solution, since, for example, the function

$$u(t) = t^2 c, \quad c \in E,$$

satisfies the homogeneous problem

$$u''(t) + \frac{1}{t}u'(t) - \frac{4}{t^2}u(t) = 0, \quad u(0) = u'(0) = 0,$$

for $\mu = -1$ and $\nu = 1$; thus, the unique solvability of the problems under consideration is violated.

Example 4. Let $\mu = -1$ and $\nu = -1/2$ in Eq. (27), and let the operator A be the generator of a cosine operator function $C(t)$. Then, according to formula (25) in Theorem 6, the unique solution of problem (26), (27) is the function

$$u_{-1,-1/2}(t) = tC(t)u_0 + \frac{1}{2}tS(t)u_1 + \frac{1}{2}t^2 u_1.$$

Remark 1. The resolving operators of the Cauchy problem (8), (9) defined in Theorems 3 and 4 represent the cosine operator functions $C(t)$ integrated in a special way.

Note that if the operator A is the generator of an integrated cosine operator function $C_{\mu+\nu+1/2}(t)$, $\mu > -1/2$, and $\mu + \nu + 1/2 > 0$, then the function

$$u(t) = \frac{\Gamma(\mu+\nu+3/2)}{t^{\mu+\nu+1/2}} C_{\mu+\nu+1/2}(t)u_0$$

is a solution of the equation

$$u''(t) + \frac{2(\mu+\nu)+1}{t}\big(u'(t) - u'(0)\big) + \frac{4(\mu+\nu)^2 - 1}{4t^2}\big(u(t) - u(0) - tu'(0)\big) = Au(t), \tag{28}$$

with the conditions

$$u(0) = u_0, \quad u'(0) = 0; \tag{29}$$

this is easily shown by a straightforward verification using relation (5).

For $\nu = \mu + 1/2$, Eqs. (28) and (8) coincide and owing to the statement about the uniqueness in Theorem 3, the solution of problem(28), (29) has the representation

$$u(t) = (2\mu+1)\int_0^1 s(1-s^2)^{\mu-1/2} Y_{2\mu+1}(ts)u_0\,ds = \frac{\Gamma(2\mu+2)}{t^{2\mu+1}} C_{2\mu+1}(t)u_0.$$

Remark 2. The resolving operators of the Cauchy problem (8), (9) for $\mu > 0$ and $\nu = \mu + 1/2$ defined in Theorems 3 and 4 are connected by the relation

$$\begin{aligned} 2\mu\int_0^t \tau^{2\mu+1}\int_0^1 s(1-s^2)^{\mu-1}\ Y_{2\mu+2}(\tau s)\,ds\,d\tau &= 2\mu\int_0^t \xi Y_{2\mu+2}(\xi)\int_\xi^t \tau(\tau^2-\xi^2)^{\mu-1}\,d\tau\,d\xi \\ &= t^{2\mu+2}\int_0^1 s(1-s^2)^{\mu} Y_{2\mu+2}(ts)\,ds. \end{aligned} \tag{30}$$

If the operator A is bounded, then, taking into account Example 2 and the integral (2.22.2.1) in [24], we write relation (30) as

$$\int\limits_0^t \tau^{2\mu+1}\,{}_1F_2\left(1;\mu+1,\mu+\frac{3}{2};\frac{\tau^2 A}{4}\right)d\tau = \frac{t^{2\mu+2}}{2\mu+2}\,{}_1F_2\left(1;\mu+\frac{3}{2},\nu+\frac{3}{2};\frac{t^2 A}{4}\right),\quad \mu>-1. \tag{31}$$

In particular, for $\mu=-1/2$ relation (31) takes the form of the well-known formula for the connection between the cosine operator function $C(t)$ and the Struve operator function $S(t)$,

$$\int\limits_0^t C(\tau)\,d\tau = \int\limits_0^t \cosh\left(\tau\sqrt{A}\right)\,d\tau = \frac{\sinh\left(t\sqrt{A}\right)}{\sqrt{A}} = S(t).$$

2. DIRICHLET PROBLEM

Generally speaking, boundary value problems for Eq. (8) are ill posed in the hyperbolic case, but at present the need to solve ill-posed problems is generally recognized (see the introduction in [31], as well as [32, 33] and an extensive bibliography therein). The monograph [31, Ch. 2] studies the well-posedness of general boundary value problems for a first-order differential operator equation and for an abstract wave equation (the case of $k=0$ in Eq. (1)).

Many ill-posed problems for differential operator equations can be reduced to operator equations of the first kind $Bx=y$, $x,y\in E$, and the main difficulty lies in establishing their solvability. Below, for $A\in G_{2\mu+1}$ in the hyperbolic case, we will solve an operator equation of the first kind and establish the conditions for the well-posedness of the Dirichlet boundary value problem for the functional-differential equation (8).

For $\nu>0$ and $\mu\geq -1/2$, we will seek a solution $u(t)\in C^2([0,1],E)\cap C((0,1],D(A))$ of Eq. (8) on an interval of finite length $t\in(0,1)$ with the boundary conditions

$$u(0)=u_0,\quad u(1)=v_1. \tag{32}$$

As was already noted, problem (8), (32) is ill posed. Let us establish conditions on the operator $A\in G_{2\mu+1}$ and the elements u_0, $v_1\in E$ ensuring the unique solvability of this problem. The case with the parameter $\nu=0$ in Eq. (8) was considered in [34].

It follows from Theorem 3 that the correct setting of the initial conditions for the functional differential equation (8) consists in specifying the initial values (9) at the point $t=0$, with the only solution of problem (8), (9) having the form (15) or (21).

Returning to the Dirichlet problem (8), (32) in question, note that, given the representation (21), we should determine an element $u_1\in D(A)$ from the equation

$$(2\nu+1)\int\limits_0^1 s(1-s^2)^{\nu-1/2}Y_{2\mu+2}(s)u_1\,ds = v_2, \tag{33}$$

where

$$v_2 = v_1 - 2\nu\int\limits_0^1 s(1-s^2)^{\nu-1}Y_{2\mu+1}(s)u_0\,ds. \tag{34}$$

We transform Eq. (33) using the following formula (see [1]) expressing the Bessel operator function $Y_{2\mu+2}(t)$ via the resolvent $R(\lambda)=(\lambda I-A)^{-1}$ of the operator A:

$$Y_{2\mu+2}(t)u_0 = \frac{2^{\mu+1/2}\Gamma(\mu+3/2)}{i\pi t^{\mu+1/2}}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\lambda^{1/2-\mu}I_{\mu+1/2}(t\lambda)R(\lambda^2)u_0\,d\lambda,\quad \sigma>\omega,\quad u_0\in D(A), \tag{35}$$

where λ^2 for $\operatorname{Re}\lambda>\omega\geq 0$ belongs to the resolvent set $\rho(A)$ of the operator A.

Substituting the expression (35) into the left-hand side of (33), after elementary transformations we obtain

$$\begin{aligned}
&(2\nu+1)\int_0^1 s(1-s^2)^{\nu-1/2}Y_{2\mu+2}(s)u_1\,ds \\
&\quad= (2\nu+1)\int_0^1 s(1-s^2)^{\nu-1/2}\frac{2^{\mu+1/2}\Gamma(\mu+3/2)}{i\pi\ s^{\mu+1/2}}\int_{\sigma-i\infty}^{\sigma+i\infty}\xi^{1/2-\mu}I_{\mu+1/2}(s\xi)R(\xi^2)u_1\,d\xi\,ds \\
&\quad= \frac{2^{\mu+1/2}(2\nu+1)\Gamma(\mu+3/2)}{i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\xi^{1/2-\mu}R(\xi^2)u_1\int_0^1 s^{1/2-\mu}(1-s^2)^{\nu-1/2}I_{\mu+1/2}(s\xi)\,ds\,d\xi \\
&\quad= \frac{1}{i\pi}\int_{\sigma-i\infty}^{\sigma+i\infty}\xi\,{}_1F_2\big(1;\mu+3/2,\nu+3/2;\xi^2/4\big)R(\xi^2)u_1\,d\xi,
\end{aligned} \tag{36}$$

where we have used the integral (2.15.2.5) in [28].

In what follows, an important role will be played by the entire function

$$\cosh i_{\mu,\nu}(\lambda) = {}_1F_2\big(1;\mu+3/2,\nu+3/2;\lambda/4\big); \tag{37}$$

applying this function and taking into account the representation (36), we write the operator equation of the first kind (33) in the form

$$Bu_1 \equiv \frac{1}{\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty}\xi\cosh i_{\mu,\nu}(\xi^2)\ R(\xi^2)u_1\,d\xi = v_2. \tag{38}$$

To establish the solvability of Eq. (38), we impose an additional condition on the resolvent of the operator A.

Condition 1. Each zero λ_j, $j\in\mathbb{N}$, defined by relation (37), of the entire function $\cosh i_{\mu,\nu}(\lambda)$ belongs to the resolvent set $\rho(A)$, and there exists a number $d>0$ such that

$$\sup_{j\in\mathbb{N}}\|R(\lambda_j)\|\le d.$$

We will assume that Condition 1 is satisfied. Since each zero λ_j, $j\in\mathbb{N}$, of the function $\cosh i_{\mu,\nu}(\lambda)$ lies in $\rho(A)$, it belongs to $\rho(A)$ together with a circular neighborhood Ω_j of radius $1/d$, whose clockwise boundary we denote by γ_j. Let Υ_0 be a contour on the complex plane consisting of the straight line $\operatorname{Re} z=\sigma_0>\omega$, let Υ_0^2 (a parabola) be the image of Υ_0 under the mapping $w=z^2$ ($z\in\Upsilon_0$, $w\in\Upsilon_0^2$), and let $\Xi=\Upsilon_0^2\bigcup_{j\in\mathbb{N}}\gamma_j$.

Let us take $\lambda_0\in\rho(A)$, $\operatorname{Re}\lambda_0>\sigma>\sigma_0$, and choose $n\in\mathbb{N}$ so that

$$n>(2\mu+\nu+3)/2. \tag{39}$$

Consider the bounded operator

$$Hq=\frac{1}{2\pi i}\int_{\Xi}\frac{R(z)q\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n},\qquad H\colon E\to E. \tag{40}$$

Let us show that the integral in (40) is absolutely convergent under certain conditions. Indeed, owing to the choice of the contour Υ_0^2, the inequality (see [2])

$$\left\|\lambda^{1/2-\mu} R(\lambda^2)\right\| \le \frac{M}{(\operatorname{Re}\lambda-\omega)^{\mu+3/2}}, \quad \operatorname{Re}\lambda > \omega,$$

and the asymptotic behavior of the hypergeometric function ${}_1F_2(a;b_1,b_2;z)$ as $|z|\to\infty$, $|\arg z|<\pi$,

$${}_1F_2(a;b_1,b_2;z) = \frac{\Gamma(b_1)\Gamma(b_2)}{2\sqrt{\pi}\Gamma(a)} z^{(a-b_1-b_2+1/2)/2} e^{2\sqrt{z}}\left(1+O\left(\frac{1}{\sqrt{z}}\right)\right),$$

the integral

$$\int\limits_{\Upsilon_0^2} \frac{R(z)\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n} = 2\int\limits_{\Upsilon_0} \frac{\lambda^{\mu+1/2}\lambda^{1/2-\mu}R(\lambda^2)\,d\lambda}{{}_1F_2(1;\mu+3/2,\nu+3/2;\lambda^2/4)(\lambda^2-\lambda_0)^n}$$

converges absolutely, since, as follows from the constraint (39), one has the inequality $2n-2\mu-\nu-2>1$, which ensures its absolute convergence.

Now consider the integral

$$\frac{1}{2\pi i}\int\limits_{\bigcup\limits_{j\in\mathbb{N}}\gamma_j} \frac{R(z)\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n} \tag{41}$$

over the remaining part of the contour Ξ.

In the general case, we do not know the asymptotics of the zeros λ_j of the function $\cosh i_{\mu,\nu}(\lambda)$; therefore, along with Condition 1, we provide the absolute convergence of the integral in (41) by the following assumption.

Condition 2. For some number n satisfying inequality (39), the series

$$\sum_{j=1}^{\infty}\int\limits_{\gamma_j} \frac{R(z)\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n}$$

converges absolutely.

Example 5. If $\mu=0$ and $\nu=1$ in Eq. (8), then

$$\cosh i_{0,1}(\lambda) = \frac{2\left(\cosh\sqrt{\lambda}-1\right)}{\lambda}, \quad \lambda_j=-4\pi^2 j^2, \quad j\in\mathbb{N}.$$

The series considered in Condition 2 converges absolutely if for some n so does the series

$$\sum_{j=1}^{\infty}\int\limits_{\gamma_j} \frac{R(z)\,dz}{\cosh i_{0,1}(z)(z-\lambda_0)^n} = \sum_{j=1}^{\infty}\int\limits_{\gamma} \frac{(\xi+2\pi ji)^3 R\big((\xi+2\pi ji)^2\big)\,d\xi}{(\cosh\xi-1)\big((\xi+2\pi ji)^2-\lambda_0\big)^n},$$

where γ is the circle of radius $1/d$ centered at the point $z=0$. Owing to Condition 1, the resolvent $R(\cdot)$ is bounded in a circular neighborhood with contour γ; therefore, the order of the integral over $j\to\infty$ is equal to j^{3-2n}; consequently, the series considered in Condition 2 converges absolutely for $n>2$.

Theorem 8. *Let $\nu>0$, $\mu\ge -1/2$, and $A\in G_{2\mu+1}$, and let a number $n\in\mathbb{N}$ be chosen so that inequality (39) holds and Conditions 1 and 2 are satisfied. If $u_0,u_2\in D(A^{n+1})$, then problem (8), (32) has a unique solution.*

Proof. As we have already found out, the proof of the existence of a unique solution of problem (8), (32) boils down to the proof of the existence of the inverse of the bounded operator B defined by relation (38). Let us show that the operator B has an inverse $B^{-1}\colon D(A^n)\to E$.

Let $q\in D(A)$ and $\sigma_0<\sigma<\operatorname{Re}\lambda$. Then, applying the operator H defined by relation (40) to Bq and taking into account Hilbert's identity

$$R(z)R(\xi^2)=\frac{R(z)-R(\xi^2)}{\xi^2-z},$$

we obtain the relation

$$\begin{aligned}HBq&=\frac{1}{2\pi i}\int\limits_{\Xi}\frac{R(z)\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n}\frac{1}{\pi i}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\xi\cosh i_{\mu,\nu}(\xi^2)R(\xi^2)q\,d\xi\\&=\frac{2}{(2\pi i)^2}\int\limits_{\Xi}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\left(\frac{\xi\cosh i_{\mu,\nu}(\xi^2)R(z)q}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n(\xi^2-z)}-\frac{\xi\cosh i_{\mu,\nu}(\xi^2)R(\xi^2)q}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n(\xi^2-z)}\right)d\xi\,dz.\end{aligned}\tag{42}$$

The integral in (42) is absolutely convergent. Changing the order of integration, we have

$$\begin{aligned}HBq={}&\frac{2}{(2\pi i)^2}\int\limits_{\Xi}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\frac{\xi\cosh i_{\mu,\nu}(\xi^2)R(z)q\,d\xi\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n(\xi^2-z)}\\&-\frac{2}{(2\pi i)^2}\int\limits_{\sigma-i\infty}^{\sigma+i\infty}\xi\cosh i_{\mu,\nu}(\xi^2)R(\xi^2)q\int\limits_{\Xi}\frac{dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n(\xi^2-z)}\,d\xi.\end{aligned}\tag{43}$$

If we close the integration contour Υ_0^2 to the left without intersecting $\bigcup_{j\in\mathbb{N}}\gamma_j$, then the inner integral in the second term in (43) vanishes due to the choice of the contour Ξ and the Cauchy theorem for a multiply connected domain. To calculate the integrals in the first term in (43), we use the Cauchy integral formula. Thus, we have the relation

$$\begin{aligned}HBq&=\frac{2}{(2\pi i)^2}\int\limits_{\Xi}\int\limits_{\Upsilon}\frac{\xi\cosh i_{\mu,\nu}(\xi^2)R(z)q\,d\xi\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n(\xi^2-z)}=\frac{2}{(2\pi i)^2}\int\limits_{\Xi}\int\limits_{\Upsilon^2}\frac{\cosh i_{\mu,\nu}(\lambda)R(z)q\,d\lambda\,dz}{\cosh i_{\mu,\nu}(z)(z-\lambda_0)^n(\lambda-z)}\\&=\frac{1}{2\pi i}\int\limits_{\Xi}\frac{R(z)q\,dz}{(z-\lambda_0)^n}=\frac{1}{2\pi i}\int\limits_{\Upsilon_0^2}\frac{R(z)q\,dz}{(z-\lambda_0)^n}=\frac{-1}{(n-1)!}R^{(n-1)}(\lambda_0)q=(-1)^nR^n(\lambda_0)q.\end{aligned}$$

The commuting operators H, B, $R^n(\lambda_0)$ are bounded, and the domain $D(A)$ is dense in E, so the equality $HBq=(-1)^nR^n(\lambda_0)q$ is also true for $q\in E$ and $HB\colon E\to D(A^n)$. It follows that the operator $B^{-1}q=(-1)^n(\lambda_0I-A)^nHq$ for $q\in D(A^n)$ is the inverse of B, $B^{-1}\colon D(A^n)\to E$. Indeed,

$$BB^{-1}q=(-1)^nB(\lambda_0I-A)^nHq=(-1)^nBH(\lambda_0I-A)^nq=R^n(\lambda_0)(\lambda_0I-A)^nq=q,\quad q\in D(A^n),$$
$$B^{-1}Bq=(-1)^n(\lambda_0I-A)^nHBq=(\lambda_0I-A)^nR^n(\lambda_0)q=q,\quad q\in E.$$

Returning to problem (8), (32), we define the initial element $u_1=(-1)^n(\lambda_0I-A)^nHv_2$ belonging to the domain $D(A)$, where v_2 is defined by relation (34), $v_2\in D(A^{n+1})$, the operator H is defined by (40), $\lambda_0\in\rho(A)$, and $\operatorname{Re}\lambda_0>\sigma_0>\omega$. Then, by virtue of the representation (21), the unique solution $u(t)$ of problem (8), (32) has the form

$$u_{\mu,\nu}(t)=2\nu\int\limits_0^1 s(1-s^2)^{\nu-1}Y_{2\mu+1}(ts)u_0\,ds+(2\nu+1)t\int\limits_0^1 s(1-s^2)^{\nu-1/2}Y_{2\mu+2}(ts)u_1\,ds.$$

The proof of the theorem is complete.

3. NEUMANN PROBLEM

Consider another case of regular boundary conditions for the hyperbolic equation (8) (Neumann problem),

$$u'(0) = u_1, \quad u'(1) = w_1. \tag{44}$$

In this case, just as in [34, 35], for the well-posed solvability of the Neumann problem it is necessary that there exists a bounded operator A^{-1}; this will be assumed when studying this problem.

Given the representation (15) and the conditions in (44), we determine the unknown element $u_0 \in D(A)$ from the equation

$$2\nu \int_0^1 s^2(1-s^2)^{\nu-1} Y'_{2\mu+1}(s) u_0\, ds + \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \int_0^1 s^2(1-s^2)^{\nu-1} L'_{2\mu+1}(s) u_1\, ds = w_1. \tag{45}$$

Using the formula for differentiation (see [1])

$$Y'_k(t)u_0 = \frac{t}{k+1} Y_{k+2}(t) A u_0,$$

we rewrite Eq. (45) as

$$\frac{\nu}{\mu+1} \int_0^1 s^3(1-s^2)^{\nu-1} Y_{2\mu+3}(s) A u_0\, ds = u_2, \tag{46}$$

where

$$u_2 = w_1 - \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \int_0^1 s^2(1-s^2)^{\nu-1} L'_{2\mu+1}(s) u_1\, ds. \tag{47}$$

Taking into account the representation (35), after elementary transformations the left-hand side of Eq. (46) takes the form

$$\begin{aligned} &\frac{\nu}{\mu+1} \int_0^1 s^3(1-s^2)^{\nu-1} Y_{2\mu+3}(s) A u_0\, ds \\ &= \frac{\nu}{\mu+1} \int_0^1 s^{2-\mu}(1-s^2)^{\nu-1} \frac{2^{\mu+1}\Gamma(\mu+2)}{i\pi s^{\mu+1}} \int_{\sigma-i\infty}^{\sigma+i\infty} \xi^{-\mu} I_{\mu+1}(s\xi) R(\xi^2) A u_0\, d\xi\, ds \\ &= \frac{2^{\mu+1}\nu\Gamma(\mu+1)}{i\pi} \int_{\sigma-i\infty}^{\sigma+i\infty} \xi^{-\mu} R(\xi^2) A u_0 \int_0^1 s^{2-\mu}(1-s^2)^{\nu-1} I_{\mu+1}(s\xi)\, ds\, d\xi \\ &= \frac{1}{2(\mu+1)\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \xi\, {}_1F_2\left(1; \mu+2, \nu+1; \frac{\xi^2}{4}\right) R(\xi^2) A u_0\, d\xi, \end{aligned} \tag{48}$$

where we have used the integral (2.15.2.5) in [28].

Let us introduce the entire function

$$\psi_{\mu,\nu}(\lambda) = \frac{1}{2(\mu+1)}\, {}_1F_2\left(1; \mu+2, \nu+1; \frac{\lambda}{4}\right) \tag{49}$$

and use it, taking into account the representation (48), to rewrite the operator equation of the first kind (46) in the form

$$B_1 A u_0 \equiv \frac{1}{\pi i} \int\limits_{\sigma-i\infty}^{\sigma+i\infty} \xi \psi_{\mu,\nu}(\xi^2) R(\xi^2) A u_0 \, d\xi = u_3. \tag{50}$$

The proof of the solvability of Eq. (50) for Au_0 is similar to the proof of Theorem 8. Let us state conditions sufficient for this, requiring the existence of the inverse operator A^{-1}.

Condition 3. The number $\theta_0 = 0$, as well as each zero θ_j, $j \in \mathbb{N}$, of the function $\psi_{\mu,\nu}(\lambda)$ defined by relation (49) belongs to the resolvent set $\rho(A)$ of the operator A, and there exists a number $d > 0$ such that

$$\sup_{j\in\mathbb{N}} \left\| R(\theta_j) \right\| \le d.$$

In what follows, we use the previously introduced contours Ξ and γ_j, but, in contrast to Theorem 8, we use the functions $\psi_{\mu,\nu}(\lambda)$ constructed from the zeros θ_j of the function $\psi_{\mu,\nu}(\lambda)$.

Condition 4. For some m satisfying the inequality $m > (2\mu + \nu + 3)/2$, the series

$$\sum_{j=1}^{\infty} \int\limits_{\gamma_j} \frac{R(z)\, dz}{\psi_{\mu,\nu}(z)(z-\lambda_0)^m}$$

is absolutely convergent.

Under Conditions 3 and 4, we introduce the bounded operator

$$H_1 q = \frac{1}{2\pi i} \int\limits_{\Xi} \frac{R(z) q \, dz}{\psi_{\mu,\nu}(z)(z-\lambda_0)^m}, \quad H_1\colon E \to E. \tag{51}$$

The following statement about the solvability of the Neumann problem is true.

Theorem 9. *Let $\nu > 0$, $\mu \ge -1/2$, $A \in G_{2\mu+1}$, and $u_1, w_1 \in D(A^{n+2})$, and let Conditions 3 and 4 be satisfied. Then problem (8), (44) is uniquely solvable, and its solution has the form*

$$u(t) = 2\nu \int\limits_0^1 s(1-s^2)^{\nu-1} Y_{2\mu+1}(ts) u_0 \, ds + \frac{4\Gamma(\nu+3/2)}{\sqrt{\pi}\Gamma(\nu)} \int\limits_0^1 s(1-s^2)^{\nu-1} L_{2\mu+1}(ts) u_1 \, ds,$$

where $u_0 = (-1)^m A^{-1} (\lambda_0 I - A)^m H_1 u_2$ and the element u_2 and the operator H_1 are defined, respectively, by relations (47) and (51).

Example 6. If A is the operator of multiplication by a number $A \ne 0$, then, according to formula (16), the solution of the Cauchy problem has the form

$$u_{\mu,\nu}(t) = {}_1F_2\big(1; \mu+1, \nu+1; t^2 A/4\big) u_0 + t\, {}_1F_2\big(1; \mu+3/2, \nu+3/2; t^2 A/4\big) u_1;$$

therefore, the Dirichlet problem with the conditions $u(0) = u_0$, $u(1) = v_1$ is solvable if

$${}_1F_2\big(1; \mu+3/2, \nu+3/2; A/4\big) \ne 0.$$

Let $\mu = -1/2$, $\nu = 1$, and $u_0 = 0$ in Eq. (8). In this particular case, the solution $u(t)$ of Eq. (8) with the conditions $u(0) = 0$ and $u(1) = v_1$ has the form

$$u(t) = \frac{\left(\sqrt{A} t \cosh\left(t\sqrt{A}\right) - \sinh\left(t\sqrt{A}\right)\right)}{t^2} u_1,$$

$$u_1 = \frac{v_1}{\sqrt{A}\cosh\sqrt{A} - \sinh\sqrt{A}}.$$

Likewise, the solution of the Neumann problem with the conditions $u'(0) = 0$ and $u'(1) = w_1$ has the form

$$u(t) = \frac{1 - \cosh\left(t\sqrt{A}\right) + \sqrt{A}t\sinh\left(t\sqrt{A}\right)}{t^2}u_0,$$

$$u_0 = \frac{w_1}{(2 + A)\cosh\sqrt{A} - 2\sqrt{A}\sinh\sqrt{A} - 2}.$$

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CONFLICT OF INTEREST

The author of this work declares that he has no conflicts of interest.

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