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PARTIAL DIFFERENTIAL EQUATIONS

Elliptic Problems and Integral Equations in Spaces of Different Smoothness in Different Variables

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Abstract—We consider a model elliptic pseudodifferential equation and the simplest boundary value problems in a quadrant in a Sobolev–Slobodetsky space of different orders of smoothness in different variables. In the case of a special representation of the symbol, we describe a general solution of the equation and consider the simplest boundary value problem with the Dirichlet and Neumann conditions on the sides of the quadrant. This boundary value problem is reduced to a system of integral equations, which, under additional assumptions about the structure of the symbol, can also be reduced to a system of first-order difference equations.

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INTRODUCTION

The theory of boundary value problems for elliptic pseudodifferential equations originates from the mid-1960s, namely, from the papers by M.I. Vishik and G.I. Eskin, the results of which are summarized in the monograph [1]. The results obtained attracted attention and were further developed by a number of researchers (see, e.g., [2, 3]). The second author of the present paper also showed interest in this topic, proposing his own approach to constructing the theory of boundary value problems for elliptic pseudodifferential equations in domains with conical points and edges of various dimensions on the boundary (see [4, 5] and the continuation in the papers [6–10]).

All studies were carried out in ordinary Sobolev–Slobodetsky spaces; however, spaces of different orders of smoothness in different variables are possible [11–13]. Here we consider the simplest case of Sobolev–Slobodetsky spaces of different order of smoothness in different variables and describe the reduction of the boundary value problem to a system of integral equations.

1. ELLIPTIC EQUATIONS

In this subsection, we give some definitions and results to be relied on in what follows.

1.1. Sobolev–Slobodetsky Spaces of Different Smoothness

Following [14] (see also [11]), we introduce some convenient notation. We represent the multidimensional Euclidean space $\mathbb{R}^M$ as an orthogonal sum of subspaces in which only some of the coordinates $x_1, x_2, \ldots, x_M$ are nonzero. More precisely, if $K \subset \{1, \ldots, M\}$ is a nonempty set, then we set

$$\mathbb{R}^K = \left\{x \in \mathbb{R}^M : x = (x_1, \ldots, x_M),\ \ x_j = 0 \ \text{ for each } \ j \notin K\right\} \subset \mathbb{R}^M.$$

Let $K_1, K_2, \ldots, K_n \subset \{1, 2, \ldots, M\}$ be some subsets such that

$$\bigcup_{j=1}^{n} K_j = \{1, 2, \ldots, M\}, \quad K_i \cap K_j = \varnothing, \quad i \neq j.$$

Then we have the representation

$$\mathbb{R}^M = \mathbb{R}^{K_1} \oplus \mathbb{R}^{K_2} \oplus \cdots \oplus \mathbb{R}^{K_n},$$

denoting an element of the space $\mathbb{R}^{K_j}$ by x_{K_j}.

For functions defined in the space $\mathbb{R}^M$, we use the standard Fourier transform

$$\tilde{u}(\xi) = \int\limits_{\mathbb{R}^M} u(x) e^{ix\cdot\xi}\, dx, \quad \xi = (\xi_1, \ldots, \xi_M).$$

Now we define the Sobolev–Slobodetsky space $H^S(\mathbb{R}^M)$, where we write $S = (s_1, \ldots, s_n)$ for simplicity, as the Hilbert space with inner product

$$(f, g) = \int\limits_{\mathbb{R}^M} f(x)\overline{g(x)}\, dx$$

and norm

$$\|f\|_S = \left(\int\limits_{\mathbb{R}^M} \left(1 + |\xi_{K_1}|\right)^{2s_1} \left(1 + |\xi_{K_2}|\right)^{2s_2} \cdots \left(1 + |\xi_{K_n}|\right)^{2s_n} \left|\tilde{f}(\xi)\right|^2 d\xi \right)^{1/2}.$$

Such H^S-spaces have a standard set of properties of Sobolev–Slobodetsky spaces [11]. In particular, the space $H^s(\mathbb{R}^M)$ is obtained by the following selection of the subsets K_j and the parameters s_j:

$$K_1 = K_2 = \cdots = K_{n-1} = \varnothing, \quad K_n = \{1, 2, \ldots, M\}, \quad S = (0, 0, \ldots, 0, s).$$

1.2. Model Equation and Solvability

In accordance with the locality principle, we focus on the study of a model pseudodifferential equation with an operator whose symbol does not depend on the space variable. Detailed proofs of the results presented here can be found in [15].

A pseudodifferential operator A is defined by the formula

$$(Au)(x) = \frac{1}{(2\pi)^M} \int\limits_{\mathbb{R}^M} \int\limits_{\mathbb{R}^M} e^{i(x-y)\cdot\xi} \tilde{A}(\xi) u(y)\, dy\, d\xi,$$

where $\tilde{A}(\xi)$ is a given measurable function called the *symbol of the operator* A.

Assume that the symbol $\tilde{A}(\xi)$ satisfies the condition

$$c_1 \prod_{j=1}^{n} \left(1 + |\xi_{K_j}|\right)^{\alpha_j} \le \left|A(\xi)\right| \le c_2 \prod_{j=1}^{n} \left(1 + |\xi_{K_j}|\right)^{\alpha_j}, \quad \alpha_j \in \mathbb{R}, \quad j = 1, \ldots, n, \tag{1}$$

with positive constant c_1 and c_2.

Denote $\alpha = (\alpha_1, \ldots, \alpha_n)$.

Lemma 1. *Let A be a pseudodifferential operator with symbol $\tilde{A}(\xi)$ satisfying condition (1). Then $A : H^S(\mathbb{R}^M) \to H^{S-\alpha}(\mathbb{R}^M)$ is a continuous linear operator.*

A simple consequence of this lemma is the following fact. If A is a pseudodifferential operator with symbol $\tilde{A}(\xi)$ satisfying condition (1), then the equation

$$(Au)(x) = v(x), \quad x \in \mathbb{R}^M, \tag{2}$$

with an arbitrary right-hand side $v \in H^{S-\alpha}(\mathbb{R}^M)$ has a unique solution $u \in H^S(\mathbb{R}^M)$, and one has the a priori estimate

$$\|u\|_S \le \text{const}\, \|v\|_{S-\alpha}.$$

Note that if we consider Eq. (2) not in the entire space $\mathbb{R}^M$ but in another canonical domain (also a cone), then such a simple corollary does not hold. Here, as before [6–10, 16–23], we will be interested in the case of a convex cone that does not contain an entire line.

Let $C_{K_j} \subset \mathbb{R}^{K_j}$ be a convex cone not containing an entire line. We set

$$C = C_{K_1} \times C_{K_2} \times \cdots \times C_{K_n}.$$

It is obvious that C is a convex cone in the space $\mathbb{R}^M$.

Now we study the question of solvability of the equation

$$(Au)(x) = 0, \quad x \in C, \tag{3}$$

in the space $H^S(C)$.

Below we present definitions and results concerning the solvability of Eq. (3).

Definition 1. The space $H^S(C)$ consists of (generalized) functions in $H^S(\mathbb{R}^M)$ whose supports are contained in $\overline{C}$.

Denote by $\tilde{H}^S(C)$ the *Fourier transform of the space* $H^S(C)$.

Definition 2. The *radial tubular domain* over a cone C is the domain in the multidimensional complex space $\mathbb{C}^M$ given by

$$T(C) \equiv \{z \in \mathbb{C}^M : z = x + iy, \ \ x \in \mathbb{R}^M, \ \ y \in C\}.$$

The *conjugate cone* $\overset{*}{C}$ is the cone formed by the points x satisfying the condition

$$x \cdot y > 0 \quad \text{for all} \quad y \in C;$$

here $x \cdot y$ is the inner product of x and y.

Definition 3. The *wave factorization* of an elliptic symbol $\tilde{A}(\xi)$ with respect to a cone C is its representation in the form

$$\tilde{A}(\xi) = A_{\neq}(\xi)A_{=}(\xi),$$

where the factors $A_{\neq}(\xi)$, $A_{=}(\xi)$ must satisfy the following conditions:

1. $A_{\neq}(\xi)$ and $A_{=}(\xi)$ are defined for all $\xi \in \mathbb{R}^M$, except possibly for the points $\xi \in \partial\overset{*}{C}$.
2. $A_{\neq}(\xi)$ and $A_{=}(\xi)$ admit analytic continuation into the radial tubular domains $T(\overset{*}{C})$ and $T(-\overset{*}{C})$, respectively, and satisfy the estimates

$$\left|A_{\neq}^{\pm 1}(\xi + i\tau)\right| \le c_1 \prod_{j=1}^{n} \left(1 + |\xi_{K_j}| + |\tau_{K_j}|\right)^{\pm\varkappa_j},$$

$$\left|A_{=}^{\pm 1}(\xi - i\tau)\right| \le c_2 \prod_{j=1}^{n} \left(1 + |\xi_{K_j}| + |\tau_{K_j}|\right)^{\pm(\alpha_j-\varkappa_j)} \quad \text{for each} \ \ \tau \in \overset{*}{C}, \quad \varkappa_j \in \mathbb{R}.$$

The vector $\varkappa = (\varkappa_1, \dots, \varkappa_n)$ is called the *wave factorization index*.

Remark 1. It should be noted that Definition 3 must be modified if some cone C_{K_j} contains an entire line, more precisely, if it has the form $\mathbb{R}^{m_j} \times C_{k_j-m_j}$, where $C_{k_j-m_j}$, $0 \le m_j \le k_j$, is a convex cone that does not contain an entire line in a $(k_j - m_j)$-dimensional space. Recall that by definition, for $m_j = 0$ we set $\mathbb{R}^0 \times C_{k_j} \equiv C_{K_j}$, for $m_j = k_j$, and accordingly, $\mathbb{R}^{k_j} \times C_0 \equiv \mathbb{R}^{K_j}$. Denoting $\sum_{j=1}^{n} = Q$, one can define Q-*wave factorization*, where the points of the Q-dimensional space $\mathbb{R}^Q = \mathbb{R}^{m_1} \times \dots \times \mathbb{R}^{m_n}$ play the role of parameters (see [4]). Then Definition 3 corresponds to the 0-wave factorization.

Theorem 1. *If the symbol $\tilde{A}(\xi)$ admits wave factorization with respect to a cone C with index $\varkappa$ such that $|\varkappa_j - s_j| < 1/2$, $j = 1, \dots, n$, then Eq. (3) has only the zero solution in the space $H^S(C)$.*

We assume that for each cone C_{K_j}, $j = 1, \dots, n$, its surface equation is written as $x_{k_j} = \varphi_j(x'_{K_j})$, where $\varphi_j : \mathbb{R}^{k_j-1} \to \mathbb{R}$, is a smooth function on $\mathbb{R}^{k_j-1} \setminus \{0\}$, $\varphi_j(0) = 0$, $x_{K_j} = (x'_{K_j}, x_{k_j})$.

Using the change of variables

$$t'_{K_j} = x'_{K_j},$$

$$t_{k_j} = x_{k_j} - \varphi_j(x'_{K_j}),$$

we define an operator $T_{\varphi_j} : \mathbb{R}^{K_j} \to \mathbb{R}^{K_j}$ as the operator of the above change of variables, while the cone C_{K_j} transforms into the upper half-space $\mathbb{R}_+^{K_j} = \{x \in \mathbb{R}^{K_j} : x_{K_j} = (x'_{K_j}, x_{k_j}),\ x_{k_j} > 0\}$.

Remark 2. Of course, this change of variables is needed only in the multidimensional case ($m \geq 2$); in the one-dimensional case there is only one cone—a ray, whose boundary is a point.

In the reasoning below, we will use the notation F_m for the Fourier transform in an m-dimensional space; therefore, F_{K_j} denotes the Fourier transform in the space $\mathbb{R}^{K_j}$.

According to the results in [8], we have the relations $F_{K_j} T_{\varphi_j} = V_{\varphi_j} F_{K_j}$.

Next, we introduce an operator $T_\varphi : \mathbb{R}^M \to \mathbb{R}^M$ using the formula $T_\varphi = \prod_{j=1}^n T_{\varphi_j}$ to obtain the operator $V_\varphi = \prod_{j=1}^n V_{\varphi_j}$ for which the identity $F_M T_\varphi = V_\varphi F_M$ holds. We also introduce the vectors $N = (n_1, \ldots, n_n)$, $L = (l_1, \ldots, l_n)$, and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$, $n_j, l_j \in \mathbb{N}$, $|\delta_j| < 1/2$, $j = 1, \ldots, n$.

Theorem 2. *If a symbol $\tilde{A}(\xi)$ admits wave factorization with respect to a cone C with an index $\varkappa$ such that $\varkappa - S = N + \varepsilon$, then the general solution of Eq.* (3) *in terms of Fourier transforms has the form*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi) V_\varphi^{-1} \left(\sum_{l_1=1}^{n_1} \sum_{l_2=1}^{n_2} \cdots \sum_{l_n=1}^{n_n} \tilde{c}_L(\xi'_K) \xi_{k_1}^{l_1-1} \xi_{k_2}^{l_2-1} \cdots \xi_{k_n}^{l_n-1} \right), \tag{4}$$

where the $c_L(x'_K) \in H^{S_L}(\mathbb{R}^{M-n})$ are arbitrary functions and

$$S_L = (s_1 - \varkappa_1 + l_1 - 1/2, \ldots, s_n - \varkappa_n + l_n - 1/2), \quad l_j = 1, \ldots, n_j, \quad j = 1, \ldots, n.$$

The following a priori estimate holds:

$$\|u\|_S \leq \text{const} \sum_{l_1=1}^{n_1} \sum_{l_2=1}^{n_2} \cdots \sum_{l_n=1}^{n_n} \|c_L\|_{S_L}.$$

2. BOUNDARY VALUE PROBLEMS

In this subsection, we consider some simple formulations of boundary value problems related to Theorem 2, which establishes the multiplicity of possible solutions of Eq. (3). Additional conditions are needed to isolate a single solution. We start with the case of a two-dimensional cone. The presence of the operator V_φ in formula (4) greatly complicates the formulation and study of boundary value problems, but the two-dimensional case is a rare exception where one can do without such an operator. This was demonstrated in the monograph [4], and a comparison of the two variants is presented in [7].

2.1. Flat Angle and General Solution

For the case of a flat angle, only one situation with different smoothness in variables is possible, namely, smoothness of the order s_1 in one variable and s_2 in the other. Our cone C has the form of a direct product of two rays; we can assume it to be the first quadrant. It is assumed that the symbol $\tilde{A}(\xi)$ admits wave factorization with respect to C with an index $\varkappa = (\varkappa_1, \varkappa_2)$ such that $\varkappa_j - s_j = n_j + \varepsilon_j$, $n_j \in \mathbb{N}$, $|\varepsilon_j| < 1/2$, $j = 1, 2$. Let us show how the general solution formula (4) looks like in this case.

Set

$$u_-(x) = -(Au)(x), \quad x \in \mathbb{R}^2.$$

By virtue of relation (3), we have $u_-(x) = 0$, $x \in C$. Let us write Eq. (3) in the form

$$(Au)(x) + u_-(x) = 0, \quad x \in \mathbb{R}^2;$$

we apply the Fourier transform and obtain

$$\tilde{A}(\xi)\tilde{u}(\xi) + \tilde{u}_-(\xi) = 0,$$

and after wave factorization of the symbol $\tilde{A}(\xi)$ with respect to C we obtain the relation

$$A_{\neq}(\xi)\tilde{u}(\xi) = -A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi).$$

By Lemma 1,

$$A_{\neq}(\xi)\tilde{u}(\xi), A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi) \in \tilde{H}^{S-\varkappa}(\mathbb{R}^2),$$

but more precise inclusions are as follows (see [4] for details):

$$\begin{gathered} A_{\neq}(\xi)\tilde{u}(\xi) \in \tilde{H}^{S-\varkappa}(C), \\ A_{=}^{-1}(\xi)\tilde{u}_{-}(\xi) \in \tilde{H}^{S-\varkappa}(\mathbb{R}^2 \setminus \overline{C}). \end{gathered} \tag{5}$$

It readily follows from the inclusions (5) that the inverse Fourier transform of these (generalized) functions, owing to them being equal, can only be a function concentrated on the boundary of the quadrant. Taking into account the structure of such functions [24], we can write

$$F^{-1}\big(A_{\neq}(\xi)\tilde{u}(\xi)\big) = \sum_{k=1}^{r_1} c_k(x_1)\delta^{k-1}(x_2) + \sum_{k=1}^{r_2} d_k(x_2)\delta^{k-1}(x_1),$$

or

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\left(\sum_{k=1}^{r_1} \tilde{c}_k(\xi_1)\xi_2^{k-1} + \sum_{k=1}^{r_2} \tilde{d}_k(\xi_2)\xi_1^{k-1}\right).$$

It remains to clarify the number of terms in the sums and the exponent s_k of the space $H^{s_k}(\mathbb{R})$ that includes the functions c_k and d_k.

We single out one term, for example, $A_{\neq}^{-1}(\xi)\tilde{c}_k(\xi_1)\xi_2^{k-1}$, and estimate it,

$$\begin{aligned} \big\|A_{\neq}^{-1}(\xi)\tilde{c}_k(\xi_1)\xi_2^{k-1}\big\|_S^2 &= \int_{\mathbb{R}^2} \big|A_{\neq}^{-1}(\xi)\big|^2 \big|\tilde{c}_k(\xi_1)\big|^2 |\xi_2|^{2(k-1)} \big(1+|\xi_1|\big)^{2s_1}\big(1+|\xi_2|\big)^{2s_2}\, d\xi \\ &\le \text{const} \int_{-\infty}^{+\infty} \big|\tilde{c}_k(\xi_1)\big|^2 \big(1+|\xi_1|\big)^{2(s_1-\varkappa_1)}\, d\xi_1 \int_{-\infty}^{+\infty} \big(1+|\xi_2|\big)^{2(s_2-\varkappa_2+k-1)}\, d\xi_2. \end{aligned}$$

The integral

$$\int_{-\infty}^{+\infty} \big(1+|\xi_2|\big)^{2(s_2-\varkappa_2+k-1)}\, d\xi_2$$

is convergent under the condition $2(s_2-\varkappa_2+k-1) < -1$, or $-n_2-\varepsilon_2+k < 1/2$. The last inequality holds for $k = 1, \ldots, n_2$. Thus, if $\tilde{c}_k \in \tilde{H}^{-n_1-\varepsilon_1}(\mathbb{R})$, then we obtain the representation

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\left(\sum_{k=1}^{n_2} \tilde{c}_k(\xi_1)\xi_2^{k-1} + \sum_{k=1}^{n_1} \tilde{d}_k(\xi_2)\xi_1^{k-1}\right)$$

and the following estimate for the solution:

$$\|u\|_S \le \text{const}\left(\sum_{k=1}^{n_2}[c_k]_{-n_1-\delta_1} + \sum_{k=1}^{n_1}[\,d_k]_{-n_2-\delta_2}\right);$$

here and in the following, $[\,\cdot\,]_s$ stands for the usual H^s-norm on the line.

Thus, the following assertion holds.

Theorem 3. *Let C be the first quadrant in the plane, and let the symbol $\tilde{A}(\xi)$ admit wave factorization with an index $\varkappa = (\varkappa_1, \varkappa_2)$ such that $\varkappa_j - s_j = n_j + \varepsilon_j$, $n_j \in \mathbb{N}$, $|\varepsilon_j| < 1/2$, $j = 1, 2$. Then the general solution of Eq. (3) in the space $H^S(C)$, $S = (s_1, s_2)$, has the form*

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi) \left(\sum_{k=1}^{n_2} \tilde{c}_k(\xi_1)\xi_2^{k-1} + \sum_{k=1}^{n_1} \tilde{d}_k(\xi_2)\xi_1^{k-1} \right).$$

The following a priori estimate holds:

$$\|u\|_S \leq \text{const} \left(\sum_{k=1}^{n_2} [c_k]_{-n_1-\delta_1} + \sum_{k=1}^{n_1} [\, d_k]_{-n_2-\delta_2} \right).$$

2.2. Dirichlet and Neumann Boundary Conditions and Integral Equations

Consider one particular case where we can restrict ourselves to the classical Dirichlet and Neumann conditions for determining arbitrary functions appearing in the structure of the general solution.

Let $n_1 = 1$ and $n_2 = 2$. According to Theorem 3, the general solution of the equation has the form

$$\tilde{u}(\xi) = A_{\neq}^{-1}(\xi)\big(c_1(\xi_1) + c_2(\xi_1)\xi_2 + \, d_1(\xi_2)\big)$$

and contains three arbitrary functions c_1, c_2, and d_1, which are to be uniquely determined to obtain a unique solution. On the sides of the corner, we set the boundary conditions

$$u_{|_{x_2=0}} = f(x_1), \quad \left(-i\frac{\partial u}{\partial x_2}\right)\bigg|_{x_2=0} = g(x_1), \quad u|_{x_1=0} = h(x_2). \tag{6}$$

In terms of Fourier transforms, conditions (6) have the form

$$\int_{-\infty}^{+\infty} \tilde{u}(\xi_1, \xi_2)\, d\xi_2 = \tilde{f}(\xi_1),$$

$$\int_{-\infty}^{+\infty} \xi_2 \tilde{u}(\xi_1, \xi_2)\, d\xi_2 = \tilde{g}(\xi_1),$$

$$\int_{-\infty}^{+\infty} \tilde{u}(\xi_1, \xi_2)\, d\xi_1 = \tilde{h}(\xi_2).$$

Substituting them into the general solution formula, we obtain the following system of linear integral equations for the three unknown functions c_1, c_2, and d_1:

$$\begin{gathered} a_1(\xi_1)c_1(\xi_1) + b_1(\xi_1)c_2(\xi_1) + \int_{-\infty}^{+\infty} A_{\neq}^{-1}(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 = \tilde{f}(\xi_1), \\ b_1(\xi_1)c_1(\xi_1) + p_1(\xi_1)c_2(\xi_1) + \int_{-\infty}^{+\infty} \xi_2 A_{\neq}^{-1}(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 = \tilde{g}_1(\xi_1), \\ \int_{-\infty}^{+\infty} A_{\neq}^{-1}(\xi_1, \xi_2)c_1(\xi_1)\, d\xi_1 + \int_{-\infty}^{+\infty} \xi_2 A_{\neq}^{-1}(\xi_1, \xi_2)c_2(\xi_1)\, d\xi_1 + p_2(\xi_2)\, d_1(\xi_2) = \tilde{h}(\xi_2), \end{gathered} \tag{7}$$

where we have introduced the notation

$$a_1(\xi_1)=\int\limits_{-\infty}^{+\infty} A_{\neq}^{-1}(\xi_1,\xi_2)\,d\xi_2,\qquad b_1(\xi_1)=\int\limits_{-\infty}^{+\infty} \xi_2 A_{\neq}^{-1}(\xi_1,\xi_2)\,d\xi_2,$$
$$p_1(\xi_1)=\int\limits_{-\infty}^{+\infty} \xi_2^2 A_{\neq}^{-1}(\xi_1,\xi_2)\,d\xi_2,\qquad p_2(\xi_2)=\int\limits_{-\infty}^{+\infty} A_{\neq}^{-1}(\xi_1,\xi_2)\,d\xi_1.$$

Thus, we have the following assertion.

Theorem 4. *Let $s_1>1/2$ and $s_2>3/2$, and let the symbol $\tilde{A}(\xi)$ admit wave factorization with respect to C with an index $\varkappa$ such that $\varkappa_1-s_1=1+\varepsilon_1$, $|\delta_1|<1/2$, $\varkappa_2-s_2=2+\varepsilon_2$, $|\delta_2|<1/2$. Then the boundary value problem* (3), (6) *is uniquely solvable in the space $H^S(C)$ if the system of integral equations* (7) *has a unique solution c_1, c_2, d_1.*

2.3. Integral and Difference Equations

The system of integral equations (7) obtained in the previous section is not simple, and it is difficult to propose any acceptable method for solving it. However, if we introduce some additional assumptions about the symbol $\tilde{A}(\xi)$, then this system can be reduced to a system of first-order difference equations. Let us describe this possibility.

Assume that the factor $A_{\neq}(\xi_1,\xi_2)$ is a positively homogeneous function of different orders in the variables ξ_1 and ξ_2, namely, of order $\varkappa_1$ in the first variable and $\varkappa_2$ in the second one, for all $t>0$, $A_{\neq}(t\xi_1,t\xi_2)=t^{\varkappa_1+\varkappa_2}A_{\neq}(\xi_1,\xi_2)$.

In this case, it is easy to verify the validity of the following homogeneity property.

Lemma 2. *The functions a_1, b_1, p_1, and p_2 possess the following homogeneity property for all $t>0$:*

$$a_1(t\xi_1)=t^{1-\varkappa_1-\varkappa_2}a_1(\xi_1),\qquad b_1(t\xi_1)=t^{2-\varkappa_1-\varkappa_2}b_1(\xi_1),$$
$$p_1(t\xi_1)=t^{3-\varkappa_1-\varkappa_2}p_1(\xi_1),\qquad p_2(t\xi_2)=t^{1-\varkappa_1-\varkappa_2}p_2(\xi_2).$$

System (7) can be written in the form

$$\begin{gathered} c_1(\xi_1)+r(\xi_1)c_2(\xi_1)+\int\limits_{-\infty}^{+\infty} K(\xi_1,\xi_2)\,d_1(\xi_2)\,d\xi_2=\tilde{F}(\xi_1),\\ c_1(\xi_1)+q(\xi_1)c_2(\xi_1)+\int\limits_{-\infty}^{+\infty} L(\xi_1,\xi_2)\,d_1(\xi_2)\,d\xi_2=\tilde{G}(\xi_1),\\ \int\limits_{-\infty}^{+\infty} M(\xi_1,\xi_2)c_1(\xi_1)\,d\xi_1+\xi_2\int\limits_{-\infty}^{+\infty} M(\xi_1,\xi_2)c_2(\xi_1)\,d\xi_1+\,d_1(\xi_2)=\tilde{H}(\xi_2), \end{gathered}\tag{8}$$

where we have introduced the notation

$$b_1(\xi_1)a_1^{-1}(\xi_1)=r(\xi_1),\quad p_1(\xi_1)b_1^{-1}(\xi_1)=q(\xi_1),\quad a_1^{-1}(\xi_1)A_{\neq}^{-1}(\xi_1,\xi_2)\equiv K(\xi_1,\xi_2),$$
$$\xi_2 b_1^{-1}(\xi_1)A_{\neq}^{-1}(\xi_1,\xi_2)=L(\xi_1,\xi_2),\quad p_2^{-1}(\xi_2)A_{\neq}^{-1}(\xi_1,\xi_2)=M(\xi_1,\xi_2),\quad \tilde{f}(\xi_1)a_1^{-1}(\xi_1)=\tilde{F}(\xi_1),$$
$$\tilde{g}(\xi_1)b_1^{-1}(\xi_1)=\tilde{G}(\xi_1),\quad \tilde{h}(\xi_2)p_2^{-1}(\xi_2)=\tilde{H}(\xi_2).$$

Lemma 3. *The functions r and q are positively homogeneous of the first degree, and the kernels K, L, and M are positively homogeneous of degree -1.*

Proof. The assertion about the functions r and q is obvious, and hence they have the form

$$r(t) = \begin{cases} r_1 t, & t > 0 \\ r_2 t, & t < 0, \end{cases} \qquad q(t) = \begin{cases} q_1 t, & t > 0 \\ q_2 t, & t < 0, \end{cases}$$

where $r_1, r_2, q_1, q_2 \in \mathbb{C}$.

Consider, for example, the kernel $M(\xi_1, \xi_2)$. We verify that

$$M(t\xi_1, t\xi_2) = p_2^{-1}(t\xi_2) A_{\neq}^{-1}(t\xi_1, t\xi_2) = t^{\varkappa_1+\varkappa_2-1} p_2(\xi_2) t^{-\varkappa_1-\varkappa_2} A_{\neq}^{-1}(\xi_1, \xi_2),$$

as desired. The proof of the lemma is complete.

Further, we write system (8) in the form

$$c_1(\xi_1) + r(\xi_1)c_2(\xi_1) + \int\limits_0^{+\infty} K(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 + \int\limits_{-\infty}^{0} K(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 = \tilde{F}(\xi_1),$$

$$c_1(\xi_1) + q(\xi_1)c_2(\xi_1) + \int\limits_0^{+\infty} L(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 + \int\limits_{-\infty}^{0} L(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 = \tilde{G}(\xi_1),$$

$$\int\limits_0^{+\infty} M(\xi_1, \xi_2)c_1(\xi_1)\, d\xi_1 + \int\limits_{-\infty}^{0} M(\xi_1, \xi_2)c_1(\xi_1)\, d\xi_1 + \xi_2 \int\limits_0^{+\infty} M(\xi_1, \xi_2)c_2(\xi_1)\, d\xi_1$$
$$+ \xi_2 \int\limits_{-\infty}^{0} M(\xi_1, \xi_2)c_2(\xi_1)\, d\xi_1 + d_1(\xi_2) = \tilde{H}(\xi_2).$$

Replacing the integration variable in the integrals over the negative half-line by a variable with the opposite sign, we obtain a new system with integrals over the positive semiaxis,

$$\begin{gathered} c_1(\xi_1) + r(\xi_1)c_2(\xi_1) + \int\limits_0^{+\infty} K(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 + \int\limits_0^{+\infty} K(\xi_1, -\xi_2)\, d_1(-\xi_2)\, d\xi_2 = \tilde{F}(\xi_1), \\ c_1(\xi_1) + q(\xi_1)c_2(\xi_1) + \int\limits_0^{+\infty} L(\xi_1, \xi_2)\, d_1(\xi_2)\, d\xi_2 + \int\limits_0^{+\infty} L(\xi_1, -\xi_2)\, d_1(-\xi_2)\, d\xi_2 = \tilde{G}(\xi_1), \\ \int\limits_0^{+\infty} M(\xi_1, \xi_2)c_1(\xi_1)\, d\xi_1 + \int\limits_0^{+\infty} M(-\xi_1, \xi_2)c_1(-\xi_1)\, d\xi_1 + \xi_2 \int\limits_0^{+\infty} M(\xi_1, \xi_2)c_2(\xi_1)\, d\xi_1 \\ + \xi_2 \int\limits_0^{+\infty} M(-\xi_1, \xi_2)c_2(-\xi_1)\, d\xi_1 + d_1(\xi_2) = \tilde{H}(\xi_2). \end{gathered} \tag{9}$$

Now we transform this system by increasing the number of unknowns and making all functions and kernels appearing in it defined only for positive values of the arguments. Let us introduce the following notation for $\xi_1, \xi_2 > 0$:

$$\begin{aligned} &c_{11}(\xi_1) = c_1(\xi_1), && c_{12}(\xi_1) = c_1(-\xi_1), && c_{21}(\xi_1) = c_2(\xi_1), && c_{22}(\xi_1) = c_2(-\xi_1), \\ &d_{11}(\xi_2) = d_1(\xi_2), && d_{12}(\xi_2) = d_1(-\xi_2), && F_1(\xi_1) = \tilde{F}(\xi_1), && F_2(\xi_1) = \tilde{F}(-\xi_1), \\ &G_1(\xi_1) = \tilde{G}(\xi_1), && G_2(\xi_1) = \tilde{G}(-\xi_1), && H_1(\xi_2) = \tilde{H}(\xi_2), && H_2(\xi_2) = \tilde{H}(-\xi_2). \end{aligned}$$

Given the kernels K, L, and M, we define new kernels for positive values of the arguments,

$$K_{11}(\xi_1,\xi_2)=K(\xi_1,\xi_2),\qquad K_{12}(\xi_1,\xi_2)=K(\xi_1,-\xi_2),$$
$$K_{21}(\xi_1,\xi_2)=K(-\xi_1,\xi_2),\qquad K_{22}(\xi_1,\xi_2)=K(-\xi_1,-\xi_2);$$

$L_{ij}(\xi_1,\xi_2)$ and $M_{ij}(\xi_1,\xi_2)$, $i,j=1,2$, are defined in a similar way.

System (9) takes the form of a 6×6 system of linear integral equations on the positive half-line,

$$c_{11}(\xi_1)+r_1\xi_1c_{21}(\xi_1)+\int\limits_0^{+\infty}K_{11}(\xi_1,\xi_2)\,d_{11}(\xi_2)\,d\xi_2+\int\limits_0^{+\infty}K_{12}(\xi_1,\xi_2)\,d_{12}(\xi_2)\,d\xi_2=F_1(\xi_1),$$

$$c_{11}(\xi_1)+q_1\xi_1c_{21}(\xi_1)+\int\limits_0^{+\infty}L_{11}(\xi_1,\xi_2)\,d_{11}(\xi_2)\,d\xi_2+\int\limits_0^{+\infty}L_{12}(\xi_1,\xi_2)\,d_{12}(\xi_2)\,d\xi_2=G_1(\xi_1),$$

$$\int\limits_0^{+\infty}M_{11}(\xi_1,\xi_2)c_{11}(\xi_1)\,d\xi_1+\int\limits_0^{+\infty}M_{21}(\xi_1,\xi_2)c_{12}(\xi_1)\,d\xi_1+\xi_2\int\limits_0^{+\infty}M_{11}(\xi_1,\xi_2)c_{21}(\xi_1)\,d\xi_1$$
$$+\,\xi_2\int\limits_0^{+\infty}M_{21}(\xi_1,\xi_2)c_{22}(\xi_1)\,d\xi_1+\,d_{11}(\xi_2)=H_1(\xi_2),$$

$$c_{12}(\xi_1)+r_2\xi_1c_{22}(\xi_1)+\int\limits_0^{+\infty}K_{21}(\xi_1,\xi_2)\,d_{11}(\xi_2)\,d\xi_2+\int\limits_0^{+\infty}K_{22}(\xi_1,\xi_2)\,d_{12}(\xi_2)\,d\xi_2=F_2(\xi_1),$$

$$c_{12}(\xi_1)+q_2\xi_1c_{22}(\xi_1)+\int\limits_0^{+\infty}L_{21}(\xi_1,\xi_2)\,d_{11}(\xi_2)\,d\xi_2+\int\limits_0^{+\infty}L_{22}(\xi_1,\xi_2)\,d_{12}(\xi_2)\,d\xi_2=G_2(\xi_1),$$

$$\int\limits_0^{+\infty}M_{12}(\xi_1,\xi_2)c_{11}(\xi_1)\,d\xi_1+\int\limits_0^{+\infty}M_{22}(\xi_1,\xi_2)c_{12}(\xi_1)\,d\xi_1+\xi_2\int\limits_0^{+\infty}M_{12}(\xi_1,\xi_2)c_{21}(\xi_1)\,d\xi_1$$
$$+\,\xi_2\int\limits_0^{+\infty}M_{22}(\xi_1,\xi_2)c_{22}(\xi_1)\,d\xi_1+\,d_{12}(\xi_2)=H_2(\xi_2).$$

We apply the Mellin transform [25]

$$\hat f(\lambda)=\int\limits_0^{+\infty}f(t)t^{\lambda-1}dt,\quad \lambda=s+i\sigma,$$

to this system and, as a result, taking into account the properties

$$\widehat{tf(t)}(\lambda)=\hat f(\lambda+1)$$

of the Mellin transform, we obtain the system of first-order difference equations

$$\hat c_{11}(\lambda)+r_1\hat c_{21}(\lambda+1)+\hat K_{11}(\lambda)\,\hat d_{11}(\lambda)+\hat K_{12}(\lambda)\,\hat d_{12}(\lambda)=\hat F_1(\lambda),$$
$$\hat c_{12}(\lambda)+r_2\hat c_{22}(\lambda+1)+\hat K_{21}(\lambda)\,\hat d_{11}(\lambda)+\hat K_{22}(\lambda)\,\hat d_{12}(\lambda)=\hat F_2(\lambda),$$
$$\hat c_{11}(\lambda)+q_1\hat c_{21}(\lambda+1)+\hat L_{11}(\lambda)\,\hat d_{11}(\lambda)+\hat L_{12}(\lambda)\,\hat d_{12}(\lambda)=\hat G_1(\lambda),$$
$$\hat c_{12}(\lambda)+q_2\hat c_{22}(\lambda+1)+\hat L_{21}(\lambda)\,\hat d_{11}(\lambda)+\hat L_{22}(\lambda)\,\hat d_{12}(\lambda)=\hat G_1(\lambda),$$

$$\begin{aligned}&\hat{M}_{11}(\lambda)\hat{c}_{11}(\lambda)+\hat{M}_{21}(\lambda)\hat{c}_{12}(\lambda)+\hat{M}_{11}(\lambda+1)\hat{c}_{21}(\lambda+1)\\&\qquad\qquad+\hat{M}_{21}(\lambda+1)\hat{c}_{22}(\lambda+1)+\hat{d}_{11}(\lambda)=\hat{H}_1(\lambda),\\&\hat{M}_{12}(\lambda)\hat{c}_{11}(\lambda)+\hat{M}_{22}(\lambda)\hat{c}_{12}(\lambda)+\hat{M}_{12}(\lambda+1)\hat{c}_{21}(\lambda+1)\\&\qquad\qquad+\hat{M}_{22}(\lambda+1)\hat{c}_{22}(\lambda+1)+\hat{d}_{12}(\lambda)=\hat{H}_2(\lambda),\end{aligned}\tag{10}$$

where $\hat{K}_{ij}(\lambda)$ and $\hat{L}_{ij}(\lambda)$ are the Mellin transforms of the functions $K_{ij}(t,1)$ and $L_{ij}(t,1)$ and the $\hat{M}_{ij}(\lambda)$ are the Mellin transforms of the functions $M_{ij}(1,t)$, $i,j=1,2$.

Thus, the following assertion holds.

Theorem 5. *If the function possesses the property of generalized positive homogeneity, i.e.,*

$$A_{\neq}(t\xi_1,t\xi_2)=t^{\varkappa_1+\varkappa_2}A_{\neq}(\xi_1,\xi_2)$$

for all $t>0$, *then the system of integral equations* (7) *can be reduced to the* 6×6 *system of first-order difference equations* (10).

CONCLUSIONS

The simplest version of the boundary value problem in the Sobolev–Slobodetsky space with different smoothness in different variables is described. Unfortunately, the formula for the general solution in the multidimensional case is too cumbersome to write down and study the general boundary value problem, but in some cases meaningful results can be obtained. The authors intend to continue work in this direction.

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