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On Applications of Integral Transforms Composition Method

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Abstract—The purpose of this paper is to address questions related to transmutation theory and applications of the integral transforms composition method (ITCM) for obtaining a general concept of generalized transmutations operators via integral transforms. This method allows us to obtain a wide range of transmutation operators. Classical integral transforms are implied in the ITCM as fundamentals blocks, among them are Hankel, Y, Mellin, Laplace, Fourier, sine- and cosine-Fourier, and some generalized transforms. In [5] the ITCM and transmutations derived by it are applied to get connection formulas for solutions of singular differential equations and more simple nonsingular ones. The main conclusion is the fact that approach via ITCM for constructing transmutations is very important and constructive tool for obtaining connection formulas and explicit representations of solutions to a wide class of singular differential equations, including those with Bessel operators.

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1. INTRODUCTION AND PRELIMINARIES

In order to fix some ideas before dealing with the main aim of this paper, we first state some basic concepts and tools that would be required for representation of our results.

Further we provide definitions and brief information on the differential Bessel operator, the transmutation theory, the special functions, some classes of functions and integral transforms.

1.1. The Differential Bessel Operator

The differential Bessel operator is given by

$$B_\nu = D^2 + \frac{\nu}{x}D, \quad \nu \geq 0, \quad D := \frac{d}{dx}, \tag{1}$$

and its fractional powers $(B_\nu)^\alpha, \alpha \in \mathbb{R}$, have been studied in many papers. However, in the majority of them, they were defined implicitly as a power function multiplication under Hankel transform. In [10] fractional powers of the Bessel operators were derived in the form

$$(B_{\nu,b-}^{-\alpha} f)(x) = \frac{1}{\Gamma(2\alpha)} \int\limits_x^b \left(\frac{y^2-x^2}{2y}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2}, \alpha; 2\alpha; 1-\frac{x^2}{y^2}\right) f(y)dy, \tag{2}$$

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for $f \in C^{[2\alpha]+1}(0,b]$, $b \in (0,+\infty)$, and

$$(B_{\nu,a+}^{-\alpha}f)(x)=\frac{1}{\Gamma(2\alpha)}\int\limits_a^x \left(\frac{y}{x}\right)^{\nu}\left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\nu-1}{2},\alpha;2\alpha;1-\frac{y^2}{x^2}\right)f(y)dy, \tag{3}$$

for $f \in C^{[2\alpha]+1}[a,+\infty)$, $a\in(0,+\infty)$. The operators (2) and (3) are called *the right- and left-sided fractional Bessel integrals*. they are given explicitly in the integral form without using integral transforms in their definitions.

The defined fractional powers of the Bessel differential operator (1) are also generalizations of Riemann–Liouville fractional integrals, as it is easy to derive that for $\nu = 0$

$$(B_{0,b-}^{-\alpha}f)(x) = (I_{b-}^{2\alpha}f)(x), \quad (B_{0,a+}^{-\alpha}f)(x)y = (I_{a+}^{2\alpha}f)(x),$$

where $I_{b-}^{2\alpha}$ and $I_{a+}^{2\alpha}$ are the right-sided and left-sided Riemann–Liouville fractional integrals given respectively by

$$I_{b^-}^{\alpha}[f](\xi) = \frac{1}{\Gamma(\alpha)}\int\limits_{\xi}^{b}(\xi-\lambda)^{\alpha-1}f(\lambda)d\lambda, \quad \alpha>0, \quad \xi\in[a,b). \tag{4}$$

and

$$I_{a^+}^{\alpha}[f](\xi) = \frac{1}{\Gamma(\alpha)}\int\limits_{a}^{\xi}(\xi-\lambda)^{\alpha-1}f(\lambda)d\lambda, \quad \alpha>0, \quad \xi\in(a,b], \tag{5}$$

$I_{a^+}^{\alpha}[f]$ and $I_{b^-}^{\alpha}[f]$ are defined on (a,b) for $f \in L^1(a,b;\mathbb{R})$, cf. [9, 15].

1.2. Transmutation Theory

Let's consider the following second order linear differential

$$\mathcal{L} := -\frac{d^2}{dx^2} + q(x), \tag{6}$$

where q is an L^2-function defined on a finite interval. The next equation is called the one-dimensional Sturm–Liouville equation

$$\mathcal{L}y(x) = \gamma y(x), \quad \gamma\in\mathbb{C}, \tag{7}$$

taking in consideration that Liouville transformation reduces a large variety of linear ordinary second order equations to this form.

An intertwining operator is sought to relate $\mathcal{L}$ to the simplest linear second order expression $\mathcal{B} = -\frac{d^2}{dx^2}$ by the formula $\mathcal{L}T = T\mathcal{B}$.

Let (A,B) be a given pair of operators. An operator T is called transmutation (or intertwining) operator if the following property is valid on elements of some functional spaces

$$TA = BT. \tag{8}$$

1.3. Some Special Functions

Here are some definitions and brief information some special functions and classes of functions.

Let ${}_2\mathcal{F}_1$ be the hypergeometric function defined by the power series

$${}_2\mathcal{F}_1(a,b;c;t) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{t^n}{n!}, \quad \text{where} \quad |t|<1. \tag{9}$$

The Bessel functions of the first and second kind of order ν, respectively J_ν and Y_ν, are defined as follows by series expansions near $t=0$

$$J_\nu(t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\,\Gamma(m+\nu+1)}\left(\frac{t}{2}\right)^{2m+\nu}, \tag{10}$$

and

$$Y_\nu(t)=\frac{\cos\pi\nu J_\nu(t)-J_{-\nu}(t)}{\sin\pi\nu} \tag{11}$$

where $\Re\nu>-1/2$ and is non-integer.

Normalized (or "small") Bessel function of the first kind j_ν is defined by the formula

$$j_\nu(x)=\frac{2^\nu\Gamma(\nu+1)}{x^\nu}\;J_\nu(x), \tag{12}$$

where J_ν is Bessel function of the first kind. Its basic property is $j_\nu(0)=1$.

We denote by $\mathbb{H}_\nu$ the Struve function given by the following power series expansion

$$\mathbb{H}_\nu(t)=\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(m+\frac{3}{2})\,\Gamma(m+\nu+\frac{3}{2})}\left(\frac{t}{2}\right)^{2m+\nu+1}. \tag{13}$$

The best reference on Bessel and connected functions is still [16]

1.4. Integral Transforms

After introducing some special functions we now consider integral transforms, which use the above defined special functions as kernels.

The one-dimentional *Hankel transform* of a function $f\in L_{1,\nu}(\mathbb{R}^1_+)$ is defined as

$$H_\nu[f](\xi)=H_\nu[f(x)](\xi)=\widehat{f}(\xi)=\int\limits_0^\infty f(x)\,j_{\frac{\nu-1}{2}}(x\xi)x^\nu dx, \tag{14}$$

where $\nu>0$, and j_ν is the normalized Bessel function of the first kind (12).

From now on, we assume $f\in\mathbb{S}$, where $\mathbb{S}$ is the space of rapidly decreasing functions on $(0,\infty)$

$$\mathbb{S}=\left\{f\in C^\infty(0,\infty):\sup_{t\in(0,\infty)}\left|t^aD^bf(t)\right|<\infty\quad\forall a,b\in Z_+\right\}.$$

Let the Hankel, $\mathcal{Y}$ and Struve transforms be the integral transforms of order ν of a function f defined as follow

$$\left(\mathcal{H}_\nu f\right)(x)=\int\limits_0^{+\infty}(xt)^{\frac{1}{2}}J_\nu(xt)f(t)dt, \tag{15}$$

$$\left(\mathcal{Y}_\nu f\right)(x)=\int\limits_0^{+\infty}(xt)^{\frac{1}{2}}Y_\nu(xt)f(t)dt, \tag{16}$$

$$\left(\mathcal{S}_\nu f\right)(x)=\int\limits_0^{+\infty}(xt)^{\frac{1}{2}}\mathbb{H}_\nu(xt)f(t)dt. \tag{17}$$

It is known that (15), (16) and (17) are invertible transforms, where $(\mathcal{H}_\nu)^{(-1)}=\mathcal{H}_\nu$, $(\mathcal{Y}_\nu)^{(-1)}=\mathcal{S}_\nu$.

1.5. The Integral Transform Composition Method

What is the integral transform composition method (ITCM), and how does it work? In transmutation theory, explicit operators were derived based on different ideas and methods, often not connecting altogether. Therefore, there is an urgent need in transmutation theory to develop a general method for obtaining known and new classes of transmutations.

In this section, we give such a method for constructing transmutation operators. We call this method Integral Transform Composition Method, or ITCM for short. The method is based on the representation of transmutation operators as compositions of basic integral transforms. The Integral Transform Composition Method (ITCM) gives the algorithm not only for constructing new transmutation operators, but also for all now explicitly known classes of transmutations, including Poisson, Sonine, Vekua–Erdelyi–Lowndes, Buschman–Erdelyi, Sonin–Katrakhov and Poisson–Katrakhov ones, cf. [1–4, 6, 7, 12–15] as well as the classes of elliptic, hyperbolic and parabolic transmutation operators introduced by R. Carroll [1–3]. This method, with many applications and examples, was essentially introduced and developed by S.M. Sitnik.

The formal algorithm of ITCM is next. Let us take as input a pair of arbitrary operators A, B, and also connect with them generalized Fourier transforms F_A, F_B, which are invertible and act by the formulas

$$F_A A = g(t) F_A, \quad F_B B = g(t) F_B, \tag{18}$$

t is a dual variable, g is an arbitrary function with suitable properties. It is often convenient to choose $g(t) = -t^2$ or $g(t) = -t^\alpha$, $\alpha \in \mathbb{R}$. Then, the essence of ITCM is to obtain formally a pair of transmutation operators P and S as the method output by the next formulas

$$S = F_B^{-1} \frac{1}{w(t)} F_A, \quad P = F_A^{-1} w(t) F_B \tag{19}$$

with arbitrary function $w(t)$. When P and S are transmutation operators intertwining A and B

$$SA = BS, \quad PB = AP. \tag{20}$$

A formal checking of (20) can be obtained by direct substitution. The main difficulty is the calculation of compositions (19) in an explicit integral form, as well as the choice of domains of operators P and S.

The main advantages of Integral Transform Composition Method (ITCM) have been listed in [5, 11].

One obstacle to which to apply ITCM is the next one: we know acting of classical integral transforms usually on standard spaces like L_2, L_p, C^k, variable exponent Lebesgue spaces and so on. However, for application of transmutations to differential equations, we usually need more conditions to hold, say, at zero or at infinity. For these problems, we may first construct a transmutation by ITCM and then expand it to the needed functional classes.

Let us stress that formulas of the type (19) of course are not new for integral transforms and its applications to differential equations. *However, ITCM is new when applied to transmutation theory!* In other fields of integral transforms and connected differential equations, theory compositions (19) for the choice of classical Fourier transform leads to famous pseudo-differential operators with symbol function $w(t)$. For the choice of the classical Fourier transform and the function $w(t) = (\pm it)^{-s}$ we get fractional integrals on the whole real axis, while for $w(t) = |x|^{-s}$ we get M. Riesz potential, also for $w(t) = (1+t^2)^{-s}$ in formulas (19) we get Bessel potential and for $w(t) = (1 \pm it)^{-s}$ – modified Bessel potentials [9].

The next choice for algorithm of ITCM

$$A = B = B_\nu, \quad F_A = F_B = H_\nu, \quad g(t) = -t^2, \quad w(t) = j_\nu(st) \tag{21}$$

leads to generalized translation operators of Delsarte [15], for this case we have to choose in the algorithm defined by (18), (19) the above values (21) in which B_ν is the Bessel operator (1), H_ν is the Hankel transform (14), j_ν is the normalized Bessel function (12).

It is possible to apply ITCM instead of classical approaches for getting fractional powers of Bessel operators [14, 15]. Therefore, we may conclude that the method we consider in the paper for obtaining transmutations – ITCM is effective; it is connected to many known methods and problems, it gives all known classes of explicit transmutations and works as a tool to construct new classes of transmutations. Application of ITCM needs the following three steps:

- For a given pair of operators A, B and connected generalized Fourier transforms F_A, F_B define and calculate a pair of transmutations P, S by basic formulas (18), (19).

- Derive exact conditions and find classes of functions for which transmutations obtained by step 1 satisfy proper intertwining properties.

- Apply now correctly defined transmutations by the first an second steps on proper classes of functions to deriving connection formulas for solutions of differential equations.

1.6. Some Previous Results

In this section we collect some of our results on the applications of ITCM from papers [5, 11]. We start by investigating composition operators of the form

$$T = \mathcal{H}_\mu\, \mathcal{Y}_\nu, \quad T = \mathcal{Y}_\mu\, \mathcal{Y}_\nu, \quad T = \mathcal{Y}_\mu\, \mathcal{H}_\nu, \quad T = \mathcal{H}_\mu\, \mathcal{H}_\nu. \tag{22}$$

where $\mathcal{H}_\mu$ and $\mathcal{Y}_\nu$ are the *Hankel* and *$\mathcal{Y}$-transforms* defined by (15) and (16). The operators T defined in (22) commute with the differential operator (23) given by

$$L_\nu = \frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} - \frac{\nu^2}{x^2}, \tag{23}$$

and obeying the transmutation property (8) in the form $T\, L_\nu = L_\mu\, T$. The compositions $\mathcal{H}_\mu\; \mathcal{Y}_\nu$ and $\mathcal{Y}_\mu\; \mathcal{H}_\nu$ are considered as generalizations of Hilbert operators.

For the special case $\nu = \mp\frac{1}{2}$, it is easy to see that the operator $\mathcal{H}_\nu\, \mathcal{Y}_\nu$ equals to Hilbert transform on semi-axes

$$\mathcal{H}_{\frac{1}{2}}\, [\mathcal{Y}_{\frac{1}{2}} f](x) = \frac{2}{\pi} \int\limits_0^\infty \frac{x f(t)}{t^2 - x^2} dt.$$

The norms of $\mathcal{H}_\nu$, $\mathcal{Y}_\nu$ and their compositions in $L_2(0,\infty)$ are also studied in this section and it is shown that $||\mathcal{H}_\nu \mathcal{Y}_\nu||_{L_2} = ||\mathcal{Y}_\nu \mathcal{H}_\nu||_{L_2}$. By applying ITCM of the form $T^{(\varphi)}_{\nu,\,\mu} = F^{-1}_\mu \Big(\varphi(t) F_\nu\Big)$, we obtain an interesting and important family of transmutations including index shift $B-$hyperbolic transmutations, "descent" operators, classical Sonine and Poisson-type transmutations, explicit integral representations for fractional powers of the Bessel operator, generalized translations of Delsarte and other transmutations or "shift operators", such that $T^{(\varphi)}_{\nu,\,\mu} B_\nu = B_\mu T^{(\varphi)}_{\nu,\,\mu}$.

For $\varphi(t) = Ct^\alpha$, $C \in \mathbb{R}$, and $T^{(\varphi)}_{\nu,\,\mu} = T^{(\alpha)}_{\nu,\,\mu}$ we proved in [5, 11] the following integral representation

$$\left(T^{(\alpha)}_{\nu,\,\mu} f\right)(x) = C\, \frac{2^{\alpha+1}\Gamma\left(\frac{\alpha+\mu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)} \left[\frac{x^{-1-\mu-\alpha}}{\Gamma\left(-\frac{\alpha}{2}\right)} \int\limits_0^x f(y)\, {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha}{2}+1; \frac{\nu+1}{2}; \frac{y^2}{x^2}\right) y^\nu dy \right.$$

$$\left. + \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\mu+1}{2}\right)\Gamma\left(\frac{\nu-\mu-\alpha}{2}\right)} \int\limits_x^\infty f(y)\, {}_2F_1\left(\frac{\alpha+\mu+1}{2}, \frac{\alpha+\mu-\nu}{2}+1; \frac{\mu+1}{2}; \frac{x^2}{y^2}\right) y^{\nu-\mu-\alpha-1} dy \right],$$

where ${}_2F_1$ is the Gauss hypergeometric function.

2. GENERALIZATIONS OF FRACTIONAL BESSEL OPERATORS

The next operators constructed by ITCM may be considered as generalized fractional powers of the Bessel operator, as they annihilate integer powers of classical Bessel operators.

Theorem 1. *Let $f \in L^2(0,\infty)$, $x, y > 0$, $Re(s) < 0$ and $Re(s+\nu+1) > 0$. Then, for the transmutation operator T_1 obtained by ITCM in the form $(T_1 f)(x) = \mathcal{H}_\nu\left[(-t^2)^s \mathcal{H}_\nu f\right](x)$ the next integral representation is true*

$$\mathcal{H}_\nu\left[(-t^2)^s \mathcal{H}_\nu f\right](x) = C_{s,\nu}\, x^{-2s-\nu-\frac{3}{2}} \int\limits_0^x {}_2F_1\left(\nu+s+1, s+1; \nu+1; \frac{y^2}{x^2}\right) y^{\nu+\frac{1}{2}}\, f(y)\, dy$$

$$+ C_{s,\nu}\, x^{\nu+\frac{1}{2}} \int\limits_x^{+\infty} {}_2F_1\left(\nu+s+1, s+1; \nu+1; \frac{x^2}{y^2}\right) y^{-2s-\nu-\frac{3}{2}}\, f(y)\, dy, \tag{24}$$

where $C_{s,\nu} = (-1)^s 2^{2s+1} \dfrac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}$ *and* ${}_2F_1$ *is the Gauss hypergeometric function.*

Proof. Applying the ITCM for the Hankel transform given by (15) and the dual variable t such that $(-(d^2))^s = (-t^2)^s$, we have

$$\mathcal{H}_\nu\left[(-t^2)^s \mathcal{H}_\nu f\right](x) = (-1)^s \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} J_\nu(xt) t^{2s} dt \int\limits_0^{+\infty} (yt)^{\frac{1}{2}} J_\nu(yt)\, f(y)\, dy$$

$$= (-1)^s \int\limits_0^{+\infty} (xy)^{\frac{1}{2}} f(y) dy \int\limits_0^{+\infty} t^{2s+1} J_\nu(xt)\, J_\nu(yt)\, dt$$

$$= (-1)^s \int\limits_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, J_\nu(yt)\, dt$$

$$= (-1)^s \int\limits_0^{x} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, J_\nu(yt)\, dt$$

$$+ (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, J_\nu(yt)\, dt.$$

Using ([1], formula 2.12.31, page 209) we derive

$$\int\limits_0^{+\infty} x^{\alpha-1}\, J_\mu(bx)\, J_\nu(cx)\, dx = A^\alpha_{\mu,\nu}, \quad b, c, Re(\alpha+\mu+\nu) > 0, Re(\alpha) < 2. \tag{25}$$

$$A^\alpha_{\mu,\nu} = 2^{\alpha-1} b^{-\nu-\alpha} c^\nu \frac{\Gamma(\frac{\nu+\mu+\alpha}{2})}{\Gamma(\frac{\mu-\nu-\alpha}{2}+1)\Gamma(\nu+1)} {}_2F_1\left(\frac{\nu+\mu+\alpha}{2}, \frac{\nu-\mu+\alpha}{2}; \nu+1; \frac{c^2}{b^2}\right), \quad 0 < c < b,$$

$$A^\alpha_{\mu,\nu} = 2^{\alpha-1} b^{\mu} c^{-\mu-\alpha} \frac{\Gamma(\frac{\alpha+\mu+\nu}{2})}{\Gamma(\frac{\nu-\mu-\alpha}{2}+1)\Gamma(\mu+1)} {}_2F_1\left(\frac{\alpha+\mu+\nu}{2}, \frac{\alpha+\mu-\nu}{2}; \mu+1; \frac{b^2}{c^2}\right), \quad 0 < b < c.$$

Thus, for $\mu = \nu,\ \alpha = 2(s+1),\ b = x,\ c = y$, we get

- for $0 < y < x$,

$$\int\limits_0^{+\infty} t^{2s+1}\, J_\nu(xt)\, J_\nu(yt)\, dt$$

$$= 2^{2s+1} x^{-\nu-2(s+1)} y^\nu \frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)} {}_2F_1\left(\nu+s+1, s+1; \nu+1; \frac{y^2}{x^2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{x}\right)^{2(s+1)}\left(\frac{y}{x}\right)^{\nu}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{y^2}{x^2}\right).$$

- for $0<x<y$,

$$\int\limits_0^{+\infty} t^{2s+1}\,J_\nu(xt)\,J_\nu(yt)\,dt$$

$$= 2^{2s+1}x^{\nu}y^{-\nu-2(s+1)}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{x^2}{y^2}\right)$$

$$= \frac{1}{2}\left(\frac{2}{y}\right)^{2(s+1)}\left(\frac{x}{y}\right)^{\nu}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{x^2}{y^2}\right).$$

Here we have $Re(2(s+1))<2$ for $Re(s)<0$, and $x,\ y,\ Re(2(s+\nu+1))>0$. Hence

$$\mathcal{H}_\nu\left[(-t^2)^s\mathcal{H}_\nu f\right](x)$$

$$= \frac{(-1)^s}{2}\int\limits_0^{x}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\left(\frac{2}{x}\right)^{2(s+1)}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{y^2}{x^2}\right)\,y\,f(y)\,dy$$

$$+ \frac{(-1)^s}{2}\int\limits_x^{+\infty}\left(\frac{x}{y}\right)^{\nu+\frac{1}{2}}\left(\frac{2}{y}\right)^{2(s+1)}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{x^2}{y^2}\right)yf(y)dy$$

$$= \frac{(-1)^s}{2}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\left(\frac{2}{x}\right)^{2(s+1)}\int\limits_0^{x}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{y^2}{x^2}\right)\,y\,f(y)\,dy$$

$$+ \frac{(-1)^s}{2}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}2^{2(s+1)}\int\limits_x^{+\infty}\left(\frac{x}{y}\right)^{\nu+\frac{1}{2}}\,{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{x^2}{y^2}\right)y^{-2s-1}f(y)dy$$

$$= (-1)^s2^{2s+1}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,x^{-2s-\nu-\frac{3}{2}}\int\limits_0^{x}{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{y^2}{x^2}\right)\,y^{\nu+\frac{1}{2}}\,f(y)\,dy$$

$$+ (-1)^s2^{2s+1}\frac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}\,x^{\nu+\frac{1}{2}}\int\limits_x^{+\infty}{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{x^2}{y^2}\right)\,y^{-2s-\nu-\frac{1}{2}}\,f(y)\,dy.$$

Thus,

$$\mathcal{H}_\nu\left[(-t^2)^s\mathcal{H}_\nu f\right](x) = C_{s,\nu}\,x^{-2s-\nu-\frac{3}{2}}\int\limits_0^{x}{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{y^2}{x^2}\right)\,y^{\nu+\frac{1}{2}}\,f(y)\,dy$$

$$+ C_{s,\nu}\,x^{\nu+\frac{1}{2}}\int\limits_x^{+\infty}{}_2F_1\left(\nu+s+1,s+1;\nu+1;\frac{x^2}{y^2}\right)\,y^{-2s-\nu-\frac{3}{2}}\,f(y)\,dy,$$

where $C_{s,\nu}=(-1)^s2^{2s+1}\dfrac{\Gamma(\nu+s+1)}{\Gamma(-s)\Gamma(\nu+1)}$. □

3. OPERATORS COMMUTING WITH THE BESSEL OPERATOR AND GENERALIZATIONS OF HILBERT OPERATORS

In this section we construct some more examples by ITCM method.

Theorem 2. *Let $f \in L^2(0,\infty)$, $x, y > 0$, $Re(s) < 0$, $-Re\,\nu < Re\,2(s+1) < 2$. We have that for an operator $(T_2 f)(x) = \mathcal{H}_\nu\left[(-t^2)^s \mathcal{Y}_\nu f\right](x)$, the next integral representation is true*

$$(T_2 f)(x) = \frac{(-1)^{s+1}}{2\pi}\left(\frac{2}{x}\right)^{2(s+1)} \cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}$$

$$\times \int\limits_0^x \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} {}_2F_1\left(s+1, s+1+\nu; 1+\nu; \frac{y^2}{x^2}\right) y\, f(y)\, dy$$

$$+\frac{(-1)^{s+1}}{2\pi}\left(\frac{2}{x}\right)^{2(s+1)} \frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)} \int\limits_0^x \left(\frac{x}{y}\right)^{\nu+\frac{1}{2}} {}_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{y^2}{x^2}\right) y\, f(y)\, dy$$

$$+\frac{(-1)^s}{2\pi} 2^{2(s+1)} \cos(s\pi)\frac{\Gamma(s+1+\nu)\Gamma(s+1)}{\Gamma(\nu+1)}$$

$$\times \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\nu+\frac{1}{2}} {}_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right) y^{-2s-1}\, f(y)\, dy. \qquad (26)$$

Proof. Applying the ITCM for the Hankel and the $\mathcal{Y}$-transforms given by (15) and (16), and the dual variable t such that $(-(d^2))^s = (-t^2)^s$, we have

$$\mathcal{H}_\nu\left[(-t^2)^s \mathcal{Y}_\nu f\right](x) = \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} J_\nu(xt)\left[(-t^2)^s \int\limits_0^{+\infty} (yt)^{\frac{1}{2}} Y_\nu(yt) f(y) dy\right] dt$$

$$= (-1)^s \int\limits_0^{+\infty} (xt)^{\frac{1}{2}} J_\nu(xt) t^{2s} dt \int\limits_0^{+\infty} (yt)^{\frac{1}{2}} Y_\nu(yt)\, f(y)\, dy$$

$$= (-1)^s \int\limits_0^{+\infty} y^{\frac{1}{2}} f(y) dy \int\limits_0^{+\infty} (x)^{\frac{1}{2}} t^{2s+1} J_\nu(xt)\, Y_\nu(yt)\, dt$$

$$= (-1)^s \int\limits_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\nu(yt)\, dt$$

$$= (-1)^s \int\limits_0^{x} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\nu(yt)\, dt$$

$$+ (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y f(y) dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\nu(yt)\, dt.$$

Using (formula 4. 2.13.15 from [1], p. 272) of the form

- for $0 < b < c$

$$\int\limits_0^{+\infty} x^{\alpha-1}\, J_\mu(bx)\, Y_\nu(cx)\, dx = -\frac{2^{\alpha-1}\, b^\mu}{\pi\, c^{\alpha+\mu}}$$

$$\times \cos\left(\frac{\alpha+\mu-\nu}{2}\pi\right)\frac{\Gamma(\frac{\alpha+\mu+\nu}{2})\Gamma(\frac{\alpha+\mu-\nu}{2})}{\Gamma(\mu+1)}\ {}_2F_1\left(\frac{\alpha+\mu+\nu}{2},\frac{\alpha+\mu-\nu}{2};\mu+1;\frac{b^2}{c^2}\right); \qquad (27)$$

- for $0 < c < b$

$$\int_0^{+\infty} x^{\alpha-1}\, J_\mu(bx)\, Y_\nu(cx)\, dx$$
$$= -\frac{2^{\alpha-1}\, c^\nu}{\pi\, b^{\alpha+\nu}}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(\frac{\alpha+\mu+\nu}{2})}{\Gamma(1-\frac{\alpha+\mu-\nu}{2})}\ {}_2F_1\left(\frac{\alpha-\mu+\nu}{2},\frac{\alpha+\mu+\nu}{2};1+\nu;\frac{c^2}{b^2}\right)$$
$$-\frac{2^{\alpha-1}\, b^{\nu-\alpha}}{\pi c^\nu}\frac{\Gamma(\nu)\Gamma(\frac{\alpha+\mu-\nu}{2})}{\Gamma(1+\frac{\nu+\mu-\alpha}{2})}\ {}_2F_1\left(\frac{\alpha-\mu-\nu}{2},\frac{\alpha+\mu-\nu}{2};1-\nu;\frac{c^2}{b^2}\right); \qquad (28)$$

where $b, c > 0$; $|Re\,\nu| - Re\,\mu < Re\,\alpha$. Thus, for $\alpha = 2(s+1)$, $\mu = \nu$, $x = t$, $b = x$, $c = y$, we obtain

$$\int_0^{+\infty} t^{2s+1}\, J_\nu(xt)\, Y_\nu(yt)\, dt$$
$$= \frac{2^{2s+1}\, x^\nu}{\pi\, y^{2(s+1)+\nu}}\cos(s\pi)\frac{\Gamma(s+1+\nu)\Gamma(s+1)}{\Gamma(\nu+1)}{}_2F_1\left(s+1+\nu, s+1;\nu+1;\frac{x^2}{y^2}\right),$$
$$\text{for } 0 < x < y;$$

and also

$$\int_0^{+\infty} t^{2s+1}\, J_\nu(xt)\, Y_\nu(yt)\, dt$$
$$= -\frac{2^{2s+1}\, y^\nu}{\pi\, x^{2(s+1)+\nu}}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}\ {}_2F_1\left(s+1, s+1+\nu;1+\nu;\frac{y^2}{x^2}\right)$$
$$-\frac{2^{2s+1}\, x^{\nu-2(s+1)}}{\pi y^\nu}\frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)}\ {}_2F_1\left(s+1, s+1-\nu;1-\nu;\frac{y^2}{x^2}\right),$$
$$\text{for } 0 < y < x;$$

$x, y > 0$; $-Re\,\nu < Re\,2(s+1) < 2$, where $Re(s) < 0$. Hence,

$$\mathcal{H}_\nu\left[(-t^2)^s\mathcal{Y}_\nu f\right](x) = (-1)^s\int_0^x\left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy\int_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\nu(yt)\, dt$$
$$+ (-1)^s\int_x^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy\int_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\nu(yt)\, dt = I_1 + I_2,$$

where

$$I_1 = (-1)^s\int_0^x\left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy\int_0^{+\infty} t^{2(s+1)-1}\, J_\nu(xt)\, Y_\nu(yt)\, dt$$
$$= \frac{(-1)^{s+1}}{2\pi}\left(\frac{2}{x}\right)^{2(s+1)}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}$$
$$\times\int_0^x\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\ {}_2F_1\left(s+1, s+1+\nu;1+\nu;\frac{y^2}{x^2}\right)\ y\, f(y)\, dy$$

$$+\frac{(-1)^{s+1}}{2\pi}\left(\frac{2}{x}\right)^{2(s+1)}\frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)}\int\limits_0^x\left(\frac{x}{y}\right)^{\nu+\frac{1}{2}}\ {}_2F_1\left(s+1,s+1-\nu;1-\nu;\frac{y^2}{x^2}\right)\ y\,f(y)\,dy,$$

and at last we get the final formula

$$I_2=(-1)^s\int\limits_x^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}}yf(y)dy\int\limits_0^{+\infty}t^{2(s+1)-1}\ J_\nu(xt)\ Y_\nu(yt)\ dt$$
$$=\frac{(-1)^s}{2\pi}2^{2(s+1)}\cos(s\pi)\frac{\Gamma(s+1+\nu)\Gamma(s+1)}{\Gamma(\nu+1)}$$
$$\times\int\limits_x^{+\infty}\left(\frac{x}{y}\right)^{\nu+\frac{1}{2}}\ {}_2F_1\left(s+1,s+1-\nu;1-\nu;\frac{x^2}{y^2}\right)\ y^{-2s-1}\ f(y)\,dy.$$

□

It is easy to see that for $\nu=\frac{1}{2}$ the operator $\mathcal{H}_\nu\,\mathcal{Y}_\nu$ equals to Hilbert transform on semi-axes

$$[\mathcal{H}_{\frac{1}{2}}\mathcal{Y}_{\frac{1}{2}}f](x)=\frac{2}{\pi}\int\limits_0^{\infty}\frac{xf(y)}{y^2-x^2}dy.$$

Now we calculate another useful composition by ITCM method, it is again a transmutation operator.

Theorem 3. *Let $f\in L^2(0,\infty)$, $x,y>0$; $Re(s)<0$, $-Re\nu<Re\,2(s+1)<2$. Define an operator*

$$T_3(f)=\mathcal{Y}_\nu(-t^2)^s\mathcal{H}_\nu(f). \tag{29}$$

Then, the next integral representation holds true

$$(T_3f)(x)=\frac{(-1)^s}{2\pi}\left(\frac{2}{x}\right)^{2(s+1)}\cos(s\pi)\frac{\Gamma(s+1+\nu)\Gamma(s+1)}{\Gamma(\nu+1)}$$
$$\times\int\limits_0^x\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\ {}_2F_1\left(s+1,s+1+\nu;1+\nu;\frac{y^2}{x^2}\right)\ y\ f(y)\ dy$$
$$+\frac{(-1)^{s+1}2^{2s+1}}{\pi}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}$$
$$\times\int\limits_x^{+\infty}\left(\frac{x}{y}\right)^{\nu+\frac{1}{2}}y^{-2(s+1)}\ {}_2F_1\left(s+1,s+1+\nu;1+\nu;\frac{x^2}{y^2}\right)\ y\ f(y)\ dy$$
$$+\frac{(-1)^{s+1}2^{2s+1}}{\pi}\ \frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)}$$
$$\times\int\limits_x^{+\infty}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}y^{-2(s+1)}{}_2F_1\left(s+1,s+1-\nu;1-\nu;\frac{x^2}{y^2}\right)\ y\ f(y)\ dy. \tag{30}$$

Proof. Applying the ITCM for the Hankel and the $\mathcal{Y}$-transforms given by (15) and (16), and the dual variable $(-(d^2))^s=(-t^2)^s$, we have

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{H}_\nu f\right](x)=\int\limits_0^{+\infty}(xt)^{\frac{1}{2}}Y_\nu(xt)\left[(-t^2)^s\int\limits_0^{+\infty}(yt)^{\frac{1}{2}}J_\nu(yt)f(y)dy\right]dt$$
$$=(-1)^s\int\limits_0^{+\infty}(x)^{\frac{1}{2}}Y_\nu(xt)t^{2s+1}dt\int\limits_0^{+\infty}(y)^{\frac{1}{2}}J_\nu(yt)\ f(y)\ dy$$

$$= (-1)^s \int\limits_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1} J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$= (-1)^s \int\limits_0^{x} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$+ (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(yt)\, Y_\nu(xt)\, dt.$$

Using formula (27) and (28) for $\alpha = 2(s+1)$, $\mu = \nu$, $x = t$, $b = y$, $c = x$, we obtain for $0 < y < x$,

$$\int\limits_0^{+\infty} t^{2s+1}\, J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$= \frac{2^{2s+1}\, y^\nu}{\pi\, x^{2(s+1)+\nu}} \cos(s\pi)\, \frac{\Gamma(s+1+\nu)\Gamma(s+1)}{\Gamma(\nu+1)}\, {}_2F_1\left(s+1, s+1+\nu; \nu+1; \frac{y^2}{x^2}\right),$$

and also the next is valid for $0 < x < y$

$$\int\limits_0^{+\infty} t^{2s+1}\, J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$= -\frac{2^{2s+1}\, x^\nu}{\pi\, y^{2(s+1)+\nu}} \cos(\nu\pi)\, \frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}\, {}_2F_1\left(s+1, s+1+\nu; 1+\nu; \frac{x^2}{y^2}\right)$$
$$- \frac{2^{2s+1}\, y^{\nu-2(s+1)}}{\pi x^\nu} \frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)}\, {}_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right).$$

$x, y > 0$; $-Re\,\nu < Re\,2(s+1)$. Thus,

$$\left[\mathcal{Y}_\nu(-t^2)^s \mathcal{H}_\nu f\right](x) = (-1)^s \int\limits_0^{x} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$+ (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(yt)\, Y_\nu(xt)\, dt = J_1 + J_2,$$

where

$$J_1 = (-1)^s \int\limits_0^{x} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$= \frac{(-1)^s 2^{2s+1}}{\pi\, x^{2(s+1)}} \cos(s\pi)\, \frac{\Gamma(s+1+\nu)\Gamma(s+1)}{\Gamma(\nu+1)}$$

$$\times \int\limits_0^{x} \left(\frac{y}{x}\right)^{\nu-\frac{1}{2}} {}_2F_1\left(s+1, s+1+\nu; \nu+1; \frac{y^2}{x^2}\right) yf(y)dy$$

and

$$J_2 = (-1)^s \int\limits_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int\limits_0^{+\infty} t^{2(s+1)-1}\, J_\nu(yt)\, Y_\nu(xt)\, dt$$

$$= \frac{(-1)^{s+1}2^{2s+1}}{\pi}\cos(\nu\pi)\frac{\Gamma(-\nu)\Gamma(s+1+\nu)}{\Gamma(-s)}$$
$$\times\int_x^{+\infty}\left(\frac{x}{y}\right)^{\nu+\frac{1}{2}}{}_2F_1\left(s+1,s+1+\nu;1+\nu;\frac{x^2}{y^2}\right)y^{-2s-1}f(y)dy$$
$$+(-1)^{s+1}\frac{2^{2s+1}}{\pi}\frac{\Gamma(\nu)\Gamma(s+1)}{\Gamma(\nu-s)}\int_x^{+\infty}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}{}_2F_1\left(s+1,s+1-\nu;1-\nu;\frac{x^2}{y^2}\right)y^{-2s-1}f(y)dy.$$

□

Remark. The special case when s tends to 0:

$$\lim_{s\to 0}\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{H}_\nu f\right](x)=\lim_{s\to 0}\left[J_1+J_2\right]\Leftrightarrow\left[\mathcal{Y}_\nu\mathcal{H}_\nu f\right](x)=\lim_{s\to 0}J_1+\lim_{s\to 0}J_2,$$

where

$$\lim_{s\to 0}J_1=\frac{2}{\pi\,x^2}\int_0^x\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}{}_2F_1\left(1,1+\nu;\nu+1;\frac{y^2}{x^2}\right)yf(y)dy=\frac{2}{\pi}\int_0^x\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\frac{yf(y)}{x^2-y^2}dy$$

and

$$\lim_{s\to 0}J_2=-\frac{2}{\pi}\int_x^{+\infty}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}{}_2F_1\left(1,1-\nu;1-\nu;\frac{x^2}{y^2}\right)y^{-1}f(y)dy=\frac{2}{\pi}\int_x^{+\infty}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\frac{yf(y)}{x^2-y^2}dy.$$

Hence

$$\left[\mathcal{Y}_\nu\mathcal{H}_\nu f\right](x)=\frac{2}{\pi}\int_0^{+\infty}\left(\frac{y}{x}\right)^{\nu-\frac{1}{2}}\frac{yf(y)}{x^2-y^2}dy=\frac{2}{\pi}\int_0^{+\infty}\left(\frac{x}{y}\right)^{\frac{1}{2}-\nu}\frac{yf(y)}{x^2-y^2}dy.$$

For $\nu=-\frac{1}{2}$ the operator $\mathcal{Y}_\nu\mathcal{H}_\nu$ equals to Hilbert transform on semi-axes

$$-[\mathcal{Y}_{-\frac{1}{2}}\mathcal{H}_{-\frac{1}{2}}f](x)=[\mathcal{H}_{\frac{1}{2}}\mathcal{Y}_{\frac{1}{2}}f](x)=\frac{2}{\pi}\int_0^{\infty}\frac{xf(y)}{y^2-x^2}dy.$$

Theorem 4. *Let f be a proper function, $|Re(\nu)|^2<Re(2(s+1)<2$ and $Re(s)<0$. Then, for the transmutation operator T_4 obtained by ITCM such that $T_4(f)=\mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\nu(f)$ the next integral representation is true*

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\nu f\right](x)=(-1)^s\frac{2^{2s+1}}{\pi^2}x^{\nu+\frac{1}{2}}\cos(\nu\pi)\cos\left((s+1)\pi\right)\Gamma(-\nu)\Gamma(s+1+\nu)\Gamma(s+1)$$
$$\times\int_x^{+\infty}y^{-2(s+1)-\nu+\frac{1}{2}}f(y)\,{}_2F_1\left(s+1+\nu,s+1;1+\nu;\frac{x^2}{y^2}\right)dy$$
$$+(-1)^s\frac{2^{2s+1}}{\pi^2}x^{-\nu+\frac{1}{2}}\cos\left((s+1-\nu)\pi\right)\Gamma(\nu)\Gamma(s+1-\nu)\Gamma(s+1)$$
$$\times\int_x^{+\infty}y^{-2(s+1)+\nu+\frac{1}{2}}f(y)\,{}_2F_1\left(s+1,s+1-\nu;1-\nu;\frac{x^2}{y^2}\right)dy,\tag{31}$$

where ${}_2F_1$ is the Gauss hypergeometric function.

Proof. We have

$$\left[\mathcal{Y}_\nu(-t^2)^s\mathcal{Y}_\nu f\right](x)=(-1)^s\int_0^{+\infty}(xt)^{\frac{1}{2}}Y_\nu(xt)t^{2s}dt\int_0^{+\infty}(yt)^{\frac{1}{2}}Y_\nu(yt)\,f(y)\,dy$$

$$= (-1)^s \int_0^{+\infty} (xy)^{\frac{1}{2}} f(y)dy \int_0^{+\infty} t^{2s+1} Y_\nu(xt)\, Y_\nu(yt)\, dt$$

$$= (-1)^s \int_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int_0^{+\infty} t^{2(s+1)-1}\, Y_\nu(xt)\, Y_\nu(yt)\, dt.$$

Using ([8], formula 2.13.20, page 278), we derive

$$\int_0^{+\infty} x^{\alpha-1}\, Y_\mu(bx)\, Y_\nu(cx)\, dx = \frac{2^{\alpha-1}b^\mu}{\pi^2 c^{\alpha+\mu}} \cos(\mu\pi) \cos\left(\frac{\alpha+\mu-\nu}{2}\pi\right)$$

$$\times \Gamma(-\mu)\Gamma(\frac{\alpha+\mu+\nu}{2})\Gamma(\frac{\alpha+\mu-\nu}{2})_2F_1\left(\frac{\alpha+\mu+\nu}{2}, \frac{\alpha+\mu-\nu}{2}; 1+\mu; \frac{b^2}{c^2}\right)$$

$$+ \frac{2^{\alpha-1}c^{\mu-\alpha}}{\pi^2 b^\mu} \cos\left(\frac{\alpha-\mu-\nu}{2}\pi\right)$$

$$\times \Gamma(\mu)\Gamma(\frac{\alpha-\mu+\nu}{2})\Gamma(\frac{\alpha-\mu-\nu}{2})_2F_1\left(\frac{\alpha-\mu+\nu}{2}, \frac{\alpha-\mu-\nu}{2}; 1-\mu; \frac{b^2}{c^2}\right),$$

$$0 < c < b, \quad |Re(\mu)|\,| + Re(\nu)| < Re(\alpha) < 2. \tag{32}$$

Thus, for $\mu = \nu$, $\alpha = 2(s+1)$, $x = t$, $b = x$, $c = y$, and $Re(s) < 0$, we obtain

$$\int_0^{+\infty} t^{2s+1}\, Y_\nu(xt)\, Y_\nu(yt)\, dt = \frac{2^{2s+1}x^\nu}{\pi^2 y^{2(s+1)+\mu}} \cos(\nu\pi) \cos\left((s+1)\pi\right)$$

$$\times \Gamma(-\nu)\Gamma(s+1+\nu)\Gamma(s+1)_2F_1\left(s+1+\nu, s+1; 1+\nu; \frac{x^2}{y^2}\right)$$

$$+ \frac{2^{2s+1}y^{\nu-2(s+1)}}{\pi^2 x^\nu} \cos\left((s+1-\nu)\pi\right)\, \Gamma(\nu)\Gamma(s+1)\Gamma(s+1-\nu)_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right),$$

$$0 < x < y,\ |Re(\nu)|^2 < Re(2(s+1)) < 2.$$

Thus,

$$\left[\mathcal{Y}_\nu(-t^2)^s \mathcal{Y}_\nu f\right](x) = (-1)^s \int_0^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} yf(y)dy \int_0^{+\infty} t^{2(s+1)-1}\, Y_\nu(xt)\, Y_\nu(yt)\, dt$$

$$= (-1)^s \frac{2^{2s+1}x^\nu}{\pi^2} \cos(\nu\pi) \cos\left((s+1)\pi\right) \Gamma(-\nu)\Gamma(s+1+\nu)\Gamma(s+1)$$

$$\times \int_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y^{-(2(s+1)+\mu)} yf(y)_2F_1\left(s+1+\nu, s+1; 1+\nu; \frac{x^2}{y^2}\right) dy$$

$$+ \frac{2^{2s+1}}{\pi^2 x^\nu} \cos\left((s+1-\nu)\pi\right) \Gamma(\nu)\Gamma(s+1)\Gamma(s+1-\nu)$$

$$\times \int_x^{+\infty} \left(\frac{x}{y}\right)^{\frac{1}{2}} y^{\nu-2(s+1)} yf(y)_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right) dy$$

$$= (-1)^s \frac{2^{2s+1}}{\pi^2} x^{\nu+\frac{1}{2}} \cos(\nu\pi) \cos\left((s+1)\pi\right) \Gamma(-\nu)\Gamma(s+1+\nu)\Gamma(s+1)$$

$$\times \int_{x}^{+\infty} y^{-2(s+1)-\nu+\frac{1}{2}} f(y)\, {}_2F_1\left(s+1+\nu, s+1; 1+\nu; \frac{x^2}{y^2}\right) dy$$

$$+ (-1)^s \frac{2^{2s+1}}{\pi^2} x^{-\nu+\frac{1}{2}} \cos\left((s+1-\nu)\pi\right) \Gamma(\nu)\Gamma(s+1-\nu)\Gamma(s+1)$$

$$3 \times \int_{x}^{+\infty} y^{-2(s+1)+\nu+\frac{1}{2}} f(y)\, {}_2F_1\left(s+1, s+1-\nu; 1-\nu; \frac{x^2}{y^2}\right) dy.$$

□

So we proved a collection of composition formulas for Hankel and Y transforms with different parameters by ITCM method, these compositions are also transmutations for differential operators connected with singular Bessel operator.

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