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REVIEW

Overview of fractional calculus and its computer implementation in Wolfram Mathematica

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Abstract

This survey aims to present various approaches to non-integer integrals and derivatives and their practical implementation within Wolfram Mathematica. It begins by short discussion of historical moments and applications related to fractional calculus. Different methods for handling non-integer powers of differentiation operators are presented, along with generalizations of fractional integrals and derivatives. The survey also delves into the diverse applications of fractional calculus in physics, engineering, medicine, and numerical calculations. Essential details of fractional integro-differentiation implemented in Wolfram Mathematica are highlighted. The Hadamard regularization of Riemann-Liouville operator is utilized as the foundation for creating the arbitrary order of integro-differential operator in Mathematica. The survey describes the application of fractional integro-differentiation to Taylor series expansions near zero using Hadamard regularization and the use of the Meijer G-function for evaluating derivatives of complex orders. We conclude with a discussion on applying fractional integro-differentiation to "differential constants" and provide generic formulas for fractional differentiation. The extensive list of references underscores the vast body of works on fractional calculus.

Keywords Fractional calculus (primary) · Wolfram language · Wolfram Mathematica · Meijer G-function

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1 Introduction

In this survey, we would like to consider some approaches to non-integer integro-differentiation and its implementation in the computer algebra system Wolfram Mathematica.

In Section 2, some historical moments and applications are briefly discussed. The ideas that initiated the era of fractional calculus are presented in Subsection 2.1. Different approaches to non-integer powers of $\frac{d}{dx}$ are provided in Subsection 2.2. Generalizations of fractional derivatives, such as operators with a power-logarithmic kernel, non-integer powers of $x\frac{d}{dx}$, and $\frac{1}{x^\gamma}\frac{d}{dx}x^\gamma\frac{d}{dx}$, fractional derivatives of a function f with respect to another function g, Erdélyi-Kober type operators, variable-order differential operators, and sequential fractional derivatives are described in Subsection 2.3. Fractional calculus has diverse and widespread applications. References to some of numerous works where fractional calculus is applied to physics, engineering, medicine, as well as some approaches to the numerical calculation of fractional derivatives are presented in Subsection 2.3.5.

In Section 3, some essential details of fractional integro-differentiation, which have been implemented in Wolfram Mathematica, are presented. This section forms a new perspective on fractional differentiation for analytic functions of complex variables. The Hadamard regularized Riemann-Liouville operator serves as the basis for creating the integro-differential operator in Wolfram Mathematica. The Riemann-Liouville-Hadamard integro-differentiation of an arbitrary function to an arbitrary symbolic order α in the Wolfram Language is described in Subsection 3.1. Subsection 3.2 presents an approach in which fractional integro-differentiation is applied to each term of Taylor series expansions of all functions near zero using Hadamard regularization. For the evaluation of the derivative of an arbitrary complex order, the Meijer G-function can be utilized. Many functions can be represented as a Meijer G-function, facilitating the finding of an α-derivative of the Meijer G-function. This approach is detailed in Subsection 3.3. Subsection 3.4 discusses how fractional integro-differentiation is applied to "differential constants", which are not constants everywhere but vary on some domains of the complex plane, exhibiting discontinuity on certain lines. This subsection also includes the description of generic formulas for fractional differentiation.

The list of references contains numerous items, but the total number of works on fractional calculus is however immense, indeed.

2 History, nowadays and applications

2.1 Background and concept

2.1.1 Gamma function and the idea of non-integer derivative

Calculus teaches us how to compute derivatives of any integer order. We can interpret the differentiation of negative integer order as repeated integration. The zeroth order of differentiation gives the function itself. The question is how to generalize derivatives to non-integer orders.

For extensive details on the history of fractional calculus, we refer readers to the handbook [115], especially to the numerous historical notes there. More information on the main milestones of fractional calculus can be found in [57, 106]. The history of fractional calculus in recent times is presented in [68, 69] and in [42], Vol. 1, 1–22 (2019). The first application of fractional calculus was discovered by Abel (see [96] for details).

The starting point for the transition from integer-order to fractional-order derivatives and integrals is Euler's introduction of the Gamma function. Namely, in 1729, Euler derived the integral representation for the factorial $n!$, leading to the Gamma function $\Gamma(n+1) = n!$, allowing the definition of $n!$ for any complex number n except $0, -1, -2, \ldots$. Besides the Gamma function, there are other ways to generalize the factorial to complex numbers (see [73], p. 35–36, [74]).

Euler introduced the Gamma function through the integral, which later became known as the Euler integral of the second kind:

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x}\, dx, \qquad \mathrm{Re}(\alpha) > 0. \tag{2.1}$$

The property $\Gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\alpha}$ can be used to uniquely extend $\Gamma(\alpha)$ to a meromorphic function defined for all complex numbers α except integers less than or equal to zero. If α=0, -1, -2, ... the Gamma function has simple poles.

The Bohr–Mollerup theorem [5] shows that the Gamma function is the only function that satisfies the properties

1. $f(1) = 1$,
2. $f(x+1) = xf(x)$,
3. for every $x \geq 0$ $\ln f$ is a convex function.

So the Euler's Gamma function is the "best" extension of the factorial function to the real (complex) numbers. The Gamma function in Wolfram Mathematica, denoted as `Gamma[z]`, is suitable for both symbolic and numerical manipulation, $z \in \mathbb{C}$.

This generalization of the factorial allowed Euler to realize that the concept of the n-th order derivative of the power function x^p acquired meaning for a non-integer n.

Namely, let $n \in \mathbb{N}$, $x > 0$ and $p \in \mathbb{R}$. It is obvious that

$$(x^p)^{(n)} = p(p-1)...(p-n+1)x^{p-n}$$

or

$$(x^p)^{(n)} = \frac{p!}{(p-n)!} x^{p-n}. \tag{2.2}$$

The expression (2.2) has sense for non-integer n when we use the Gamma function (2.1). The derivative of x^p of non-integer order α can be defined by

$$\frac{d^\alpha}{dx^\alpha}\,x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)}\,x^{p-\alpha}. \tag{2.3}$$

It should be noted that formula (2.3) is valid not for all values p and α. Therefore, we need to exclude the arguments that result in poles of the Gamma function in the numerator. That means that in (2.3)

$$-(p+1) \notin \mathbb{N} \cup \{0\}.$$

We know that for $\alpha = 0$ we just get the $\frac{d^0}{dx^0}\,x^p = x^p$ and for $\alpha = -1$ we get integral and for $\alpha = -2, -3, ...$ iterated integrals. For example, if $p = -1$ and $\alpha = -1$ we obtain

$$\frac{d^{-1}}{dx^{-1}}\,x^{-1} = \int \frac{dx}{x} = \log(x) + C.$$

This case already results in a logarithm rather than a power function. Let us discuss how we get a logarithm here. We can say that this is obtained by regularizing the integral $\int x^p dx$ and then taking a limit $p \to -1$.

Let us see how to obtain the logarithm from formula below, including an arbitrary constant C:

$$\int x^p dx = \frac{x^{p+1}}{p+1} + C$$

by taking the limit.

We can change this constant to another constant $-\frac{1}{p+1}+C_1$, including parameter p and a constant C_1.

After evaluating the integral and taking the limit, we get

$$\int x^p dx = \frac{x^{p+1}}{p+1} - \frac{1}{p+1} + C_1 = \frac{x^{p+1}-1}{p+1} + C_1,$$

and also

$$\int \frac{dx}{x} = \lim_{p\to -1} \int x^p dx = \lim_{p\to -1} \frac{x^{p+1}-1}{p+1} + C_1 = \log(x) + C_1.$$

Here, we applied regularization using the term $\left(-\frac{1}{p+1}\right)$. Integration implemented in Wolfram Mathematica also uses regularization when necessary. The general approach to finding an arbitrary order derivative of a power function is considered in Subsection 3.2.

Therefore, if function $f(x)$ is locally given by a convergent power series or $f(x)$ is an analytic function:

$$f(x) = \sum_{p=0}^{\infty} a_p x^p, \qquad a_p = \frac{f^{(p)}(0)}{p!},$$

then the derivative of order $\alpha > 0$ can be formally defined as

$$\frac{d^\alpha f(x)}{dx^\alpha} = \sum_{p=0}^{\infty} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} a_p x^{p-\alpha}.$$

To ensure the existence of such a fractional derivative, it is necessary to verify the convergence of the specified series. Here, $p \geq 0$ allow to avoid the poles of the Gamma function in the numerator. The general case when α and p can be negative is considered in Subsection 3.2.

In 1823, the first application of fractional calculus was discovered by Abel (see for details [57, 96]). Abel was searching for a curve in the plane such that the time required for a particle to slide down the curve to its lowest point under the influence of gravity is independent of its initial position on the curve. Such a curve is called the tautochrone. Abel found that to find this curve, it is necessary to solve the equation

$$\int_0^x \frac{f(t)dt}{(x-t)^{1/2}} = \varphi(x) = T = \text{const}, \qquad x > 0.$$

It is worth noting that Abel solved a more general equation

$$\int_0^x \frac{f(t)dt}{(x-t)^\alpha} = \varphi(x), \qquad x > 0,$$

where $0 < \alpha < 1$.

Furthermore, in 1832, Liouville formally extended formula for the integer derivative of the exponent $\frac{d^n}{dx^n} e^{bx}$ (b is some number) to derivatives of arbitrary order $\frac{d^\alpha}{dx^\alpha} e^{bx}$. Namely,

$$\frac{d^\alpha e^{bx}}{dx^\alpha} = b^\alpha e^{bx}. \tag{2.4}$$

Based on formula (2.4), one can formally write the derivative of order $\alpha \in \mathbb{R}$ of an arbitrary function f represented by the series

$$\frac{d^\alpha f(x)}{dx^\alpha} = \sum_{k=0}^{\infty} c_k b_k^\alpha e^{b_k x}, \qquad \text{where} \qquad f(x) = \sum_{k=0}^{\infty} c_k e^{b_k x}.$$

The limitation of this definition is related to the convergence of the series.

2.1.2 Formal definition of fractional integro-differentiation

So, starting from the 17th century, the need for a formal definition of fractional integro-differentiation gradually formed. We present here such a definition following [106].

Let $z \in \mathbb{C}, x \in \mathbb{R}$ be variables. The starting point in fractional integro-differentiation is 0, $\nu \in \mathbb{C}$ or $\nu \in \mathbb{R}$ is an order of integro-differentiation. Fractional integro-differentiation D^ν acts to $f(z)$ by z and acts to $f(x)$ by x. In [106], one can find the following criteria of fractional integro-differentiation.

Criteria 1 *Operator $D^\nu f(z)$ is the integro-differential operator of order $\nu \in \mathbb{C}$ if and only if*

1. *If $f(z)$ is an analytic function of the complex variable z, the derivative $D^\nu f(z)$ is an analytic function of ν and z.*
2. *The operation $D^\nu f(z)$ must produce the same result as ordinary differentiation when ν is a positive integer. If ν is a negative integer, say $\nu = -n$, then $D^{-n} f(z)$ must produce the same result as ordinary n-fold integration and $D^{-n} f(z)$ must vanish along with its $(n-1)$-derivatives at $z = 0$.*
3. *The operation of order zero leaves the function unchanged:*

$$D^0 f(z) = f(z).$$

4. *The fractional operators must be linear:*

$$D^\nu[af(z) + bg(z)] = aD^\nu f(z) + bD^\nu g(z).$$

5. *The semigroup property of arbitrary order for suitable function f holds:*

$$D^\nu D^\mu f(z) = D^{\nu+\mu} f(z).$$

For fractional integral operators, the semigroup property is most often satisfied for all functions for which this integral exists. However, for fractional derivatives, it is necessary to construct a suitable class of functions to satisfy the semigroup property (see, for example, [115], p. 34 and p. 45 and further).

In the next Subsections 2.2 and 2.3, we examine different operators that realize non-integer powers of differentiations and integrations.

2.2 Different approaches to classical fractional calculus

In this subsection, we consider different classical approaches to introduce a non-integer power $\left(\frac{d}{dx}\right)^\alpha$ of the differentiation operator $\frac{d}{dx}$. If an operator has two forms, left-sided and right-sided, we will consider only the left-sided.

2.2.1 Riemann-Liouville type fractional integrals and derivatives

The most well known definitions of operators giving arbitrary real (complex) powers of the $\frac{d}{dx}$ which fulfill Criteria 1 are

$$(I^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad a < x \leq b, \quad \alpha > 0, \tag{2.5}$$

and

$$(D^{\alpha}_{a+}f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_a^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \quad a < x \leq b, \quad \alpha > 0, \tag{2.6}$$

where $n = [\alpha] + 1$, when α in not integer and $n = \alpha$, when $\alpha \in \mathbb{Z}_+$. The definition of the operator (2.5) is valid for $\mathrm{Re}(\alpha) > 0$ and the definition of the operator (2.6) is valid for $n - 1 \leq \mathrm{Re}(\alpha) < n$.

Operators (2.5) and (2.6) are called left-sided Riemann-Liouville integral and derivative on the segment $[a, b]$, respectively. The definition of fractional integral is based on a generalization of the formula for an n-fold integral

$$\underbrace{\int\limits_a^x dx ... \int\limits_a^x dx \int\limits_a^x}_{n} f(x)dx = \frac{1}{(n-1)!} \int\limits_a^x (x-t)^{n-1} f(t)dt, \qquad x > a. \tag{2.7}$$

Then the fractional derivative (2.6) is obtained as a left inverse operator to (2.5) which is constructed as a solution to the Abel equation. It is necessary to note that the operator (2.6) is one of infinitely many different left-inverse operators to the fractional Riemann-Liouville integral (2.5).

If we consider $D^{\alpha}_{a+}x^p$, $x > 0$, $p \in \mathbb{R}$ we obtain

$$\begin{aligned} D^{\alpha}_{a+}x^p &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \left(\int\limits_a^x \frac{t^p dt}{(x-t)^{\alpha-n+1}} \right) \\ &= \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n x^{p-\alpha+n} \left(\frac{\Gamma(p+1)\Gamma(n-\alpha)}{\Gamma(p+n-\alpha+1)} - B_{\frac{a}{x}}(p+1, n-\alpha) \right) \\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} - \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n x^{p-\alpha+n} B_{\frac{a}{x}}(p+1, n-\alpha), \end{aligned}$$

where $B_z(a, b)$ is the incomplete Beta function

$$B_z(a,b) = \int\limits_0^z t^{a-1}(1-t)^{b-1}dt.$$

It is easy to see that in order for the resulting formula to correspond to (2.3), we need to set $a = 0$, when the last term disappears:

$$D_{0+}^{\alpha} x^p = \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}.$$

Because of this, in Section 3 we will consider only fractional derivatives and integrals tied to the origin, i.e. with $a = 0$.

Liouville fractional integral of order α on the semiaxis $(0,\infty)$ for $f \in L_1(0,\infty)$ has the form

$$(I_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad x \in (0,\infty). \tag{2.8}$$

Liouville fractional integral of order α on the whole real axis is

$$(I_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{-\infty}^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \qquad x \in \mathbb{R}. \tag{2.9}$$

Liouville fractional derivatives of order α on the semi-axis and on the whole real axis are, respectively

$$(\mathcal{D}_{0+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_0^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \qquad x \in (0,+\infty), \tag{2.10}$$

$$(\mathcal{D}_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dx}\right)^n \int\limits_{-\infty}^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}}, \qquad x \in \mathbb{R}, \tag{2.11}$$

where $n = [\alpha] + 1$, when α in not integer and $n = \alpha$, when $\alpha \in \mathbb{Z}_+$.

Next, we consider a modification of the Riemann-Liouville integration, for which lower and upper integral limits are symmetric. For fixed arbitrary point $c \in \mathbb{R}$ and $\alpha > 0$ operator

$$(I_c^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \begin{cases} \int\limits_c^x (x-t)^{\alpha-1} f(t)dt & \text{if } x > c; \\ \int\limits_x^c (t-x)^{\alpha-1} f(t)dt & \text{if } x < c \end{cases} \tag{2.12}$$

is called the Chen fractional integral.

Let $\alpha > 0$ be not integer, $n = [\alpha] + 1$ then Chen fractional derivative is

$$(D_c^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \begin{cases} \left(\frac{d}{dx}\right)^n \int\limits_c^x \frac{f(t)dt}{(x-t)^{\alpha-n+1}} & \text{if } x > c; \\ \left(-\frac{d}{dx}\right)^n \int\limits_x^c \frac{f(t)dt}{(t-x)^{\alpha-n+1}} & \text{if } x < c. \end{cases} \tag{2.13}$$

One of the most popular fractional derivative that is used now is the Caputo fractional derivative which has the form

$$({}^C D_{0+}^\alpha f)(x) = \frac{1}{\Gamma(n-\alpha)} \int\limits_0^x \frac{f^{(n)}(y)dy}{(x-y)^{\alpha-n+1}}, \qquad n = [\alpha] + 1, \qquad x > 0. \tag{2.14}$$

Operator (2.14) with $n = 1$ was introduced already by Abel, see the paper [96]. Original works of Italian physicist M. Caputo are [11] and [12].

In 1948 (see [37], submitted in 1947) the Soviet mechanics A.N. Gerasimov introduced fractional derivative of the form

$$\frac{1}{\Gamma(1-\alpha)} \int\limits_{-\infty}^x \frac{f'(y)dy}{(x-y)^\alpha}, \quad y > 0, \quad x \in \mathbb{R}, \quad 0 < \alpha < 1. \tag{2.15}$$

In the same work, A.N. Gerasimov studied two new problems in viscoelasticity theory. He reduced this problem to differential equations with partial fractional derivative.

Let us notice that in the article [96] Abel's contribution to the theory of fractional calculus was carefully analyzed, in particular the appearance of the derivative (2.14) for $\alpha \in (0, 1)$.

The popularity of the fractional derivative (2.14) in applications is explained as follows. If we consider the fractional differential equation with Riemann–Liouville fractional derivative of the form

$$(D_{0+}^\alpha f)(x) = \lambda f(x), \qquad x > 0, \qquad 0 < \alpha < 1, \qquad \lambda \in \mathbb{R}$$

we should add the initial condition

$$(D_{0+}^{\alpha-1} f)(0+) = 1$$

and the solution of this problem is (see [49])

$$f(x) = x^{\alpha-1} E_{\alpha,\alpha}(\lambda x^\alpha) \tag{2.16}$$

where $E_{\alpha,\beta}(\lambda x^\alpha) = \sum\limits_{k=0}^\infty \frac{(\lambda x^\alpha)^k}{\Gamma(\alpha k+\beta)}$ is the Mittag–Leffler function (see [38], [42], Vol. 1, 269–296 (2019)). Note that we have a singularity at zero in (2.16). Therefore, it is not possible to consider the classical Cauchy problem at zero initial point with Riemann–Liouville fractional derivative.

From the other side, the solution to the Cauchy problem for fractional differential equation with fractional derivative (2.14)

$$\begin{aligned}({}^C D^\alpha_{0+} f)(x) &= \lambda f(x), \qquad x > 0, \qquad 0 < \alpha \leq 1, \qquad \lambda \in \mathbb{R}\\ f(0+) &= 1,\end{aligned}$$

is (see [49])

$$f(x) = E_{\alpha,1}(\lambda x^\alpha). \tag{2.17}$$

Now it is bounded at zero and so the classical Cauchy problem is correct.

Let us find the Riemann–Liouville derivative of (2.17):

$$\begin{aligned}D^\alpha_{0+} E_{\alpha,1}(\lambda x^\alpha) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} D^\alpha_{0+} x^{\alpha k}\\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k+1)} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k-\alpha+1)} x^{\alpha k-\alpha}\\ &= \sum_{k=0}^{\infty} \frac{\lambda^k x^{\alpha(k-1)}}{\Gamma(\alpha k-\alpha+1)}\\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \lambda \sum_{k=1}^{\infty} \frac{(\lambda x^\alpha)^{k-1}}{\Gamma(\alpha(k-1)+1)}\\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \lambda \sum_{k=0}^{\infty} \frac{(\lambda x^\alpha)^{k}}{\Gamma(\alpha k+1)}\\ &= \frac{x^{-\alpha}}{\Gamma(1-\alpha)} + \lambda E_{\alpha,1}(\lambda x^\alpha).\end{aligned}$$

So we see that the term $\frac{x^{-\alpha}}{\Gamma(1-\alpha)}$ has a singularity at $x = 0$.

Then, consider ${}^C D^\alpha_{0+} x^p$, $p \in \mathbb{R}$:

$$\begin{aligned}{}^C D^\alpha_{0+} x^p &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)\Gamma(n-\alpha)} \int\limits_0^x \frac{y^{p-n}dy}{(x-y)^{\alpha-n+1}}\\ &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)\Gamma(n-\alpha)} \frac{\Gamma(n-\alpha)\Gamma(p-n+1)x^{p-\alpha}}{\Gamma(p-\alpha+1)}\\ &= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}.\end{aligned}$$

Let us note that integration can only be done with $p > n-1$, where $n = [\alpha] + 1$. Therefore, we have serious limitations and cannot, generally speaking, apply the derivative (2.14) to term by term to each member of the power series. In order to get around this problem, we put

$$^C D^{\alpha}_{0+} x^p = \begin{cases} \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha} & \text{if } p > n-1; \\ 0 & \text{if } p \le n-1 \end{cases}$$

then we can find $^C D^{\alpha}_{0+} E_{\alpha,1}(\lambda x^{\alpha})$ by the term-by-term differentiation of the series

$$\begin{aligned} ^C D^{\alpha}_{0+} E_{\alpha,1}(\lambda x^{\alpha}) &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma(\alpha k + 1)} \, ^C D^{\alpha}_{0+} x^{\alpha k} \\ &= \lambda \sum_{k=1}^{\infty} \frac{(\lambda x^{\alpha})^{k-1}}{\Gamma(\alpha(k-1)+1)} = \lambda E_{\alpha,1}(\lambda x^{\alpha}). \end{aligned}$$

So here we saw significant limitations when using a fractional derivative (2.14).

In addition to the Caputo derivative, various other modifications of the Riemann-Liouville derivative appeared. We present here some of them.

Although we only consider left-sided operators, we will mention one right-sided fractional integral of the form

$$(I^{\alpha}_{-} f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (y-x)^{\alpha-1} f(y) dy.$$

Some authors call it Weyl fractional order integral of function $f(y)$ (see [135]).

Cossar in [15] presented the following fractional derivative

$$(\mathcal{D}^{\alpha}_{-} f)(x) = -\frac{1}{\Gamma(1-\alpha)} \lim_{N\to\infty} \frac{d}{dx} \int_x^N \frac{f(y)}{(y-x)^{\alpha}} dy, \qquad 0 < \alpha < 1.$$

Osler fractional derivative (see [85], formula 2.2)

$$(^O D^{\alpha}_{0+} f)(x) = \frac{\Gamma(1+\alpha)}{2\pi i} \int_{C[a,z^+]} \frac{f(t)}{(t-z)^{\alpha+1}} dt, \qquad 0 < \alpha < 1.$$

where the contour $C[a, z^+]$ starts and ends at $t = 0$.

2.2.2 Finite differences approach to fractional calculus

Here we consider the finite differences approach to fractional calculus. This is an approach based on classical limit definition of derivative.

Let us consider how to generalize the derivative of order n in the form

$$f^{(n)}(x) = \lim_{h \to 0} \frac{(\Delta_h^n f)(x)}{h^n}, \tag{2.18}$$

where $(\Delta_h^n f)(x)$ is a finite difference

$$(\Delta_h^n f)(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(x-kh), \qquad \binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{2.19}$$

to the non-integer order.

Since the Gamma function is a generalization of the factorial function to non-integer values then for $\alpha > 0$ a fractional derivative can be given as a generalization of (2.18) by

$$f^{(\alpha)}(x) = \lim_{h \to +0} \frac{(\Delta_h^\alpha f)(x)}{h^\alpha}, \tag{2.20}$$

where

$$(\Delta_h^\alpha f)(x) = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} f(x-kh), \quad \binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)}. \tag{2.21}$$

For $h > 0$ and $\alpha > 0$ (2.20) is the left-hand sided Grünwald–Letnikov derivative. We can also consider the case when $h < 0$ in (2.21). The left-hand sided Grünwald–Letnikov fractional integral in a real line is

$$f^{(-\alpha)}(x) = \lim_{h \to +0} h^\alpha (\Delta_h^{-\alpha} f)(x). \tag{2.22}$$

Let us find Grünwald–Letnikov derivative of the power function x^p, $x > 0$, $p \in \mathbb{R}$. Since the definition (2.20) was given for whole real line, we consider a function

$$f_p(x) = \begin{cases} x^p, & x > 0; \\ 0, & x \leqslant 0. \end{cases}$$

Then f_p can be written as an inverse Laplace transform

$$f_p(x) = \frac{\Gamma(p+1)}{2\pi i} \int_{c-i\infty}^{c+i\infty} s^{-p-1} e^{xs}\, ds \qquad (c > 0),$$

then

$$\begin{aligned}
f_p^{(\alpha)}(x) &= \frac{\Gamma(p+1)}{2\pi i} \lim_{h\to+0} \frac{1}{h^\alpha} (\Delta_h^\alpha)_x \int\limits_{c-i\infty}^{c+i\infty} s^{-p-1} e^{xs}\, ds \\
&= \frac{\Gamma(p+1)}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} e^{xs} s^{-p-1} \lim_{h\to+0} h^{-\alpha} \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} e^{-khs} ds \\
&= \frac{\Gamma(p+1)}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} e^{xs} s^{-p-1} \lim_{h\to+0} h^{-\alpha} (1-e^{-hs})^\alpha ds \\
&= \frac{\Gamma(p+1)}{2\pi i} \int\limits_{c-i\infty}^{c+i\infty} e^{xs} s^{\alpha-p-1} ds \\
&= \frac{\Gamma(p+1)}{\Gamma(p-\alpha+1)} x^{p-\alpha}
\end{aligned}$$

that coincides with $D_{0+}^\alpha x^p$ for $x > 0$. Here we consider first $\int\limits_{c-iA}^{c+iA}$, then change limits, then take $A \to \infty$. Also we used Lebesgue's dominated convergence theorem.

The definition of Marchaud derivative based on finite differences approach (2.21) is

$$(\mathbf{D}_+^\alpha f)(x) = -\frac{1}{\Gamma(-\alpha) A_l(\alpha)} \int\limits_0^\infty \frac{(\Delta_t^l f)(x)}{t^{\alpha+1}} dt, \quad A_l(\alpha) = \sum_{k=0}^{l} (-1)^{k-1} \binom{l}{k} k^\alpha, \tag{2.23}$$

where $0 < \operatorname{Re} \alpha < l$, $l \in \mathbb{N}$ or $l = [\alpha] + 1$ when $\alpha \in \mathbb{R}$.

Other modifications to fractional derivatives constructed from fractional differences are presented in Chapter 4 of book [115]. For example, Grünwald–Letnikov–Riesz fractional derivative of order $\alpha > 0$ was defined by

$$({}^{GLR}D_x^\alpha f)(x) = \frac{1}{2\cos(\alpha\pi/2)} \lim_{h\to 0+} \frac{(\Delta_h^\alpha f)(x) + (\Delta_{-h}^\alpha f)(x)}{|h|^\alpha} \tag{2.24}$$

where the difference $(\Delta_h^\alpha f)(x)$ of a fractional order $\alpha > 0$ is defined by the series (2.21).

Grünwald–Letnikov operators (2.20) and (2.22), Marchaud derivative (2.23) and Grünwald–Letnikov–Riesz fractional derivative (2.24) are considered in [60, 84, 115].

2.2.3 One-dimensional Riesz and Bessel potentials and their inversions

Now we consider fractional operators defined as the convolution of a function with the Riesz and Bessel kernels and their inversions. Such concept comes from the field of potential theory and harmonic analysis.

The integral

$$(I^\alpha f)(x) = \frac{1}{2\Gamma(\alpha)\cos\left(\frac{\pi}{2}\alpha\right)} \int\limits_{-\infty}^{\infty} \frac{f(t)dt}{|t-x|^{1-\alpha}}, \qquad \operatorname{Re}\alpha > 0, \qquad \alpha \neq 1, 3, 5, ... \tag{2.25}$$

is called the Riesz potential. Along with (2.25), we consider its modification of the form

$$H^\alpha f(x) = \frac{1}{2\Gamma(\alpha)\sin\left(\frac{\pi}{2}\alpha\right)} \int\limits_{-\infty}^{\infty} \frac{\operatorname{sgn}(x-t)}{|t-x|^{1-\alpha}} f(t)dt, \quad \operatorname{Re}\alpha > 0, \quad \alpha \neq 2, 4, 6, ... \tag{2.26}$$

For $0 < \alpha < 1$ operators inverse to I^α and H^α may be constructed in the following forms:

$$((I^\alpha)^{-1} f)(x) = \frac{1}{2\Gamma(-\alpha)\cos\left(\frac{\pi}{2}\alpha\right)} \int\limits_{-\infty}^{\infty} \frac{f(x-t)-f(x)}{|t|^{1+\alpha}} dt,$$

$$((H^\alpha)^{-1} f)(x) = \frac{1}{2\Gamma(-\alpha)\sin\left(\frac{\pi}{2}\alpha\right)} \int\limits_{-\infty}^{\infty} \frac{f(x-t)-f(x)}{|t|^{1+\alpha}} \operatorname{sgn} t\, dt.$$

Next we introduce the convolution operator

$$(G^\alpha f)(x) = \int\limits_{-\infty}^{\infty} G_\alpha(x-t) f(t)dt,$$

which is defined using the Fourier transform by the equality

$$F[G^\alpha f](x) = \frac{1}{(1+|x|^2)^{\alpha/2}} F[f](x), \qquad \operatorname{Re}(\alpha) > 0. \tag{2.27}$$

The function $G_\alpha(x)$, whose Fourier transform is $(1+|x|^2)^{-\alpha/2}$, is evaluated in terms of Bessel functions. This is why the operator $G_\alpha(x)$ is referred to as a Bessel fractional integration operator or Bessel potential.

Bessel potential or Bessel fractional integral can be written in the form

$$(G^{\alpha} f)(x) = \frac{2^{\frac{1-\alpha}{2}}}{\sqrt{\pi}\Gamma\left(\frac{\alpha}{2}\right)} \int_{-\infty}^{\infty} f(t)|x-t|^{\frac{\alpha-1}{2}} K_{\frac{1-\alpha}{2}}(|x-t|)\, dt, \qquad \mathrm{Re}(\alpha) > 0.$$

2.2.4 Fractional integro-differentiation of analytic functions

Analytic functions are functions that can be represented by convergent power series in a neighborhood of each point in their domain. Fractional integro-differentiation of analytic functions is particularly relevant in the field of fractional calculus, because we can use series expansion.

Let the function $f(z) = \sum_{k=0}^{\infty} f_k z^k$, be analytic in the unit disc, so

$$(\mathfrak{D}_0^{\alpha} f)(z) = z^{-\alpha} \sum_{k=0}^{\infty} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} f_k z^k. \tag{2.28}$$

A natural way to generalize the (2.28) is to replace the factor $\frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)}$ in (2.28) by a more general one. One of such generalization is Gel'fond–Leont'ev differentiation of the form (see [35])

$$\mathfrak{D}^n(a; f) = \sum_{k=n}^{\infty} \frac{a_{k-n}}{a_k} f_k z^{k-n}, \tag{2.29}$$

where $a(z) = \sum_{k=0}^{\infty} a_k z^k$. The operator in (2.29) is called the Gel'fond–Leont'ev operator of generalized differentiation. It is obvious that $\mathfrak{D}^n(a; f) = \frac{d^n f}{dz^n}$ in the case $a(z) = e^z$.

The operator

$$\mathfrak{I}^n(a; f) = \sum_{k=0}^{\infty} \frac{a_{k+n}}{a_k} f_k z^{k+n},$$

which is the right inverse to (2.29), will be called the Gel'fond–Leont'ev operator of generalized integration. Operators $\mathfrak{D}^n$ and $\mathfrak{I}^n$ are introduced as direct generalizations concerning integer order n of integro-differentiation, they contain that for fractional order as well. To show this, let us consider the following special case when $a(z) = E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k+1)}$, $\alpha > 0$ is the Mittag-Leffler function (see Ch. 2, [52] and [38]). The corresponding operator for generalized integration of order $n = 1$ is

$$(\mathfrak{I}_{\alpha} f)(z) = \mathfrak{I}^1(E_{1/\alpha}; f) = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+\alpha+1)} f_k z^{k+1}.$$

This operator admits the following integral representation

$$(\mathfrak{I}_\alpha f)(z) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t)^{\alpha-1} f(zt^\alpha)dt.$$

Corresponding differential operator for $0 < \alpha < 1$ is

$$(\mathfrak{D}_\alpha f)(z) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{g'(z)} \frac{d}{dz} \int_0^z \frac{f(t)g'(t)}{[g(z)-g(t)]^\alpha} dt,$$

where $g(z) = z^{1/\alpha}$. The operator $\mathfrak{D}_\alpha$ corresponds to the expansion

$$(\mathfrak{D}_\alpha f)(z) = \sum_{k=1}^{\infty} \frac{\Gamma(\alpha k+1)}{\Gamma(\alpha k+1-\alpha)} f_k z^{k-1}.$$

If we consider the analytic in the unit disc function $b(z) = \sum\limits_{k=0}^{\infty} b_k z^k$ then the Hadamard product composition of functions $b(z)$ and $f(z)$ is

$$\mathfrak{D}\{b; f\} = b \circ f = \sum_{k=0}^{\infty} b_k f_k z^k. \tag{2.30}$$

The operator (2.30) is a very wide generalization of the differentiation. Under the assumption $b_k \to \infty$ a generalized integration is

$$\mathfrak{I}\{b; f\} = \sum_{k=0}^{\infty} \frac{f_k}{b_k} z^k. \tag{2.31}$$

Choosing various functions $b(z)$ in (2.30) and (2.31) we obtain integro-differentiation operations of various types. Then $b(z) = \frac{\Gamma(\alpha+1)}{(1-z)^{\alpha+1}}$ (2.30) gives Riemann-Liouville fractional differentiation of the function $z^\alpha f(z)$. If we take $b(z) = \frac{\Gamma(\alpha+1)z}{(1-z)^{\alpha+1}}$ in (2.30) we obtain the Ruscheweyh fractional derivative. Function $b(z) = \sum\limits_{k=1}^{\infty} (ik)^\alpha z^k$ gives the fractional differentiation by Weyl.

2.2.5 Weyl fractional derivative of a periodic function

The Weyl fractional derivative of a periodic function is a mathematical concept that extends the traditional notion of differentiation to non-integer orders for periodic functions.

Let $f(x)$ be a 2π-periodic function on $\mathbb{R}$ and let

$$f(x) \sim \sum_{k=-\infty}^{\infty} f_k e^{-ikx}, \qquad f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikx} f(x)dx$$

be its Fourier series. Here we will consider functions having zero mean value:

$$\int_0^{2\pi} f(x)dx = 0.$$

For periodic functions definition fractional integro-differentiation, suggested by Weyl, is

$$(I_+^{(\alpha)} f)(x) \sim \sum_{k=-\infty}^{\infty} (ik)^{-\alpha} f_k e^{ikx}, \quad f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt, \quad f_0 = 0. \tag{2.32}$$

Similarly fractional differentiation is defined:

$$(\mathcal{D}_+^{(\alpha)} f)(x) \sim \sum_{k=-\infty}^{\infty} (ik)^{\alpha} f_k e^{ikx}, \quad f_k = \frac{1}{2\pi}\int_0^{2\pi} e^{-ikt} f(t)dt, \quad f_0 = 0. \tag{2.33}$$

The definition (2.32) may be interpreted as

$$(I_+^{(\alpha)} f)(x) = \frac{1}{2\pi}\int_0^{2\pi} f(x-t)\Psi_+^{\alpha}(t)dt, \qquad \alpha > 0, \tag{2.34}$$

where

$$\Psi_+^{\alpha}(t) = 2\sum_{k=1}^{\infty} \frac{\cos(kt - \alpha\pi/2)}{k^{\alpha}} = e^{-\frac{1}{2}i\pi\alpha}\left(e^{i\pi\alpha}\mathrm{Li}_{\alpha}\left(e^{-it}\right) + \mathrm{Li}_{\alpha}\left(e^{it}\right)\right), \tag{2.35}$$

$\mathrm{Li}_{\alpha}(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^{\alpha}}$ is the fractional order polylogarithm function. The right-hand side in (2.34) is called the Weyl fractional integral of order α.

The Marchaud–Weyl derivative is defined as

$$(D_{+}^{(\alpha)} f)(x) = \frac{1}{2\pi} \int\limits_{0}^{2\pi} (f(x) - f(x-t)) \frac{d}{dt} \Psi_{+}^{\alpha}(t) dt, \qquad 0 < \alpha < 1. \quad (2.36)$$

It is known that for 2π-periodic function f, such that $f \in L_1(0, 2\pi)$ and $\int\limits_{0}^{2\pi} f(x)dx = 0$ the Weyl fractional integrals $I_{+}^{(\alpha)}$ for $0 < \alpha < 1$ coincide with the Liouville integrals on the real line I_{+}^{α} (2.9): $(I_{+}^{(\alpha)} f)(x) = (I_{+}^{\alpha} f)(x)$. Weyl approach to fractional derivative of a periodic function was considered in [115].

2.3 Generalizations of fractional derivatives and applied aspects

2.3.1 Operators with power-logarithmic kernels. Fractional powers of operators

Operators with power-logarithmic kernels are a class of integral operators that combine power and logarithmic singularities in their kernels. These operators arise in various mathematical contexts, particularly when studying integral equations of the first kind with power-logarithmic kernels. Fractional powers of operators extend the concept of raising an operator to a power.

Let $[a, b]$ be a segment, $\alpha > 0$, $\beta \geq 0$, $\gamma > b - a$. One of the direct generalizations of the fractional integral I_{a+}^{α} defined by (2.5) has the form

$$(I_{a+}^{(\alpha,\beta)} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{a}^{x} \ln^{\beta}\left(\frac{\gamma}{x-t}\right)(x-t)^{\alpha-1} f(t)dt. \quad (2.37)$$

Operators with a power-logarithmic kernel (2.37) were studied in [50] for integer β and $\gamma = 1$. In the general case these operators were considered in [51].

Riemann-Liouville fractional integro-differentiation is formally a fractional power $\left(\frac{d}{dx}\right)^{\alpha}$ of the differentiation operator $\frac{d}{dx}$ and is invariant relative to translation if considered on the whole axis. Hadamard [41] suggested a construction of fractional integro-differentiation which is a fractional power of the type $\left(x\frac{d}{dx}\right)^{\alpha}$. This construction is well suited to the case of the half-axis, and is invariant relative to dilation.

The Hadamard fractional integral has the form

$$(\mathfrak{F}_{+}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_{0}^{x} \frac{f(t)dt}{t\left(\ln\frac{x}{t}\right)^{1-\alpha}}, \qquad x > 0, \qquad \alpha > 0. \quad (2.38)$$

It is easily seen that operator $\mathfrak{F}_{+}^{\alpha}$ is connected with Liouville operators I_{+}^{α} of the form (2.9)

$$(\mathfrak{F}_{+}^{\alpha} f)(x) = A^{-1} I_{+}^{\alpha} A f, \qquad (Af)(x) = f(e^{x}).$$

For $0 < \alpha < 1$ Hadamard fractional derivative has the form

$$(\mathfrak{D}^{\alpha}_{+} f)(x) = \frac{1}{\Gamma(1-\alpha)}\, x \frac{d}{dx} \int\limits_0^x \frac{f(t)dt}{t\left(\ln \frac{x}{t}\right)^{\alpha}}.$$

We may also consider Hadamard fractional integral and derivative on a finite segment $[a, b]$.

Next we briefly consider the fractional powers $(B_\gamma)^\alpha$, $\alpha \in \mathbb{R}$ of differential Bessel operator in the form

$$B_\gamma = D^2 + \frac{\gamma}{x} D = \frac{1}{x^\gamma}\frac{d}{dx} x^\gamma \frac{d}{dx}, \qquad \gamma \geq 0, \qquad D := \frac{d}{dx}. \tag{2.39}$$

For fractional powers of (2.39), explicit formulas were derived in [76] as compositions of simpler operators. An important step was taken in [126], where explicit definitions were obtained in terms of the Gauss hypergeometric functions, with various applications to PDEs. The most comprehensive study was conducted by I. Dimovski and V. Kiryakova [21–23, 52] for a more general class of hyper-Bessel differential operators related to the Obrechkoff integral transform.

The left–sided fractional Bessel integral $B^{-\alpha}_{\gamma,a+}$ on a segment $[a, b]$ for a function $f \in L_1(a, b)$, $a, b \in (0, \infty)$, is defined by formula

$$\begin{aligned}(B^{-\alpha}_{\gamma,a+} f)(x) &= (IB^{\alpha}_{\gamma,a+} f)(x)\\ &= \frac{1}{\Gamma(2\alpha)} \int\limits_a^x \left(\frac{y}{x}\right)^{\gamma} \left(\frac{x^2-y^2}{2x}\right)^{2\alpha-1} {}_2F_1\left(\alpha+\frac{\gamma-1}{2}, \alpha; 2\alpha; 1-\frac{y^2}{x^2}\right) f(y)dy.\end{aligned} \tag{2.40}$$

Let $\alpha > 0$, $n = [\alpha] + 1$, $f \in L_1(a, b)$, $IB^{n-\alpha}_{\gamma,b-} f$, $IB^{n-\alpha}_{\gamma,a+} f \in C^{2n}(a, b)$ and even. The left-sided fractional Bessel derivatives on a segment of the Riemann-Liouville type for $\alpha \neq 0, 1, 2, \ldots$ is defined as

$$(B^{\alpha}_{\gamma,a+} f)(x) = (DB^{\alpha}_{\gamma,a+} f)(x) = B^n_\gamma (IB^{n-\alpha}_{\gamma,a+} f)(x), \qquad n = [\alpha] + 1.$$

When $\alpha = n \in \mathbb{N} \cup \{0\}$, then

$$(B^0_{\gamma,a+} f)(x) = f(x), \qquad (B^n_{\gamma,a+} f)(x) = B^n_\gamma f(x),$$

where B^n_γ is an iterated Bessel operator (2.39). Fractional Bessel integrals and derivatives were studied in [118].

2.3.2 Erdelýi-Kober-type operators and fractional integrals and derivatives of a function with respect to another function

The Erdélyi-Kober-type operators often arise in the context of transmutation operators, integral transforms and functional equations, playing a crucial role in the analysis of differential equations and in the study of special functions such as Bessel functions, hypergeometric functions, and others. Fractional integrals and derivatives with respect to another function extend the concept of fractional calculus to a more generalized setting, where the differentiation or integration operation is performed with respect to a given function rather than a variable. Some basic results concerning Erdélyi-Kober operators can be found in the book by Sneddon [125].

Let $0 \le a < x < b \le \infty$ for any $\sigma \in \mathbb{R}$ or $-\infty \le a < x < b \le \infty$ for $\sigma > 0$. Erdelýi-Kober-type operators are

$$I^{\alpha}_{a+;\,\sigma,\,\eta} f(x) = \frac{\sigma x^{-\sigma(\alpha+\eta)}}{\Gamma(\alpha)} \int\limits_a^x (x^{\sigma} - t^{\sigma})^{\alpha-1} t^{\sigma\eta+\sigma-1} f(t)\,dt, \tag{2.41}$$

for $\alpha > 0$, and

$$I^{\alpha}_{a+;\,\sigma,\,\eta} f(x) = x^{-\sigma(\alpha+\eta)} \left(\frac{d}{\sigma x^{\sigma-1} dx}\right)^{n} x^{\sigma(\alpha+n+\eta)} I^{\alpha+n}_{a+;\,\sigma,\,\eta} f(x), \tag{2.42}$$

for $\alpha > -n$, $n \in \mathbb{N}$.

After the change of variables $x^{\sigma} = y$, $t^{\sigma} = \tau$ (2.41)–(2.42) are reduced to the usual Riemann-Liouville fractional integrals and derivatives

$$I^{\alpha}_{a+;\,\sigma,\,\eta} f(x) = y^{-\alpha-\eta} (I^{\alpha}_{a^{\sigma}+}\varphi)(y), \qquad \varphi(y) = y^{\eta} f(x), \qquad x^{\sigma} = y. \tag{2.43}$$

Erdélyi-Kober operators are essential and important in transmutation theory. For example, the most well–known transmutations of Sonine and Poisson are of this class when $\sigma = 2$ (see for details [48, 124]). In the monographs by Sneddon [125] and by Kiryakova [52] a comprehensive theory of these operators was given. Important properties of Erdélyi-Kober operators were studied in the monographs [48, 119].

Let $\mathrm{Re}\,\alpha > 0$, $-\infty \le a < b \le \infty$. The fractional integral of a function f with respect to another function g is (see [44])

$$(I^{\alpha}_{a+,g} f)(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (g(x) - g(t))^{\alpha-1}\, g'(t) f(t) d\,t, \tag{2.44}$$

where $g(x)$ is a strictly increasing function. Fractional integrals of a function by another in the complex plane were studied by Osler [85, 86]. Integral (2.44) is defined for every function $f(t) \in L_1(a, b)$ and for any monotone function $g(t)$, having a continuous derivative.

If $g'(x) \neq 0$, $a \leq x \leq b$, then operators $I^{\alpha}_{a+,g}$, $I^{\alpha}_{b-,g}$ are expressed via the usual Riemann-Liouville (or Liouville) fractional integration after the corresponding changes of variables

$$I^{\alpha}_{a+,g} f = Q T^{\alpha}_{c+} Q^{-1} f, \quad I^{\alpha}_{b-,g} f = Q T^{\alpha}_{d-} Q^{-1} f, \quad c = g(a), \quad d = g(b),$$

where $(Qf)(x) = f[g(x)]$.

The fractional derivative of a function f with respect to another function g of the order $\alpha \in (0, 1)$ is

$$(D^{\alpha}_{a+,g} f)(x) = \frac{1}{\Gamma(1-\alpha)} \frac{1}{g'(x)} \frac{d}{dx} \int\limits_a^x \frac{f(t)}{(g(x) - g(t))^{\alpha}} g'(t) dt. \tag{2.45}$$

It is also possible to consider the Marchaud and other forms of the fractional derivatives of a function f with respect to another function g.

The operational calculus approach for fractional calculus with respect to functions is given in [34].

Riemann-Liouville fractional integral is obtained by choosing $g(x) = x$ in (2.44). If we take in (2.44) the function $g(x) = x^{\sigma}$ we obtain Erdelýi-Kober type operator (2.41); if we take $g(x) = \ln x$ in (2.44) we get Hadamard fractional integral, and a choice $g(x) = \exp(-x)$ with its applications was considered in [26].

As A.M. Djrbashian pointed out to us operators of fractional derivatives of a function with respect to another function (2.44) even in some more general setting were introduced and studied by his father M.M. Djrbashian, cf. [24, 25, 27]. In these papers were studied integral representations of this operator class, their inversion and corresponding integro-differential equations of fractional order.

2.3.3 Variable-order differential operators and sequential fractional derivatives

In [116] fractional integration and differentiation when the order is not a constant but a function was presented. In order to define variable-order differential operators in [116] two ways were given: the first way is a direct one and the second uses Fourier transforms. For the Riemann–Liouville fractional integrals of variable order, we have for $\mathrm{Re}(\alpha(x)) > 0$, $x > a$ (see [116], formula 2)

$$(I^{\alpha(x)}_{a+} f)(x) = \frac{1}{\Gamma(\alpha(x))} \int\limits_a^x (x-y)^{\alpha(x)-1} f(y) dy. \tag{2.46}$$

If $a > -\infty$ we have the Riemann definition and for $a = -\infty$ we have the Liouville definition.

The Riemann-Liouville derivative is also extended to the case of variable order (see [116], formula 3):

$$(D_{a+}^{\alpha(x)} f)(x) = \frac{1}{\Gamma(1-\alpha(x))} \frac{d}{dx} \int_a^x (x-y)^{-\alpha(x)} f(y)dy, \tag{2.47}$$

where $0 < \mathrm{Re}(\alpha(x)) < 1$, $x > a$.

We observe that the fractional operators (2.46) and (2.47) are not inverse to each other as in the case of constant order, as it will be seen below. So, it will not be correct to introduce $D_{a+}^{-\alpha(x)}$ as $[D_{a+}^{\alpha(x)}]^{-1}$.

For $0 \le \mathrm{Re}(\alpha(x)) < 1$ Coimbra derivative [14] is

$$({}^{CV}D_0^{\alpha(x)} f)(x) = \frac{1}{\Gamma(1-\alpha(x))} \int_0^x \frac{f'(y)}{(x-y)^{\alpha(x)}}\, dy + \frac{f(0+)-f(0-)}{x^{\alpha(x)}(\Gamma(1-\alpha(x)))}.$$

Now let us consider fractional order depending of two variables $\alpha(x, y)$. Let $f : [a, b] \to \mathbb{R}$ be a function. The left–sided Riemann–Liouville fractional integral of order $\alpha(\cdot, \cdot)$ is defined by (see [72])

$$({}_aI_x^{\alpha(\cdot,\cdot)} f)(x) = \int_a^x \frac{(x-y)^{\alpha(x,y)-1}}{\Gamma(\alpha(x, y))} f(y)dy, \qquad x > a. \tag{2.48}$$

Left–sided Riemann–Liouville derivative of order $\alpha(x, y) \in (0, 1)$ is

$$({}_aD_x^{\alpha(\cdot,\cdot)} f)(x) = \frac{d}{dx} \int_a^x \frac{(x-y)^{-\alpha(x,y)}}{\Gamma(1-\alpha(x, y))} f(y)dy, \qquad x > a. \tag{2.49}$$

Similarly introduced are Caputo and Marchaud derivatives of variable fractional order (see [114, 116]).

Next we consider so called sequential derivatives. Djrbashian–Nersesyan fractional derivatives, associated with a sequence $\{\gamma_0, \gamma_1, \ldots, \gamma_m\}$ of order σ, where $\sigma = \gamma_0 + \gamma_1 + ... + \gamma_m$, are defined by

$$D_{DN}^{\sigma} = D^{\gamma_0} D^{\gamma_1} \cdots D^{\gamma_m}, \tag{2.50}$$

where D^{γ_k} are fractional integrals and derivatives of Riemann–Liouville with some endpoint. These operators were introduced in [27–30] and then studied and applied in [31]. The original definitions demand $-1 \le \gamma_0 \le 0$, $0 \le \gamma_k \le 1$, $1 \le k \le m$, as in the above papers integrodifferential equations under such conditions were studied for operators (2.50). But Djrbashian–Nersesyan fractional operators may be defined and considered for any parameters γ_k if appropriate definitions of Riemann–Liouville operators are used.

Djrbashian–Nersesyan operators were patterns for introducing in the book of Miller and Ross [78] more general sequential operators of fractional integrodifferentiation for which compositions in definitions of the form (2.50) are consisted of any fractional operators, cf. an useful discussion in [91].

2.3.4 Generalized fractional integrals

There are a lot of generalizations of fractional integro–differential operators connected with combinations and compositions of more standard fractional operators. For example, averaged or distributed order fractional operator, associated with any given fractional operator R^t, is introduced by the next formula

$$I_{MR}^{(a,b)} f = \int_a^b R^t f(t)d\,t, \tag{2.51}$$

where R^t is a given fractional operator of order t of any kind. In particular, when R^t represents the fractional Riemann-Liouville operator, terms such as 'continued' or 'distributed-order' fractional integrals are often used. Such operators were studied in [100, 101].

Fractional integrals containing the Gaussian hypergeometric function in the kernel are the Saigo fractional integrals (see [113]) which are defined by

$$J_x^{\gamma,\beta,\eta} f(x) = \frac{1}{\Gamma(\gamma)} \int_x^\infty (t-x)^{\gamma-1} t^{-\gamma-\beta}\,{}_2F_1\left(\gamma+\beta, -\eta; \gamma; 1-\frac{x}{t}\right) f(t)dt, \tag{2.52}$$

and

$$I_x^{\gamma,\beta,\eta} f(x) = \frac{x^{-\gamma-\beta}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1}\,{}_2F_1\left(\gamma+\beta, -\eta; \gamma; 1-\frac{t}{x}\right) f(t)dt, \tag{2.53}$$

where $\gamma > 0$, β, θ are real numbers. Similar class of hypergeometric fractional integrals have been introduced also by Love. Generalized fractional calculus operators with more general special functions in the kernels, as Meijer G- and Fox H-functions have been also studied, see for example [46] and [52].

We consider the case of the interval [0, 1]. Following Hadamard [41] and M.M. Djrbashian [24, 25] we introduce the operator

$$(L^{(\omega)} f)(x) = -\int_0^1 f(xt)\omega'(t)dt, \tag{2.54}$$

where the function $\omega \in C([0, 1])$ is supposed to satisfy the following assumptions:

1. $\omega(x)$ is monotone,
2. $\omega(0) = 1, \omega(1) = 0, \omega(x) \neq 0$ as $0 < x < 1$,
3. $\omega'(x) \in L_1(0, 1)$.

If $\omega(x) = \frac{(1-x)^\alpha}{\Gamma(1+\alpha)}$, then obviously $(L^{(\omega)} f)(x) = x^{-\alpha}(I^\alpha_{0+} f)(x)$, where I^α_{0+} is (2.8).

M.M. Djrbashian (also Mkhitar Dzhrbashjan, M. M. Jerbashian (see [104] about his papers)) in [24, 25] considered the operators $L^{(\omega)}$ in a more general form

$$(L^{(\omega)} f)(x) = -\frac{d}{dx}\left(x\int\limits_0^1 f(xt)dp(t)\right), \qquad p(t) = t\int\limits_t^1 \frac{\omega(x)}{x^2}dx.$$

2.3.5 Applied aspects of fractional derivatives and integrals and numerical calculus

The reasons for the large number of applications of fractional calculus are as follows.

1. The nonlocality of the fractional derivative makes it possible to use it for mathematical modeling of media with memory.
2. In many models of physics, biology and medicine, differential equations of fractional orders describe the phenomenon under consideration more accurately, since the order of the fractional derivative gives an additional degree of freedom.
3. Fractional equations describe non-Markovian processes, which opens up new horizons in probability theory and statistics.

The use of fractional calculus in theoretical physics and mechanics is described, for example, in the books [1, 43, 91, 129]. The mail idea of using fractional derivatives and integrals in physics is based on nonlocal effects which may occur in space and time. Thus, the fractional derivative by time in the model is interpreted by physicists as the presence of the memory of the described process. A fractional derivative with respect to a spatial variable indicates the presence of movement restrictions.

The paper [131] introduces scientists who started to apply fractional calculus to scientific and engineering problems during the nineteenth and twentieth centuries. How to use the Mellin integral transform in fractional calculus was described in [66, 67]. In [40] fundamental theorem of fractional calculus is presented.

The use of fractional derivatives in continuum mechanics is due to the emergence of new polymeric materials that have both the property of viscosity and the property of elasticity. Describing the behavior of solids using viscoelastic models that contain fractional order operators and relate stresses to strains was began in 1948 from papers of Rabotnov Yu. N. (see translation [102]) and of Gerasimov A. N. [37]. Let mention here the paper [82] where there are many biographical facts and papers of Gerasimov A. N., including his paper [37] were given. Models of viscoelasticity with fractional operator were described by Rossikhin Yu. A. and Shitikova M. V. in Encyclopedia of Continuum Mechanics [108]. The mathematical modeling of a stress-strain state of the viscoelastic periodontal membrane using fractional calculus is given in [8]. The fractional derivative of strain with time to the ratio between stress and strain is used in this case to describe the intermediate state of the material between elastic and liquid [117, 120] and [42], Vol. 7, 139–158 (2019). Monograph [70] provides an overview of application of fractional calculus and waves in linear viscoelastic media, which includes fractional viscoelastic models, dispersion, dissipation and diffusion. Diethelm K. in [19] described numerical methods for the fractional differential equations of viscoelasticity.

In [13] it was noticed that in the modelling of the energy dissipation and dispersion in the propagation of elastic waves, introduction of memory mechanisms in their constitutive equations is needed. It turned out that the most successful memory mechanism used to represent variance and energy dissipation is a fractional derivative. The fractional derivative generalizing the stress strain relation of anelastic media was introduced in [12, 13].

Models with fractional order operators used in dynamic problems of rigid body mechanics first appeared in [36, 37, 117] and described in reviews [107, 120–122]. Fractional calculus in the problems of mechanics of solids was given in [109–111] and [42], Vol. 7, 159–192 (2019).

In problems of thermodynamics, solutions for unsteady flows near the boundary of a semi-infinite region are expressed by combinations of fractional derivatives in [2]. In paper [77] thermodynamics was generalized in terms of fractional derivatives, while the results of traditional thermodynamics by Carnot, Clausius and Helmholtz were obtained in the particular case when the exponent of the fractional derivative is equal to one. In [98] was described how fractional calculus used in thermoelasticity.

Application of fractional modelling in rheology providing by Scott Blair was presented in [105]. V.V. Uchaikin in [130] presented fractional models in hydromechanics.

Applications of fractional calculus in electromagnetic including fractional multipoles, fractional solutions for Helmholtz equations, and fractional-image methods were studied and briefly reviewed in [32, 33].

In [88] fractional-order chaotic systems were studied. Namely, the model of chaotic system considered in [88] as three single differential equations, where the equations contain the fractional order derivatives.

Application of the fractional calculus in the control theory can be found in [90]. It was shown that for fractional-order systems the most suitable way is to use fractional-order controllers involving an integrator and differentiator of fractional order.

The works [133, 134] deal with the fractional generalization of the diffusion equation, fractality of the chaotic dynamics and kinetics, and also includes material on non-ergodic and non-well-mixing Hamiltonian dynamics.

Physical interpretation of a fractional integral can be found in [112]. Authors of [47] gave interpretation of fractional-order operators in fracture mechanics.

Multidisciplinary presentation of fractional calculus for researchers in fields such as electromagnetism, control engineering, and signal processing were given in the book [83]. In [7] was overviewed how to use fractional calculus in biomechanics.

Epidemiological SIR model in which classic first order ODE's were replaced by fractional derivatives is gaining popularity due to Covid 19 (see, for example, [80]). The presence of fractional derivatives in the model allows you to increase the number of degrees of freedom.

The book [132] shows how the fractional calculus can be used to model the statistical behavior of complex systems, gives general strategies for understanding wave propagation through random media, the nonlinear response of complex materials and the fluctuations of heat transport in heterogeneous materials.

Review [56] is fully described advances in the analysis of the fractional and generalized fractional derivatives from the probabilistic point of view. The theory of fractional Brownian motion and other long-memory processes is describes in [79], where was particularly obtained different forms of the Black-Scholes equation for fractional market. In [6] Wiener-transformable markets, where the driving process is given by an adapted transformation of a Wiener process including processes with long memory, like fractional Brownian motion and related processes were studied. Probabilistic interpretation of a fractional integration is given in [127].

Fractional derivatives and integrals are complex objects, the calculation of which is often technically difficult. In this case, numerical methods are applied. Since the fractional derivative is non-local approximation of it is more complicated than the integer derivatives. C. Li with coauthors made a great contribution to the development of the numerical calculus of fractional integrals and derivatives and their applications. Almost all existing numerical approximations for fractional integrals and derivatives were systematically examined in [62–64].

Difference schemes for solutions to equation with fractional time derivative was first used in [123]. In a series of articles [94, 95, 97] triangular strip matrices are used for approximating fractional derivatives and solving fractional differential equations.

Monte Carlo method is introduced for numerical evaluation of fractional-order derivatives in [58, 59]. A feature of these works is that the points at which the function is calculated are distributed unevenly and their distribution depends on the order of the derivative.

Group of scientists as Igor Podlubny, Ivo Petras, Blas Vinagre and others deeply advanced in applications of fractional derivatives in physics including fractional order controller and distributed-order dynamic systems (see [10, 45, 88–92]).

V. Kiryakova devoted Section 9 of the article [53] to information on numerical calculations of special functions of fractional calculus. Recent results on numerical algorithms for evaluation of special functions like Mittag–Leffler and Wright can be found in [20, 39, 93] and in [42], Vol. 1, 269–296 (2019). Numerical algorithms and results on the Wright function and its special cases can be found in [65, 71].

2.3.6 Publications after 1987

After 1987 till nowadays, a lot of monographs by authors from different countries had been published, wherein various aspects of fractional calculus, and its applications were considered.

Nowadays, the list of publications wholly or partly devoted to fractional calculus and its applications is endless. Information about current state of fractional calculus can be found in [9, 16–18, 69, 103, 128, 131, 135].

Digest of articles [42] represents a valuable and reliable reference work. The book [3] provides mathematical tools for working with fractional models and solving fractional differential equations. The books [54, 55] provide insights into multidimensional fractional integral and differential operators and their associated spaces.

We should mention the great impact of Virginia Kiryakova with her coauthors. She published a book about generalized fractional derivatives [52] and a lot of papers, including popularizing fractional derivatives [23, 131]. Also she is an editor-in-chief of the most popular journal specialized in fractional calculus: "Fractional Calculus and Applied Analysis" (FCAA; Fract. Calc. Appl. Anal.).

A more detailed list of books on fractional integro-differential calculus is presented in [42], Vol. 1, 1–22 (2019).

We have discussed various approaches to fractional integro-differentiation. It is essential to apply these methods to fundamental functions such as z^λ, e^z, etc., which are widely used in Taylor and Fourier series.

As demonstrated, there are diverse methods for constructing fractional derivatives and integrals. While the technique of analytic continuation is well-known, it cannot be directly employed to extend analytically the derivative concerning the order of differentiation, as the initial order of the derivative is restricted to natural numbers only. Instead, one can identify an analytic function that aligns with the n-th derivative of a function for natural numbers and subsequently extend this function analytically across the entire or almost entire complex plane. Consequently, fractional derivatives and integrals can be constructed in a non-unique manner.

On the other hand, if two distinct integro-differential operators meet the criteria 1, there exists a set of functions where these operators coincide. For instance, when $\alpha > 0$ and $f \in L_1(a, b)$, the left-sided Grünwald–Letnikov fractional integral (2.22) is equivalent to the Riemann–Liouville fractional integral (2.5) when $a = 0$ for nearly all x (refer to [91]).

3 Fractional-order differentiation in the Wolfram language

3.1 Riemann-Liouville-Hadamard integro-differentiation in the Wolfram language

3.1.1 Definition

We describe the Riemann-Liouville integro-differentiation of an arbitrary function to arbitrary symbolic order α in the Wolfram Language. Using techniques described below, this operation, hereafter referred to as "fractional differentiation", has been published in the Wolfram Function Repository as `ResourceFunction["FractionalOrderD"]`. Defined via an integral transform, fractional differentiation is an analytic function of α which coincides with the usual α-th order derivative when α is a positive integer and with repeated indefinite integration for negative integer α.

We will use notation $\frac{d^\alpha f(z)}{z^\alpha}$ for Riemann-Liouville integro-diffirentiation for all $\alpha \in \mathbb{C}$.

Definition. By definition of $\frac{d^\alpha f(z)}{dz^\alpha}$ we put

$$\frac{d^\alpha f(z)}{dz^\alpha} = \begin{cases} f(z), & \alpha = 0; \\ f^{(\alpha)}(z), & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha > 0; \\ \underbrace{\int\limits_0^z dt ... \int\limits_0^t dt \int\limits_0^t}_{-\alpha} f(t)dt, & \alpha \in \mathbb{Z} \quad \text{and} \quad \alpha < 0; \\ \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dz^n} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha-n+1}}, & n = \lfloor\alpha\rfloor + 1 \quad \text{and} \quad \mathrm{Re}(\alpha) > 0; \\ \frac{1}{\Gamma(-\alpha)} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha) < 0 \quad \text{and} \quad \alpha \notin \mathbb{Z}; \\ \frac{1}{\Gamma(1-\alpha)} \frac{d}{dz} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha) = 0 \quad \text{and} \quad \mathrm{Im}(\alpha) \neq 0, \end{cases} \tag{3.1}$$

where in the case of divergent integral we use Hadamard finite part. Such integro-differentiation (3.1) is called *Riemann-Liouville-Hadamard derivative*.

Above we separated cases of symbolic positive integer n-th order derivatives from generic result integro-differentiation of fractional order. In particular for $\alpha = -1, -2, ...$ we have

$$\frac{1}{\Gamma(-\alpha)} \int\limits_0^z \frac{f(t)dt}{(z-t)^{\alpha+1}} = \underbrace{\int\limits_0^z dt ... \int\limits_0^t dt \int\limits_0^t}_{-\alpha} f(t)dt.$$

So we can combine the third and fifth formulas. Function `FractionalOrderD` realized regularized Riemann-Liouville integro-diffirentiation. That means if any of the integrals in (3.1) diverges we use Hadamard regularization of this integral (see Subsection 3.2).

Function `FractionalOrderD` which calculating $\frac{d^\alpha}{dz^\alpha}$ is presented in Wolfram Function Repository. In order to get this function we should write

```
ResourceFunction["FractionalOrderD"].
```

3.1.2 Simplest examples

Let us consider how the function `FractionalOrderD` acts to simple functions. For example, `FractionalOrderD`[x^2,{x, α}] gives

$$\frac{2x^{2-\alpha}}{\Gamma(3-\alpha)}.$$

and `FractionalOrderD`[sin[z],{z, α}] gives

$$\begin{cases} \sin\left(z+\frac{\pi\alpha}{2}\right) & \text{if } \alpha\in\mathbb{Z},\ \alpha\geq 0; \\ \frac{2^{\alpha-1}\sqrt{\pi}z^{1-\alpha}}{\Gamma(1-\frac{\alpha}{2})\Gamma\left(\frac{3-\alpha}{2}\right)}\,{}_1F_2\left(1;1-\frac{\alpha}{2},\frac{3-\alpha}{2};-\frac{z^2}{4}\right) & \text{in other cases.} \end{cases}$$

If we put $\alpha = 3$ in previous result we obtain $(\sin(z))''' = -\cos(z)$.

When $\alpha = -3$, $f(z) = e^z$ we obtain

$$\begin{aligned} \frac{d^{-3}e^z}{dz^{-3}} &= \frac{1}{\Gamma(3)}\int_0^z (z-t)^2 e^t\,dt \\ &= \int_0^z \left(\int_0^{t_3}\left(\int_0^{t_2} e^{t_1}dt_1\right)dt_2\right)dt_3 \\ &= e^z - \frac{z^2}{2} - z - 1. \end{aligned}$$

For $\alpha = -1/2$ and for $\alpha = 1/2$ we can write

$$\begin{aligned} \frac{d^{-1/2}e^z}{dz^{-1/2}} &= \frac{1}{\Gamma(1/2)}\int_0^z \frac{e^t}{(z-t)^{1/2}}dt = e^z \operatorname{erf}(\sqrt{z}), \\ \frac{d^{1/2}e^z}{dz^{1/2}} &= \frac{1}{\Gamma(1/2)}\frac{d}{dz}\int_0^z \frac{e^t}{(z-t)^{1/2}}dt = e^z\left(\frac{\Gamma\left(-\frac{1}{2},z\right)}{2\sqrt{\pi}}+1\right), \end{aligned}$$

where $\operatorname{erf}(z)$ is the integral of the Gaussian distribution, given by

$$\operatorname{erf}(z) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}dt,$$

$\Gamma(\alpha, z)$ is the incomplete Gamma function, given by

$$\Gamma(\alpha, z) = \int_z^\infty t^{\alpha-1}e^{-t}dt. \tag{3.2}$$

Now we can evaluate `FractionalOrderD` operator for more then 100000 functions, analytical in variable z. Let show some examples starting with the "simplest" mathematical functions, involving only one or two letters $\frac{1}{z}$, $\sqrt{z}$, z^b, a^z, e^z, z^z. Next, we see "named functions" with head-titles like $\log(z)$, $\sin(z)$, $J_\nu(z)$, $J_z(b)$. At last we see "composed functions" $\sqrt{z^2}$, $(z^a)^b$, a^{z^c}, $\arcsin(z^3)$. If we apply differentiation or indefinite integration

$$\left(\frac{1}{z}\right)' = -\frac{1}{z^2}, \qquad (\sqrt{z})' = \frac{1}{2\sqrt{z}}, \qquad (z^b)' = bz^{b-1},$$
$$(a^z)' = a^z \log(a), \qquad (e^z)' = e^z,$$
$$(z^z)' = z^z(\log(z)+1), \qquad (\log(z))' = \frac{1}{z}, \qquad (\sin(z))' = \cos(z),$$
$$(J_\nu(z))' = \frac{1}{2}(J_{\nu-1}(z) - J_{\nu+1}(z)), \qquad (\sqrt{z^2})' = \frac{z}{\sqrt{z^2}},$$
$$((z^a)^b)' = abz^{a-1}\left(z^a\right)^{b-1}, \qquad (a^{z^c})' = c\log(a)z^{c-1}a^{z^c},$$
$$(\arcsin(z^3))' = \frac{3z^2}{\sqrt{1-z^6}}, \qquad \int \frac{dz}{z} = \log(z) + C,$$
$$\int \sqrt{z}dz = \frac{2z^{3/2}}{3} + C, \qquad \int z^b dz = \frac{z^{b+1}}{b+1} + C,$$
$$\int a^z dz = \frac{a^z}{\log(a)} + C, \qquad \int e^z dz = e^z + C,$$
$$\int \log(z)dz = -z\log(z) - z + C, \qquad \int \sin(z) = -\cos(z) + C,$$
$$\int J_\nu(z)dz = \frac{2^{-\nu-1}z^{\nu+1}\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma(\nu+1)\,\Gamma\left(\frac{\nu+3}{2}\right)}\,{}_1F_2\left(\frac{\nu+1}{2}; \nu+1, \frac{\nu+3}{2}; -\frac{z^2}{4}\right) + C,$$
$$\int \sqrt{z^2}dz = \frac{z\sqrt{z^2}}{2} + C, \qquad \int (z^a)^b dz = \frac{z\,(z^a)^b}{ab+1} + C,$$
$$\int a^{z^c}dz = -\frac{z\,(\log(a)\,(-z^c))^{-1/c}\,\Gamma\left(\frac{1}{c}, -z^c\log(a)\right)}{c} + C,$$
$$\int \arcsin(z^3)dz = z\arcsin(z^3) - \frac{3}{4}z^4\,{}_2F_1\left(\frac{1}{2}, \frac{2}{3}; \frac{5}{3}; z^6\right) + C.$$

Here C is arbitrary constant. We find that not each integral and even derivative can be evaluated. Here there are no results for

$$(J_z(b))', \qquad \int z^z dz, \qquad \int J_z(b)\,dz.$$

Also in many cases we see special functions as result of integration.

Operation `FractionalOrderD` evaluates repeatable α-integer order derivatives and indefinite integrals and extends results for arbitrary complex or real order α. For example:

$$\frac{d^\alpha}{dz^\alpha}\frac{1}{z} = z^{-\alpha}\left(\frac{1}{\Gamma(1-\alpha)}\,{}_2F_2(1,1;2,1-\alpha;-z) + G_{1,2}^{1,1}\left(z\,\middle|\,\begin{matrix}0\\-1,\alpha\end{matrix}\right)\right),$$

$$\frac{d^\alpha \sqrt{z}}{dz^\alpha} = \frac{\sqrt{\pi} z^{\frac{1}{2}-\alpha}}{2\Gamma\left(\frac{3}{2}-\alpha\right)}, \qquad \frac{d^\alpha z^b}{dz^\alpha} = \frac{\Gamma(b+1)z^{b-\alpha}}{\Gamma(b-\alpha+1)},$$

$$\frac{d^\alpha a^z}{dz^\alpha} = \begin{cases} a^z \log^\alpha(a), & \alpha \in \mathbb{Z}, \alpha \geq 0; \\ a^z \log^\alpha(a)(1-Q(-\alpha, z\log(a))), & \alpha \in \mathbb{Z}, \alpha < 0; \\ a^z z^{-\alpha}(z\log(a))^\alpha(1-Q(-\alpha, z\log(a))), & \text{in other cases,} \end{cases}$$

$$\frac{d^\alpha e^z}{dz^\alpha} = \begin{cases} e^z, & \alpha \in \mathbb{Z}, \alpha \geq 0; \\ e^z(1-Q(-\alpha, z)), & \text{in other cases,} \end{cases}$$

$$\frac{d^\alpha \sin(z)}{dz^\alpha} = \begin{cases} \sin\left(\frac{\pi\alpha}{2}+z\right), & \alpha \in \mathbb{Z}, \alpha \geq 0; \\ \frac{z^{1-\alpha}\,{}_1F_2\left(1;1-\frac{\alpha}{2},\frac{3}{2}-\frac{\alpha}{2};-\frac{z^2}{4}\right)}{\Gamma(2-\alpha)}, & \text{in other cases,} \end{cases}$$

$$\frac{d^\alpha J_\nu(z)}{dz^\alpha} = \frac{\sqrt{\pi}2^{\alpha-2\nu}\Gamma(\nu+1)z^{\nu-\alpha}}{\Gamma\left(\frac{\nu-\alpha+1}{2}\right)\Gamma\left(\frac{\nu-\alpha+2}{2}\right)} \times$$

$$\times\, {}_2F_3\left(\frac{\nu+1}{2}, \frac{\nu+2}{2}; \frac{\nu-\alpha+1}{2}, \frac{\nu-\alpha+2}{2}, \nu+1; -\frac{z^2}{4}\right),$$

$$\frac{d^\alpha \sqrt{z^2}}{dz^\alpha} = \frac{\sqrt{z^2}z^{-\alpha}}{\Gamma(2-\alpha)}, \qquad \frac{d^\alpha (z^a)^b}{dz^\alpha} = \frac{z^{-\alpha}(z^a)^b\,\Gamma(ab+1)}{\Gamma(ab-\alpha+1)},$$

$$\frac{d^\alpha \arcsin(z^3)}{dz^\alpha} = \frac{6z^{3-\alpha}}{\Gamma(4-\alpha)} \times$$

$$\times\, {}_7F_6\left(\frac{1}{2}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}, 1, \frac{7}{6}, \frac{4}{3}; \frac{2}{3} - \frac{\alpha}{6}, \frac{5}{6}-\frac{\alpha}{6}, 1-\frac{\alpha}{6}, \frac{7}{6}-\frac{\alpha}{6}, \frac{4}{3}-\frac{\alpha}{6}, \frac{3}{2}-\frac{\alpha}{6}; z^6\right),$$

where $Q(\alpha, z) = \frac{\Gamma(\alpha,z)}{\Gamma(\alpha)}$ is the regularized incomplete Gamma function, $\Gamma(\alpha, z)$ is the incomplete Gamma function (3.2). Values $\frac{d^\alpha z^z}{dz^\alpha}$, $\frac{d^\alpha \log(z)}{dz^\alpha}$, $\frac{d^\alpha J_z(b)}{dz^\alpha}$, $\frac{d^\alpha a^{z^c}}{dz^\alpha}$ are also calculated but formulas too much complicated.

We would like to compare Mathematica operations `FractionalOrderD`, `D` and `Integrate`. The direct computation from definition in Mathematica gives

$$D^7 \frac{1}{z^2} = \texttt{D[1/z}^2\texttt{, z, 7]} = -\frac{40320}{z^9},$$

$$\frac{d^7}{dz^7}\frac{1}{z^2} = \texttt{ResourceFunction["FractionalOrderD"][1/z}^2\texttt{, z, 7]}$$

$$= -\frac{40320}{z^9},$$

$$\underbrace{\int_0^z dt \ldots \int_0^t dt \int_0^t}_{7} \frac{dt}{t^2} = \texttt{Nest}\left[\int \texttt{\#dz \&, 1/z}^2\texttt{, 7}\right] = \frac{137z^5}{7200} - \frac{1}{120}z^5\log(z),$$

$$\frac{d^{-7}}{dz^{-7}}\frac{1}{z^2} =$$
$$\texttt{ResourceFunction["FractionalOrderD"]}[1/z^2, z, -7]$$
$$= -\frac{1}{120}z^5\left(\log(z) - \frac{77}{60}\right) = \frac{137z^5}{7200} - \frac{1}{120}z^5\log(z).$$

Therefore `FractionalOrderD` generalizes `D` and `Integrate`.

We should mention, that differentiation by parameters of functions (for example, $\frac{d^\alpha J_z(b)}{dz^\alpha}$) as rule can not produce well known functions. But we can say the following: "The first derivative with respect to an upper "parameter" a_k, and all derivatives of symbolic integer order m with respect to a "lower" parameter b_k of the generalized hypergeometric function ${}_pF_q(a_1, ..., a_p; b_1, ..., b_q; z)$, can be expressed in terms of the Kampé de Fériet hypergeometric function of two variables" (see [152]).

Let us describe how `FractionalOrderD` works. For evaluation of $\frac{d^\alpha}{dz^\alpha}$ we are using three approaches.

1. Generic formulas exist for single simple functions; these are converted into pattern matching rules. Here power series, generalized power series and Hadamard regularization are used.
2. Convert function to the Meijer G-function, using powerful new operator `MeijerGReduce` and rule-based `MeijerGForm` (applies to hypergeometric type functions). Then use formula for the fractional derivative of the Meijer G-function.
3. Generic formulas for fractional differentiation, which produce Appell function F_1 of two variables (in future Lauricella function D of several variables and Humbert function Φ of two variables will be used).

In the following sections we consider these three approaches.

3.2 Calculation of fractional derivatives and integrals by series expansion

3.2.1 Basic power–logarithmic examples

Let us consider the first approach to calculating $\frac{d^\alpha}{dz^\alpha}$. The `FractionalOrderD` function allows us to find all these and many other fractional derivatives because this operation applied to each term of Taylor series expansions of all functions near zero. So if

$$f(z) = z^b \sum_{n=0}^{\infty} c_n z^n, \tag{3.3}$$

then

$$\frac{d^\alpha f(z)}{dz^\alpha} = \sum_{n=0}^{\infty} c_n \frac{d^\alpha z^{b+n}}{dz^\alpha}. \tag{3.4}$$

Sum representations by formula (3.3) we meet for functions like

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \qquad J_\nu(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(n+\nu+1)} \left(\frac{z}{2}\right)^{2n+\nu},$$

but sometimes series expansions include $\log(z)$ function as in the logarithmic case of $K_0(z)$:

$$K_0(z) = -\left(\log\left(\frac{z}{2}\right) + \gamma\right) I_0(z) + \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} \left(\frac{z}{2}\right)^{2n}, \tag{3.5}$$

with n-th harmonic number $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ and γ is Euler–Mascheroni constant

$$\gamma = \lim_{n\to\infty} \left(-\log n + \sum_{k=1}^{n} \frac{1}{k}\right) = -\int_0^\infty e^{-x} \log(x) dx. \tag{3.6}$$

It means, that we should consider more general series

$$f_L(z) = z^b \log^k(z) \sum_{n=0}^{\infty} c_n z^n, \tag{3.7}$$

and evaluate for arbitrary b, α and integer $k = 0, 1, 2, \ldots$ the following values

$$\frac{d^\alpha f_L(z)}{dz^\alpha} = \sum_{n=0}^{\infty} c_n \frac{d^\alpha}{dz^\alpha} z^{b+n} \log^k(z). \tag{3.8}$$

Let $\lambda = b + n$. In order to calculate (3.4) we should find $\frac{d^\alpha z^\lambda}{dz^\alpha}$ by formula (3.1). Since $\frac{d^0 z^\lambda}{dz^0} = z^\lambda$ and for $\alpha \in \mathbb{Z}$ and $\alpha \geq 0$ we have $\alpha = 0, 1, 2, \ldots$ and integer derivative gives

$$\begin{aligned} \frac{d^\alpha z^\lambda}{dz^\alpha} &= \lambda(\lambda-1)\ldots(\lambda-\alpha+1) z^{\lambda-\alpha} = (\lambda-\alpha+1)_\alpha\, z^{\lambda-\alpha} \\ &= (-1)^\alpha (-\lambda)_\alpha\, z^{\lambda-\alpha}, \end{aligned} \tag{3.9}$$

where $(\lambda)_\alpha$ is a Pochhammer symbol.

Therefore we obtain

$$\frac{d^\alpha z^\lambda}{dz^\alpha}=\begin{cases} z^\lambda, & \alpha=0;\\ (\lambda-\alpha+1)_\alpha z^{\lambda-\alpha}, & \alpha\in\mathbb{Z}\quad\text{and}\quad \alpha>0;\\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dz^n}\int\limits_0^z\frac{t^\lambda dt}{(z-t)^{\alpha-n+1}}, & n{=}\lfloor \mathrm{Re}(\alpha)\rfloor{+}1\quad\text{and}\quad \mathrm{Re}(\alpha){>}0;\\ \frac{1}{\Gamma(-\alpha)}\int\limits_0^z\frac{t^\lambda dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha)<0;\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int\limits_0^z\frac{t^\lambda dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha)=0\quad\text{and}\quad \mathrm{Im}(\alpha)\neq 0. \end{cases}\tag{3.10}$$

In order to calculate (3.8) we should find $\frac{d^\alpha}{dz^\alpha}z^\lambda\log^k(z)$ by formula (3.1)

$$\frac{d^\alpha}{dz^\alpha}z^\lambda\log^k(z){=}\begin{cases} z^\lambda\log^k(z), & \alpha=0;\\ (z^\lambda\log^k(z))^{(\alpha)}, & \alpha\in\mathbb{Z}\quad\text{and}\quad \alpha>0;\\ \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dz^n}\int\limits_0^z\frac{t^\lambda\log^k(t)dt}{(z-t)^{\alpha-n+1}}, & n{=}\lfloor \alpha\rfloor{+}1\text{ and }\mathrm{Re}(\alpha){>}0;\\ \frac{1}{\Gamma(-\alpha)}\int\limits_0^z\frac{t^\lambda\log^k(t)dt}{(z-t)^{\alpha+1}}, & \mathrm{Re}(\alpha)<0;\\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dz}\int\limits_0^z\frac{t^\lambda\log^k(t)dt}{(z-t)^{\alpha}}, & \mathrm{Re}(\alpha)=0\text{ and }\mathrm{Im}(\alpha)\neq 0. \end{cases}\tag{3.11}$$

Here for $\alpha\in\mathbb{Z}$ and $\alpha>0$

$$(z^\lambda\log^k(z))^{(\alpha)}=\sum_{j=0}^{\alpha}\binom{\alpha}{j}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda-j+1)}z^{\lambda-j}\frac{d^{\alpha-j}\log^k(z)}{dz^{\alpha-j}}.$$

If some of integrals

$$\int\limits_0^z\frac{t^\lambda dt}{(z-t)^{\alpha+1}},\qquad \int\limits_0^z\frac{t^\lambda dt}{(z-t)^{\alpha-n+1}},$$
$$\int\limits_0^z\frac{t^\lambda\log^k(t)dt}{(z-t)^{\alpha+1}},\qquad \int\limits_0^z\frac{t^\lambda\log^k(t)dt}{(z-t)^{\alpha-n+1}}\tag{3.12}$$

in (3.10) or (3.11) diverges we take Hadamard finite part of this integral.

3.2.2 Hadamard finite part

The concept of the "finite part" of a singular integral introduced by Hadamard based on dropping some divergent terms and keeping the finite part.

Let a function $f = f(x)$ be integrable in an interval $\varepsilon < x < A$ for any $0 < \varepsilon$, $\varepsilon < A < \infty$ and the representation

$$\int\limits_{\varepsilon<x<A} f(x)dx = \sum_{k=1}^{N} a_k \varepsilon^{-\lambda_k} + h \ln \frac{1}{\varepsilon} + J_\varepsilon \tag{3.13}$$

hold valid, where a_k, h, λ_k are some constant positive numbers independent of A. If the limit $\lim\limits_{\varepsilon\to 0} J_\varepsilon$ exists, then it is called the Hadamard finite part of the singular integral of the function f. The function $f = f(x)$ is said to possess the Hadamard property at the origin. The standard notation for the finite part of the Hadamard singular integral is as follows

$$f.p. \int\limits_{x<A} f(x)dx = \lim_{\varepsilon\to 0} J_\varepsilon. \tag{3.14}$$

In the case $h = 0$ in the representation (3.13), the function $f = f(x)$ is said to possess the non-logarithmic-type Hadamard property at the origin.

For example, let consider the integral

$$\int\limits_0^\infty x^{-1} f(x)dx,$$

where $f(x)$ is an analytic function on the half infinite interval $[0, \infty)$ such that $f(0) \neq 0$ and $f(x) = O(x^{-\delta})$, $x \to \infty$ with $\delta > 0$. It is easy to see, that this integral is divergent. But if we consider the integral

$$\int\limits_\varepsilon^\infty x^{-1} f(x)dx, \qquad \varepsilon > 0,$$

then by integrating by part we get

$$\int\limits_\varepsilon^\infty x^{-1} f(x)dx = -f(\varepsilon)\log(\varepsilon) - \int\limits_\varepsilon^\infty \log(x) f'(x)dx.$$

Then, the limit

$$\lim_{\varepsilon\to 0}\left(\int\limits_\varepsilon^\infty x^{-1} f(x)dx + f(\varepsilon)\log(\varepsilon)\right)$$

exists and is finite. This limit is the Hadamard finite-part of $\int\limits_0^\infty x^{-1} f(x)dx$ and

$$f.p. \int\limits_0^\infty x^{-1} f(x)dx = \lim_{\varepsilon\to 0}\left(\int\limits_\varepsilon^\infty x^{-1} f(x)dx + f(\varepsilon)\log(\varepsilon)\right).$$

In general, for an analytic function $f(x)$ such that $f(0)\neq 0$ and $f(x)=O(x^{n-\delta-1})$, $x\to\infty$ with $\delta>0$ the $f.p.$ of the integral $\int\limits_0^\infty x^{-n} f(x)dx$ is defined by

$$f.p. \int\limits_0^\infty x^{-n} f(x)dx = \lim_{\varepsilon\to 0}\left(\int\limits_\varepsilon^\infty x^{-n} f(x)dx - \sum_{k=0}^{n-2} \frac{\varepsilon^{k+1-n}}{k!(n-1-k)!} f^{(k)}(0) + \frac{\log(\varepsilon)}{(n-1)!} f^{(n-1)}(0)\right). \quad (3.15)$$

3.2.3 Hadamard finite part for basic power-logarithmic examples

Let consider how to take Hadamard finite part of $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}}$. For the fourth case in (3.10) we have $\mathrm{Re}(\alpha) < 0$ so we do not have a singular point at $t = z$. When $\mathrm{Re}(\lambda) > -1$ we also do not have a singular point at $t = 0$ and can just to calculate integral, so

$$\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}} = \frac{\Gamma(-\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} z^{\lambda-\alpha} \qquad \text{for} \qquad \mathrm{Re}(\lambda) > -1.$$

If $-2 < \mathrm{Re}(\lambda) < -1$ we can consider

$$f.p. \int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}} = \int\limits_0^z \frac{t^\lambda}{(z-t)^\alpha}\left(\frac{1}{z-t} - \frac{1}{z}\right)dt + \frac{\Gamma(1-\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)} z^{\lambda-\alpha}.$$

Integral

$$\int\limits_0^z \frac{t^\lambda}{(z-t)^\alpha}\left(\frac{1}{z-t} - \frac{1}{z}\right)dt = \frac{1}{z}\int\limits_0^z \frac{t^{\lambda+1}dt}{(z-t)^{\alpha+1}} = \frac{\Gamma(-\alpha)\Gamma(\lambda+2)}{\Gamma(\lambda+2-\alpha)} z^{\lambda-\alpha}$$

converges for $\mathrm{Re}(\lambda) > -2$, then

$$
\begin{aligned}
f.p.\int_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}} &= \frac{\Gamma(-\alpha)\Gamma(\lambda+2)}{\Gamma(\lambda+2-\alpha)} z^{\lambda-\alpha} + \frac{\Gamma(1-\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)} z^{\lambda-\alpha} \\
&= [\Gamma(-\alpha)\Gamma(\lambda+2) + \Gamma(1-\alpha)\Gamma(\lambda+1)] \frac{z^{\lambda-\alpha}}{\Gamma(\lambda+2-\alpha)} \\
&= [\Gamma(-\alpha)(\lambda+1)\Gamma(\lambda+1) \\
&\quad +(-\alpha)\Gamma(-\alpha)\Gamma(\lambda+1)] \frac{z^{\lambda-\alpha}}{\Gamma(\lambda+2-\alpha)} \\
&= \frac{(\lambda+1-\alpha)\Gamma(-\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+2-\alpha)} z^{\lambda-\alpha} \\
&= \frac{\Gamma(-\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} z^{\lambda-\alpha}.
\end{aligned}
$$

For $-n-1 < \mathrm{Re}(\lambda) < -n, \lambda \neq -1, -2, ..., -n$ we will use the next regularisation

$$
\begin{aligned}
f.p.\int_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}} &= \int_0^z \frac{t^\lambda}{(z-t)^\alpha} \left(\frac{1}{z-t} - \frac{1}{z} \sum_{k=1}^n \frac{t^{k-1}}{z^{k-1}} \right) dt \\
&\quad + \sum_{k=1}^n \frac{\Gamma(1-\alpha)\Gamma(\lambda+k)}{\Gamma(\lambda+k+1-\alpha)} z^{\lambda-\alpha}.
\end{aligned}
$$

Let us consider the integral

$$
\begin{aligned}
I &= \int_0^z \frac{t^\lambda}{(z-t)^\alpha} \left(\frac{1}{z-t} - \frac{1}{z} \sum_{k=1}^n \frac{t^{k-1}}{z^{k-1}} \right) dt \\
&= \int_0^z \frac{t^\lambda}{(z-t)^\alpha} \left(\frac{1}{z-t} - \frac{1}{z} - \frac{t}{z^2} - .. - \frac{t^{n+1}}{z^n} \right) dt \\
&= \int_0^z \frac{t^\lambda}{(z-t)^\alpha} \left(\frac{z^n - z^{n-1}(z-t) - z^{n-2}t(z-t) - ... - t^{n-1}(z-t)}{z^n(z-t)} \right) dt \\
&= \int_0^z \frac{t^\lambda}{(z-t)^\alpha} \left(\frac{z^n - z^n + z^{n-1}t - z^{n-1}t + z^{n-2}t^2 - ... - zt^{n-1} + t^n}{z^n(z-t)} \right) dt \\
&= \frac{1}{z^n} \int_0^z \frac{t^{\lambda+n} dt}{(z-t)^{\alpha+1}} = \frac{\Gamma(-\alpha)\Gamma(n+\lambda+1)z^{\lambda-\alpha}}{\Gamma(n-\alpha+\lambda+1)}.
\end{aligned}
$$

It converges for $\mathrm{Re}(\lambda) > -n - 1$. Then

$$f.p. \int_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}} = \frac{\Gamma(-\alpha)\Gamma(n+\lambda+1)}{\Gamma(n-\alpha+\lambda+1)} z^{\lambda-\alpha}$$
$$+ \sum_{k=1}^n \frac{\Gamma(1-\alpha)\Gamma(\lambda+k)}{\Gamma(\lambda+k+1-\alpha)} z^{\lambda-\alpha}$$
$$= \left(\frac{\Gamma(n+\lambda+1)}{\Gamma(n-\alpha+\lambda+1)} \right.$$
$$\left. -\alpha \sum_{k=1}^n \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+k+1-\alpha)} \right) \Gamma(-\alpha) z^{\lambda-\alpha}.$$

Calculating sum we get

$$\sum_{k=1}^n \frac{\Gamma(\lambda+k)}{\Gamma(\lambda+k+1-\alpha)} = \frac{1}{\alpha} \left(\frac{\Gamma(n+\lambda+1)}{\Gamma(n-\alpha+\lambda+1)} - \frac{\Gamma(\lambda+1)}{\Gamma(-\alpha+\lambda+1)} \right).$$

Therefore for $\mathrm{Re}(\lambda) > -n - 1$, $\lambda \neq -1, -2, ..., -n$

$$f.p. \int_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}} = \frac{\Gamma(-\alpha)\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} z^{\lambda-\alpha}.$$

So for all α and λ such that $\lambda \neq -1, -2, ..., -n$ we have

$$\frac{d^\alpha z^\lambda}{dz^\alpha} = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} z^{\lambda-\alpha}. \tag{3.16}$$

Now let consider the case when $\lambda \in \mathbb{Z}$ and $\lambda < 0$ i.e. $\lambda = -1, -2, ..., -n$. Here we have two variants. The first one it is the case when $\alpha \in \mathbb{Z}$ and $\lambda < \alpha$. When $\alpha \in \mathbb{Z}$ and $\alpha < 0$ we have $\alpha = -1, -2, ...$ and

$$\frac{d^{-1} z^\lambda}{dz^{-1}} = f.p. \int_0^z t^\lambda dt.$$

Since for $\varepsilon > 0$

$$\int_\varepsilon^z t^\lambda dt = \frac{z^{\lambda+1}}{\lambda+1} - \frac{\varepsilon^{\lambda+1}}{\lambda+1}$$

then

$$\frac{d^{-1}z^{\lambda}}{dz^{-1}} = f.p.\int_0^z t^{\lambda}dt = \frac{z^{\lambda+1}}{\lambda+1} \qquad \text{for} \qquad \lambda = -2, -3, ...$$

Similarly, for $\alpha = -2, -3, ...$ and $\lambda < \alpha$, $\lambda \in \mathbb{Z}$

$$\frac{d^{\alpha}z^{\lambda}}{dz^{\alpha}} = f.p.\underbrace{\int_0^z dt ... \int_0^t dt \int_0^t}_{-\alpha} t^{\lambda}dt$$
$$= \frac{z^{\lambda-\alpha}}{(\lambda+1)(\lambda+2)...(\lambda-\alpha)}.$$

Let $\alpha = -n$, $\lambda = -m$. Since $m > n$ we can write

$$\begin{aligned}\frac{1}{(\lambda+1)(\lambda+2)...(\lambda-\alpha)} &= \frac{1}{(\lambda+1)(\lambda+2)...(\lambda+n)} \\ &= \frac{(-1)^n(-1-n-\lambda)!}{(-1-\lambda)!} \\ &= \frac{\lambda!}{(\lambda-\alpha)!} = (-1)^{\alpha}(-\lambda)_{\alpha}.\end{aligned}$$

So for all α and λ such that $\alpha \in \mathbb{Z}$, $\lambda \in \mathbb{Z}$, $\lambda < 0$ and $\lambda < \alpha$ taking into account (3.9) we have

$$\frac{d^{\alpha}z^{\lambda}}{dz^{\alpha}} = (-1)^{\alpha}(-\lambda)_{\alpha}z^{\lambda-\alpha}. \tag{3.17}$$

In the case $\lambda \in \mathbb{Z}$, $\lambda < 0$ and $\lambda \geq \alpha$ the integration of z^{λ} can produce $\log(z)$ (for negative integer λ when $\lambda \geq \alpha$). For example, when $\lambda = -1$ and $\alpha = -1$

$$\int_{\varepsilon}^{z} \frac{dt}{t} = \log(z) - \log(\varepsilon)$$

so by (3.13) and (3.14) we get

$$\frac{d^{-1}z^{-1}}{dz^{-1}} = f.p.\int_0^z \frac{dt}{t} = \log(z).$$

The same regularisation we can obtain if we consider the Laurent series expansion by λ at $\lambda = -1$:

$$\begin{aligned}\frac{d^{-1}z^{\lambda}}{dz^{-1}} &= \int\limits_0^z t^{\lambda}dt\\ &= \frac{z^{\lambda+1}}{\lambda+1} = \frac{1}{\lambda+1}\left[1+(\lambda+1)\log(z)+O\left((\lambda+1)^2\right)\right]\\ &= \frac{1}{\lambda+1}+\log(z)+O(\lambda+1).\end{aligned}$$

Then throwing away the main part $\frac{1}{\lambda+1}$ we get

$$\frac{d^{-1}z^{-1}}{dz^{-1}} = f.p.\int\limits_0^z \frac{dt}{t} = \lim_{\lambda\to-1}[\log(z)+O(\lambda+1)] = \log(z).$$

In general case when $\lambda = -n$, $n \in \mathbb{Z}$, $\alpha \leq -n$ the Laurent series expansion of (3.16) by λ at $\lambda = -n$ is

$$\begin{aligned}\frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}z^{\lambda-\alpha} &= \frac{(-1)^{\lambda-1}z^{\lambda-\alpha}}{(-1-\lambda)!\,\Gamma(1+\lambda-\alpha)(\lambda+n)}\\ &\quad+\frac{(-1)^{\lambda-1}(\psi(-\lambda)-\psi(\lambda-\alpha+1)+\log(z))}{(-\lambda-1)!\Gamma(\lambda-\alpha+1)}z^{\lambda-\alpha}\\ &\quad+O(\lambda+n).\end{aligned}$$

Here ψ is the diGamma function, given by $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$. Then throwing away the main part of the Laurent series expansion we obtain

$$\begin{aligned}\frac{d^{\alpha}z^{-n}}{dz^{\alpha}} &= f.p.\int\limits_0^z \frac{t^{\lambda}dt}{(z-t)^{\alpha+1}}\\ &= \lim_{\lambda\to-n}\left[\frac{(-1)^{\lambda-1}(\psi(-\lambda)-\psi(\lambda-\alpha+1)+\log(z))}{(-\lambda-1)!\Gamma(\lambda-\alpha+1)}z^{\lambda-\alpha}+O(\lambda+n)\right]\\ &= \frac{(-1)^{-n-1}(\psi(n)-\psi(1-n-\alpha)+\log(z))}{(n-1)!\Gamma(1-n-\alpha)}z^{-n-\alpha}.\end{aligned}\tag{3.18}$$

So we put

$$\frac{d^{\alpha}z^{\lambda}}{dz^{\alpha}} = \frac{(-1)^{\lambda-1}(\psi(-\lambda)-\psi(\lambda-\alpha+1)+\log(z))}{(-\lambda-1)!\Gamma(\lambda-\alpha+1)}z^{\lambda-\alpha}\tag{3.19}$$

under certain restrictions.

Finally, combining formulas (3.16), (3.17) and (3.19) we get

$$\frac{d^\alpha z^\lambda}{dz^\alpha} = z^{\lambda-\alpha} \begin{cases} (-1)^\alpha (-\lambda)_\alpha, & \alpha \in \mathbb{Z}, \lambda \in \mathbb{Z}, \lambda < 0, \lambda < \alpha; \\ \frac{(-1)^{\lambda-1}(\log(z)+\psi(-\lambda)-\psi(1-\alpha+\lambda))}{(-\lambda-1)!\Gamma(1-\alpha+\lambda)}, & \lambda \in \mathbb{Z}, \lambda < 0; \\ \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)}, & \text{in other cases.} \end{cases} \tag{3.20}$$

The second integral $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha-n+1}}$ in (3.12) is calculated in same way since $\mathrm{Re}(\alpha-n) < 0$.

We also need to calculate other integrals in (3.12). Calculation and taking the Hadamard finite part of $\int\limits_0^z \frac{t^\lambda \log^k(t)dt}{(z-t)^{\alpha+1}}$ and $\int\limits_0^z \frac{t^\lambda \log^k(t)dt}{(z-t)^{\alpha-n+1}}$ based on the same ideas as calculation of $\int\limits_0^z \frac{t^\lambda dt}{(z-t)^{\alpha+1}}$ but much more complicated. For example, when $\mathrm{Re}(\alpha) < 0$, $\mathrm{Re}(z) > 0$ and $\mathrm{Re}(\lambda) > -1$ we have

$$\frac{d^\alpha}{dz^\alpha} z^\lambda \log(z) = z^{\lambda-\alpha} \frac{\Gamma(\lambda+1)\,(H_\lambda - H_{\lambda-\alpha} + \log(z))}{\Gamma(\lambda-\alpha+1)}, \tag{3.21}$$

where $H_s = \gamma + \psi(s+1)$ is the harmonic number, $H_n = \sum\limits_{k=1}^{n} \frac{1}{k}$ for integer n, $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$ is the diGamma function, γ is Euler–Mascheroni constant. However, in general

$$\frac{d^\alpha}{dz^\alpha}\left(z^\lambda \log(z)\right) = z^{\lambda-\alpha} \begin{cases} (-1)^\alpha(-\lambda)_\alpha(\psi(-\lambda)-\psi(\alpha-\lambda)+\log(z)); & \alpha, \lambda \in \mathbb{Z}, \lambda < 0, \lambda < \alpha, \\ \frac{(-1)^{\lambda+1}}{2\Gamma(-\lambda)\Gamma(\lambda-\alpha+1)}\Big(\log^2(z)+ & \\ +\frac{\pi^2}{3} - \psi^{(1)}(\lambda-\alpha+1) - \psi^{(1)}(-\lambda)+ & \lambda \in \mathbb{Z}, \lambda < 0, \\ +(\psi(-\lambda)-\psi(\lambda-\alpha+1))\times & \\ \times(\psi(-\lambda) - H_{\lambda-\alpha} + 2\log(z) + \lambda)\Big); & \\ \frac{\Gamma(\lambda+1)(H_\lambda - H_{\lambda-\alpha}+\log(z))}{\Gamma(\lambda-\alpha+1)}; & \text{in other cases,} \end{cases} \tag{3.22}$$

Here $\psi^{(m)}(z) = \frac{\mathrm{d}^{m+1}}{\mathrm{d}z^{m+1}} \ln \Gamma(z)$ is the polyGamma function of order m.

Expanding basic functions in Mathematica to sums of the forms (3.3) or (3.7) allowed us to build fractional derivatives of order α for more than 100000 test cases, involving basic functions and their compositions.

3.3 The Meijer G-function and fractional calculus

3.3.1 Definitions and main properties

All known special functions can be divided on several large groups:

- basic special functions,
- hypergeometric type functions,
- Riemann Zeta and related functions,
- elliptic and related functions,
- number theoretic functions,
- generalized functions and other standard special functions.

Basic special functions refer to a class of mathematical functions that have well-established properties and play a fundamental role in various areas of mathematics, physics, and engineering. These functions often arise in the solutions of differential equations, integrals, and other mathematical problems. Some of the most commonly encountered basic special functions include Gamma function, Beta function, Bessel functions, hypergeometric functions. Informations about some special functions can be found in [81].

The majority of the most important special functions are the analytic functions, which by definitions typically can be presented through infinite series:

$$(z-z_0)^\alpha \log^k(z-z_0)\sum_{n=0}^{\infty} c_n (z-z_0)^n, \tag{3.23}$$

where $k\in\mathbb{Z}$, $k\geq 0$, $c_j,\alpha\in\mathbb{C}$, $j=0,1,2....$ Such series can be analytically continued (re-expanded) or represented through integrals. Very often they are solutions of corresponding differential equations, which came from applications and define special functions.

Hypergeometric type functions can be defined as the functions, which generically can be represented through linear combinations of Meijer G-function which is a very general special function of the form [143], see also handbook [99]:

$$\begin{aligned} G_{p,q}^{m,n} &= G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix} a_1,\ldots,a_n,a_{n+1},\ldots,a_p \\ b_1,\ldots,b_m,b_{m+1},\ldots,b_q\end{matrix}\right) \\ &= \frac{1}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{k=1}^{m}\Gamma(b_k-s)\prod_{k=1}^{n}\Gamma(1-a_k+s)}{\prod_{k=n+1}^{p}\Gamma(a_k-s)\prod_{k=m+1}^{q}\Gamma(1-b_k+s)} z^s\,ds, \end{aligned} \tag{3.24}$$

where contour $\mathcal{L}$ represents the path to be followed while integrating. This integral is of the so-called Mellin–Barnes type, and may be viewed as a generalization of inverse Mellin transform. Information about using of Mellin-Barnes integrals in asymptotic analysis can be found in [87]. There are three different paths $\mathcal{L}$ of integration:

- $\mathcal{L}$ runs from $-i\infty$ to $+i\infty$ so that all poles of $\Gamma(b_i - s)$, $i = 1, ..., m$, are to the right, and all the poles of $\Gamma(1 - a_k + s)$, $k = 1, ..., n$, to the left, of $\mathcal{L}$. This contour can be a vertical straight line $(\gamma - i\infty, \gamma + i\infty)$ if $\mathrm{Re}(b_i - a_k) > -1$ for $i = 1, ..., m$ and $k = 1, ..., n$, (then $\mathrm{Re}(a_k) - 1 < \gamma < \mathrm{Re}(b_i)$). The integral converges if $p+q < 2(m+n)$ and $|\arg z| < \left(m + n - \frac{p+q}{2}\right)\pi$. If $m+n-\frac{p+q}{2} = 0$, then z must be real and positive and additional condition $(q - p)\gamma + \mathrm{Re}(\mu) < 0$, $\mu = \sum\limits_{i=1}^{q} b_i - \sum\limits_{k=1}^{p} a_k + \frac{p-q}{2} + 1$, should be added.
- $\mathcal{L}$ is a loop starting and ending at $+\infty$ and encircling all poles of $\Gamma(b_i - s)$, $i = 1, ..., m$, once in the negative direction, but none of the poles of $\Gamma(1 - a_k + s)$, $k = 1, ..., n$. The integral converges if $q \geq 1$ and either $p < q$ or $p = q$ and $|z| < 1$ or $q = p$ and $|z| = 1$ and $m + n - \frac{p+q}{2} \geq 0$ and $\mathrm{Re}(\mu) < 0$.
- $\mathcal{L}$ is a loop starting and ending at $-\infty$ and encircling all poles of $\Gamma(1 - a_k + s)$, $k = 1, ..., n$, once in the positive direction, but none of the poles of $\Gamma(b_i - s)$, $i = 1, ..., m$. The integral converges if $p \geq 1$ and either $p > q$ or $p = q$ and $|z| > 1$ or $q = p$ and $|z| = 1$ and $m + n - \frac{p+q}{2} \geq 0$ and $\mathrm{Re}(\mu) < 0$.

Above definition of the Meijer G-function holds under the following assumptions:

- $0 \leq m \leq q$ and $0 \leq n \leq p$, where m, n, p and q are integer numbers,
- $a_k - b_j \neq 1, 2, 3, ...$ for $k = 1, 2, ..., n$ and $j = 1, 2, ..., m$, which implies that no pole of any $\Gamma(b_j - s)$, $j = 1, 2, ..., m$, coincides with any pole of any $\Gamma(1 - a_k + s)$, $k = 1, 2, ..., n$.
- $z \neq 0$.

Remark 1 A different from (3.24) but equivalent to it form was used in the books [4, 73, 99]:

$$G^{m,n}_{p,q} = G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right)$$
$$= \frac{1}{2\pi i} \int\limits_{\mathcal{L}} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k + s) \prod\limits_{k=1}^{n} \Gamma(1 - a_k - s)}{\prod\limits_{k=n+1}^{p} \Gamma(a_k + s) \prod\limits_{k=m+1}^{q} \Gamma(1 - b_k - s)} z^{-s}\, ds \tag{3.25}$$

where

$$m, n, p, q \in \mathbb{N} \cup \{0\}, \qquad m \leq q, \qquad n \leq p.$$

Both forms can be transformed each to other by changing variable of integration $s \to -s$.

In the system Mathematica for standard form of the Meijer G-function the following notation is used

$$\texttt{MeijerG}[\{\{\mathrm{a}_1, ..., \mathrm{a}_\mathrm{n}\}, \{\mathrm{a}_{\mathrm{n}+1}, ..., \mathrm{a}_\mathrm{p}\}\}], \{\{\mathrm{b}_1, ..., \mathrm{b}_\mathrm{m}\}, \{\mathrm{b}_{\mathrm{m}+1}, ..., \mathrm{b}_\mathrm{q}\}\}, \mathrm{z}] =$$
$$= \frac{1}{2\pi i}\texttt{ContourIntegrate}\left(\frac{\prod_{\mathrm{k}=1}^{\mathrm{n}} \Gamma(1-\mathrm{a_k}-\mathrm{s}) \prod_{\mathrm{k}=1}^{\mathrm{m}} \Gamma(\mathrm{b_k}+\mathrm{s})}{\prod_{\mathrm{k}=\mathrm{n}+1}^{\mathrm{p}} \Gamma(\mathrm{a_k}+\mathrm{s}) \prod_{\mathrm{k}=\mathrm{m}+1}^{\mathrm{q}} \Gamma(1-\mathrm{b_k}-\mathrm{s})}\mathrm{z}^{-\mathrm{s}}, \{\mathrm{s}, \mathcal{L}\}\right);$$
$$\mathrm{m} \in \mathbb{Z} \wedge \mathrm{m} \geq 0 \wedge \mathrm{n} \in \mathbb{Z} \wedge \mathrm{n} \geq 0 \wedge \mathrm{p} \in \mathbb{Z} \wedge \mathrm{p} \geq 0 \wedge \mathrm{q} \in \mathbb{Z} \wedge \mathrm{q} \geq 0 \wedge \mathrm{m} \leq \mathrm{q} \wedge \mathrm{n} \leq \mathrm{p}.$$

In many classical special functions (like Bessel functions) instead of z the construction $cz^{1/r}$ can be used and under integrand factor z^{-s} becomes $(cz^{1/r})^{-s}$ which is not equal to the expression $c^{-s}z^{-s/r}$. Correct formulas for these transformation one can find at Wolfram Function Site, for example in [138].

Described properties are supported in Mathematica by command PowerExpand with or without assumption option:

{PowerExpand[($(z^a)^b$), Assumptions → True], PowerExpand[$(z^a)^b$]}

$$\left\{e^{2i\pi b\left\lfloor \frac{1}{2}-\frac{\operatorname{Im}(a\log(z))}{2\pi}\right\rfloor}z^{ab}, z^{ab}\right\}$$

This situation stimulated us to define generalized form of the Meijer G-function with additional real parameter r, which allowed effectively without restrictions work in the full complex plane with functions like $J_a(z)$.

Remark 2 The generalized form of the Meijer G-function with additional real parameter r is defined in [137] by similar to (3.24) integral

$$G^{m,n}_{p,q}\left(z, r\,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right)$$

$$= \frac{r}{2\pi i}\int_{\mathcal{L}} \frac{\prod_{k=1}^{m} \Gamma(b_k+s) \prod_{k=1}^{n} \Gamma(1-a_k-s)}{\prod_{k=n+1}^{p} \Gamma(a_k+s) \prod_{k=m+1}^{q} \Gamma(1-b_k-s)} z^{-\frac{s}{r}}\, ds, \tag{3.26}$$

where $r \in \mathbb{R}$, $r \neq 0$, $m \in \mathbb{Z}$, $m \geq 0$, $n \in \mathbb{Z}$, $n \geq 0$, $p \in \mathbb{Z}$, $p \geq 0$, $q \in \mathbb{Z}$, $q \geq 0$, $m \leq q$, $n \leq p$.

Evidently that for default case $r = 1$ we have equality

$$G^{m,n}_{p,q}\left(z, 1\,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right) = G^{m,n}_{p,q}\left(z\,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right).$$

This Meijer G-function with parameter r satisfies two important properties [139, 140]:

$$G_{q,p}^{n,m}\left(\frac{1}{z}, r \,\middle|\, \begin{matrix} 1-b_1, ..., 1-b_m, 1-b_{m+1}, ..., 1-b_q \\ 1-a_1, ..., 1-a_n, 1-a_{n+1}, ..., 1-a_p \end{matrix}\right)$$

$$= G_{p,q}^{m,n}\left(z, r \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right), \tag{3.27}$$

$$G_{p,q}^{m,n}\left(z, r \,\middle|\, \begin{matrix} \alpha+a_1, ..., \alpha+a_n, \alpha+a_{n+1}, ..., \alpha+a_p \\ \alpha+b_1, ..., \alpha+b_m, \alpha+b_{m+1}, ..., \alpha+b_q \end{matrix}\right)$$

$$= z^{\alpha/r} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right). \tag{3.28}$$

3.3.2 Representability through Meijer *G*-functions

Using classical and generalized Meijer G-functions we can represent Bessel function $J_\nu(z)$ by formulas

$$J_\nu(z) = G_{0,2}^{1,0}\left(\frac{z^2}{4} \,\middle|\, \begin{matrix} - \\ \frac{\nu}{2}, -\frac{\nu}{2} \end{matrix}\right), \qquad -\frac{\pi}{2} < \arg(z) \le \frac{\pi}{2}, \tag{3.29}$$

$$J_\nu(z) = G_{0,2}^{1,0}\left(\frac{z}{2}, \frac{1}{2} \,\middle|\, \begin{matrix} - \\ \frac{\nu}{2}, -\frac{\nu}{2} \end{matrix}\right). \tag{3.30}$$

First formula includes parameter $r = 1/2$ but it is valid for all complex z-plane. Second formula including classical G-function is correct only for half plane. For finding last representation through Meijer G-function we can use commands `ResourceFunction["MeijerGForm"]` and `MeijerGReduce`:

```
ResourceFunction["MeijerGForm"][BesselJ[ν, z], z],
MeijerGReduce[BesselJ[ν, z], z].
```

For best understanding Meijer G-function we can suggest to use several internal commands `MeijerGInfo`, `MeijerGToSums`, `SlaterForm`, which operate with context `System`MeijerGDump``, for example,

`System`MeijerGDump`SlaterForm[MeijerG[{{a}, {b}}, {{c}, {d}}, z, 2/3], s]`

gives

$$\frac{2z^{-s}\Gamma\left(1-a-\frac{2s}{3}\right)\Gamma\left(c+\frac{2s}{3}\right)}{3\Gamma\left(b+\frac{2s}{3}\right)\Gamma\left(1-d-\frac{2s}{3}\right)}.$$

`SlaterForm` applied to the Meijer G-function reveals the internal structure of the Meijer G-function, including the ratio of products of groups with the Gamma function corresponding to their lengths $m, n, p-n, q-m$.

Through Meijer G-function we can represent more than 100 named well known functions such as log, exp, arctan, Bessel, Airy and Legendre functions. In Mathematica `TraditionalForm` of these functions form the following list with 127 functions to which we can apply `ResourceFunction["MeijerGForm"]`. See below List `MeijerG127` in `TraditionalForm`.

$$\texttt{MeijerG127=}\{a^z, e^z, \sqrt{z}, z^b, \mathrm{Ai}\,(z), \mathrm{Ai}'(z), \mathrm{Bi}(z), \mathrm{Bi}'(z), \boldsymbol{J}_a(z), \boldsymbol{J}_a^b(z), \\ \operatorname{arccos}(z), \operatorname{arcosh}(z), \operatorname{arccot}(z), \operatorname{arcoth}(z), \operatorname{arccsc}(z), \operatorname{arcsch}(z), \operatorname{arcsec}(z), \\ \operatorname{arsech}(z), \arcsin(z), \operatorname{arsinh}(z), \arctan(z), \arctan(a,z), \arctan(z,a), \operatorname{artanh}(z), \\ I_a(z), J_a(z), K_a(z), Y_a(z), B_z(a,b), B_{(c,z)}(a,b), B_{(z,c)}(a,b), I_z(a,b), I_{(c,z)}(a,b), \\ I_{(z,c)}(a,b), R_C(x,z), R_C(z,y), R_E(x,z), R_E(y,z), R_K(x,z), R_K(y,z), T_a(z), \\ U_a(z), \cos(z), \cosh(z), \mathrm{Chi}(z), \mathrm{Ci}(z), F(z), E(z), K(z), \operatorname{erf}(z), \operatorname{erf}(a,z), \operatorname{erf}(z,b), \\ \operatorname{erfc}(z), \operatorname{erfi}(z), E_a(z), \mathrm{Ei}(z), F_z, F_a(z), C(z), F(z), G(z), S(z), \Gamma(a,z), \Gamma(a,b,z), \\ \Gamma(a,z,b), Q(a,z), Q(a,b,z), Q(a,z,b), C_a^{(b)}(z), H_a^{(1)}(z), H_a^{(2)}(z), \mathrm{hav}(z), H_a(z), \\ {}_0F_1(;a;z),\ {}_0\tilde{F}_1(;a;z),\ {}_1F_1(a;b;z),\ {}_1\tilde{F}_1(a;b;z),\ {}_2F_1(a,b;c;z),\ {}_2\tilde{F}_1(a,b;c;z), \\ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z),\ {}_p\tilde{F}_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z), U(a,b,z), \mathrm{hav}^{-1}(z), \\ \mathrm{bei}_0(z), \mathrm{bei}_a(z), \mathrm{ber}_0(z), \mathrm{ber}_a(z), \mathrm{kei}_0(z), \mathrm{kei}_a(z), \mathrm{ker}_0(z), \mathrm{ker}_a(z), L_a(z), L_a^b(z), \\ P_\nu(z), P_a^b(z), Q_\nu(z), Q_a^b(z), \log(z), L_z, L_a(z), D_a(z), \mathrm{Li}_2(z), \mathrm{Li}_a(z), \mathrm{Gi}(z), \mathrm{Gi}'(z), \\ \mathrm{Hi}(z), \mathrm{Hi}'(z), \sin(z), \mathrm{sinc}(z), \sinh(z), \mathrm{Shi}(z), \mathrm{Si}(z), j_a(z), y_a(z), h_a^{(1)}(z), h_a^{(2)}(z), \\ \boldsymbol{H}_\nu(z), \boldsymbol{L}_\nu(z), \theta(z), \boldsymbol{E}_\nu(z), \boldsymbol{E}_\nu^a(z), M_{a,b}(z), W_{a,b}(z)\ \}$$

Different combinations, including these functions also sometimes can be presented through Meijer G-function. Long time work in this direction allowed to build basic collection of such functions with about 3000 cases and more than 90000 cases were tested.

3.3.3 Fractional integro-differentiation of Meijer G–functions

For all above functions one can find different representations of the fractional integro-derivatives searching Wolfram Functions Site [142]. In particular, definition formula (3.1) for fractional integro-derivative of the order α of the Meijer G-function can be converted to the following representation (see also [143]):

$$\frac{d^\alpha}{dz^\alpha} G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix}\right)$$

$$= G_{p+1,q+1}^{m,n+1}\left(z \,\middle|\, \begin{matrix} -\alpha, a_1-\alpha, \ldots, a_n-\alpha, a_{n+1}-\alpha, \ldots, a_p-\alpha \\ b_1-\alpha, \ldots, b_m-\alpha, 0, b_{m+1}-\alpha, \ldots, b_q-\alpha \end{matrix}\right). \tag{3.31}$$

We can convert the right hand side of this formula to Fox H-function if use the following input in `Wolfram Mathematica`:

`ResourceFunction`["FractionalOrderD"][MeijerG[{Table[Subscript[a,i], {i, 1, n}], Table[Subscript[a,i], {i, n + 1, p}]}, {Table[Subscript[b,i], {i, 1, m}], Table[Subscript[b,i], {i, m + 1, q}]}, z], {z, α}]

Below we demonstrate more general formula for generalized Meijer G-function with argument a z^r and parameter r_1:

$$\frac{d^\alpha}{dz^\alpha} G_{p,q}^{m,n}\left(az^r, r_1 \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right)$$

$$= r_1 z^{-\alpha}(az^r)^{\alpha/r} H_{p+1,q+1}^{m,n+1}\left(az^r \,\middle|\, \begin{matrix} \{-\alpha, r\}, \left\{a_1 - \frac{r_1}{r}\alpha, r_1\right\}, ..., \left\{a_p - \frac{r_1}{r}\alpha, r_1\right\} \\ \left\{b_1 - \frac{r_1}{r}\alpha, r_1\right\}, ..., \left\{b_q - \frac{r_1}{r}\alpha, r_1\right\}, \{0, r\} \end{matrix}\right). \tag{3.32}$$

In general (non logarithmic) cases the Meijer G-function can be represented through one or finite combination of the series (3.23) with $z_0 = 0$ and $k = 0$.

Assume $p \le q$, no two of the bottom parameters b_j, $j = 1, ..., m$, differ by an integer, and $(a_j - b_k)$ is not a positive integer when $j = 1, 2, ..., n$ and $k = 1, 2, ..., m$. Then double sums of "left" residues for $G_{p,q}^{m,n}$ can be written as finite sum of generalized hypergeometric functions (see [141]):

$$G_{p,q}^{m,n}\left(z \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right) =$$

$$= \sum_{k=1}^{m} A_{p,q,k}^{m,n}\, z^{b_k}\, {}_pF_{q-1}\left(\begin{matrix} 1 + b_k - a_1, ..., 1 + b_k - a_p \\ 1 + b_k - b_1, ... * ....1 + b_k - b_q \end{matrix}; (-1)^{p-m-n} z\right), \tag{3.33}$$

where " $*$ " indicates that the entry $(1 + b_k - b_k)$ is omitted. Also,

$$A_{p,q,k}^{m,n} = \frac{\prod\limits_{j=1, j\neq k}^{m} \Gamma(b_j - b_k) \prod\limits_{j=1}^{n} \Gamma(1 + b_k - a_j)}{\prod\limits_{j=n+1}^{p} \Gamma(a_j - b_k) \prod\limits_{j=m+1}^{q} \Gamma(1 + b_k - b_j)}.$$

The Function Site [142] includes the most complete collection of formulas for asymptotics of Meijer G-functions. In more complicated logarithmic cases, the Meijer G-function can be represented through a finite combination of the above series (3.33) with $z_0 = 0$ and $k > 0$. It means that we can evaluate Meijer G-function, as infinite sums including terms $z^b \log^k(z)$ with already uniquely defined basic elementary functions: power and logarithmic.

So if we have a hypergeometric type function $f(z)$ we first can write $f(z)$ as $z^b g(z)$ or $z^b \log^k(z) g(z)$ where $g(x)$ is supposed to be simpler than $f(z)$. Using the series expansion of the function $g(z)=\sum\limits_{n=0}^{\infty}c_n z^n$ and formula (3.33) we can rewrite $f(z)$ as a single G-function. In more complex cases, the function $f(z)$ can be written as a finite sum of G-functions. Next applying formula (3.31) we can find fractional integral or derivative of $f(z)$ in the form of the Meijer G-function. Then we can write the G-function as a simpler function if possible.

3.3.4 Fractional integro-differentiation of e^z and $K_\nu(z)$

In Wolfram Mathematica there is a function `MeijerGForm[g(z),z]` which reduces $g(z)$ to the Meijer G–function as a function of z. In order to use function `MeijerGForm` we should write

`ResourceFunction["MeijerGForm"].`

For example, `ResourceFunction["MeijerGForm"][`e^z`, z]` gives

$$e^z=\mathrm{MeijerG}(\{\{\},\{\}\},\{\{0\},\{\}\},-z,1)=G^{1,0}_{0,1}\left(-z,1\,\middle|\,\begin{matrix}-\\0\end{matrix}\right)=G^{1,0}_{0,1}\left(-z\,\middle|\,\begin{matrix}-\\0\end{matrix}\right)$$

and `ResourceFunction["MeijerGForm"][BesselK[`ν`, z], z]` gives

$$K_\nu(z)=\frac{1}{2}\mathrm{MeijerG}\left(\{\{\},\{\}\},\left\{\left\{\frac{\nu}{2},-\frac{\nu}{2}\right\},\{\}\right\},\frac{z}{2},\frac{1}{2}\right)=G^{2,0}_{0,2}\left(\frac{z}{2},\frac{1}{2}\,\middle|\,\begin{matrix}-\\\frac{\nu}{2},-\frac{\nu}{2}\end{matrix}\right),$$

where formula (3.26) was used.

Let us compare the approach of finding the fractional operator $\frac{d^\alpha}{dz^\alpha}$ through series expansion and through the G-function representation. Direct calculation through the G-function representation gives

$$\frac{d^\alpha e^z}{dz^\alpha}=\begin{cases}e^z, & \alpha\in\mathbb{Z}\text{ and }\alpha\geq 0;\\ e^z(1-Q(-\alpha,z)), & \text{in other cases,}\end{cases}\tag{3.34}$$

where $Q(a,z)=\frac{\Gamma(a,z)}{\Gamma(a)}$ is the regularized incomplete Gamma function, $\Gamma(a,z)=\int\limits_z^\infty t^{a-1}e^{-t}dt$ is the incomplete Gamma function. The exponential function e^z has Taylor series at $z=0$ of the form

$$e^z=\sum_{n=0}^{\infty}\frac{z^n}{n!}.$$

Since $n\in\mathbb{N}\cup\{0\}$ using the third line in (3.20) we obtain

$$\frac{d^\alpha e^z}{dz^\alpha}=\sum_{n=0}^{\infty}\frac{1}{n!}\frac{d^\alpha z^n}{dz^\alpha}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \frac{\Gamma(n+1) z^{n-\alpha}}{\Gamma(n-\alpha+1)} = \sum_{n=0}^{\infty} \frac{z^{n-\alpha}}{\Gamma(n-\alpha+1)}$$
$$= e^z \left(1 + \frac{\alpha \Gamma(-\alpha, z)}{\Gamma(1-\alpha)}\right).$$

For $\alpha = 0, 1, 2, ...$ we have $\frac{\alpha \Gamma(-\alpha,z)}{\Gamma(1-\alpha)} = 0$ and

$$\frac{\alpha \Gamma(-\alpha, z)}{\Gamma(1-\alpha)} = \frac{\alpha \Gamma(-\alpha, z)}{-\alpha \Gamma(-\alpha)} = -\frac{\Gamma(-\alpha, z)}{\Gamma(-\alpha)} = -Q(a, z),$$

therefore, the result coincides with formula (3.34).

For $K_0(z)$ we have

$$\frac{d^\alpha}{dz^\alpha} K_0(z) = \frac{1}{2} G_{2,4}^{2,2} \left(\frac{z}{2}, \frac{1}{2} \middle| \begin{matrix} \frac{1-\alpha}{2}, -\frac{\alpha}{2} \\ -\frac{\alpha}{2}, -\frac{\alpha}{2}, 0, \frac{1}{2} \end{matrix} \right). \tag{3.35}$$

On the other hand by (3.5)

$$\frac{d^\alpha}{dz^\alpha} K_0(z) = \frac{d^\alpha}{dz^\alpha} \left(-(\log(z) - \log(2) + \gamma) \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{z}{2}\right)^{2n} + \sum_{n=1}^{\infty} \frac{H_n}{(n!)^2} \left(\frac{z}{2}\right)^{2n} \right),$$

where γ is Euler–Mascheroni constant (3.6). Applying (3.20) and (3.21) we obtain

$$\begin{aligned} &\frac{d^\alpha}{dz^\alpha} K_0(z) \\ &\quad = \frac{1}{2} \sum_{k=0}^{\infty} \frac{2^{1-2k} (2k)! \psi(k+1)}{(k!)^2 \Gamma(2k-\alpha+1)} z^{2k-\alpha} \\ &\quad - \sum_{k=0}^{\infty} \frac{(2k)! \left(H_{2k} - H_{2k-\alpha} + \log\left(\frac{z}{2}\right)\right)}{2^{2k} (k!)^2 \Gamma(2k-\alpha+1)} z^{2k-\alpha}. \end{aligned} \tag{3.36}$$

Using Mathematica one can check that (3.35) equals to (3.36).

3.3.5 Riemann-Liouville integral of generalized Meijer G–function

Let us now evaluate the following "main" Riemann-Liouville integral

$$\frac{1}{\Gamma(\beta)} \int_0^z (z-\tau)^{\beta-1} \tau^{\alpha-1} G_{p,q}^{m,n} \left(w\tau^g, r \middle| \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix} \right) d\tau \tag{3.37}$$

which defines fractional order β integral of generalized Meijer G-function (3.26), multiplied by $\tau^{\alpha-1}$, with $r \in \mathbb{R}$, $r \neq 0$, $m \in \mathbb{Z}$, $m \geq 0$, $n \in \mathbb{Z}$, $n \geq 0$, $p \in \mathbb{Z}$, $p \geq 0$,

$q \in \mathbb{Z}, q \geq 0, m \leq q, n \leq p$. In formula (3.26) for simplicity we use vertical contour $\mathfrak{L} = \{\gamma - i\infty, \gamma + i\infty\}$.

The integral (3.37) with particular parameters $g = \frac{\ell}{k} \in \mathbb{Q}$ and $r = 1$ has value, shown at [4], p. 535, formula 3.36.2.1. Below we derive representations of this integral through Fox H-function or Meijer G-function and demonstrate in detail how to build the corresponding sets of conditions for convergence of this integral.

After substitution of above definition of Meijer G-function to Riemann-Liouville integral and changing the order of integration with respect to s and τ we come to the following chain operations where we met and evaluated internal integral

$$
\begin{aligned}
I &= \frac{1}{\Gamma(\beta)} \int_0^z (z-\tau)^{\beta-1} \tau^{\alpha-1} G_{p,q}^{m,n} \left(w\tau^g, r \,\middle|\, \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix} \right) d\tau \\
&= \frac{1}{2\pi i \Gamma(\beta)} \int_0^z (z-\tau)^{\beta-1} \tau^{\alpha-1} \\
&\times \left(\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k+s) \prod\limits_{k=1}^{n} \Gamma(1-a_k-s)}{\prod\limits_{k=n+1}^{p} \Gamma(a_k+s) \prod\limits_{k=m+1}^{q} \Gamma(1-b_k-s)} (w\tau^g)^{-\frac{s}{r}} \, ds \right) d\tau \\
&= \frac{1}{2\pi i \Gamma(\beta)} \int_{\gamma-i\infty}^{\gamma+i\infty} w^{-\frac{s}{r}} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k+s) \prod\limits_{k=1}^{n} \Gamma(1-a_k-s)}{\prod\limits_{k=n+1}^{p} \Gamma(a_k+s) \prod\limits_{k=m+1}^{q} \Gamma(1-b_k-s)} \times \\
&\times \left(\int_0^z (z-\tau)^{\beta-1} \tau^{\alpha-g\frac{s}{r}-1} d\tau \right) ds \\
&= \frac{z^{\alpha+\beta-1}}{2\pi i} \int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\prod\limits_{k=1}^{m} \Gamma(b_k+s) \prod\limits_{k=1}^{n} \Gamma(1-a_k-s) \Gamma\left(\alpha - \frac{g}{r}s\right)}{\prod\limits_{k=n+1}^{p} \Gamma(a_k+s) \prod\limits_{k=m+1}^{q} \Gamma(1-b_k-s) \Gamma\left(\alpha+\beta-\frac{g}{r}s\right)} \\
&\times \left(w^{\frac{1}{r}} z^{\frac{g}{r}} \right)^{-s} ds.
\end{aligned} \tag{3.38}
$$

The last Mellin-Barnes integral (3.38) can be written through the corresponding Fox H-function by definition of Fox H-function. As a result we have the following representation of the Riemann-Liouville fractional order β integral of the generalized Meijer G-function through Fox H-function

$$
\frac{1}{\Gamma(\beta)} \int_0^z (z-\tau)^{\beta-1} \tau^{\alpha-1} G_{p,q}^{m,n} \left(w\tau^g, r \,\middle|\, \begin{matrix} a_1, \ldots, a_n, a_{n+1}, \ldots, a_p \\ b_1, \ldots, b_m, b_{m+1}, \ldots, b_q \end{matrix} \right) d\tau
$$

$$= z^{\alpha+\beta-1} H_{p,q}^{m,n} \left[w^{\frac{1}{r}} z^{\frac{g}{r}} \middle| \begin{matrix} \left(1-\alpha, \frac{g}{r}\right), (a_1, 1), ..., (a_n, 1), (a_{n+1}, 1), ..., (a_p, 1) \\ (b_1, 1), ..., (b_m, 1), (b_{m+1}, 1), ..., (b_q, 1), \left(1-\alpha-\beta, \frac{g}{r}\right) \end{matrix} \right]. \tag{3.39}$$

We should mention that operations like $(w\tau^g)^{-\frac{s}{r}} \to \left(w^{\frac{1}{r}} z^{\frac{g}{r}}\right)^{-s}$ with complex variables demand special conditions for their correctness.

The formula (3.38) with products of Gamma functions includes gammas with variable of integration s which has coefficients $+1$ or -1 or $-\frac{g}{r}$. Definition of Meijer G-function does not include coefficients not equal to $+1$ or -1 but after some special transformations above Mellin-Barnes integral can be re-written as Meijer G-function.

Let consider the Mellin-Barnes integral from formula (3.38), which we denote as MB:

$$MB = \frac{z^{\alpha+\beta-1}}{2\pi i} \int\limits_{\gamma-i\infty}^{\gamma+i\infty} \frac{\Gamma\left(\alpha - \frac{g}{r}s\right)}{\Gamma\left(\alpha+\beta-\frac{g}{r}s\right)} \cdot$$
$$\cdot \frac{\prod\limits_{k=1}^{m} \Gamma(b_k+s)}{\prod\limits_{k=n+1}^{p} \Gamma(a_k+s)} \cdot \frac{\prod\limits_{k=1}^{n} \Gamma(1-a_k-s)}{\prod\limits_{k=m+1}^{q} \Gamma(1-b_k-s)} \left(w^{\frac{1}{r}} z^{\frac{g}{r}}\right)^{-s} ds.$$

Suppose that $\frac{g}{r}$ is the rational value with positive integers g and r (that do not have common factors). Making change of variable $s = r\zeta$ we obtain

$$MB = \frac{z^{\alpha+\beta-1}}{2\pi i} \int\limits_{\delta-i\infty}^{\delta+i\infty} \frac{\Gamma(\alpha - g\zeta)}{\Gamma(\alpha+\beta-g\zeta)} \cdot$$
$$\cdot \frac{\prod\limits_{k=1}^{m} \Gamma(b_k+r\zeta)}{\prod\limits_{k=n+1}^{p} \Gamma(a_k+r\zeta)} \cdot \frac{\prod\limits_{k=1}^{n} \Gamma(1-a_k-r\zeta)}{\prod\limits_{k=m+1}^{q} \Gamma(1-b_k-r\zeta)} \left(w^{\frac{1}{r}} z^{\frac{g}{r}}\right)^{-r\zeta} r d\zeta, \quad \delta = \frac{\gamma}{r}.$$

Since $r, g \in \mathbb{N}$ for each Gamma function in the previous Mellin-Barnes integral, we can apply known Gauss multiplication formulas of the forms (see [144])

$$\Gamma(b+\zeta r) = (2\pi)^{\frac{1-r}{2}} r^{b+\zeta r-\frac{1}{2}} \prod_{j=0}^{r-1} \Gamma\left(\frac{b+j}{r}+\zeta\right), \qquad r \in \mathbb{N}, \tag{3.40}$$

$$\Gamma(\alpha-\zeta g) = (2\pi)^{\frac{1-g}{2}} g^{\alpha-\zeta g-\frac{1}{2}} \prod_{j=0}^{g-1} \Gamma\left(\frac{\alpha+j}{g}-\zeta\right), \qquad g \in \mathbb{N} \tag{3.41}$$

for each Gamma function. We get

$$\mathcal{A}(\alpha,\beta,r)=\frac{\Gamma(\alpha-g\zeta)}{\Gamma(\alpha+\beta-g\zeta)}\cdot\frac{\prod\limits_{k=1}^{m}\Gamma\left(b_k+r\zeta\right)}{\prod\limits_{k=n+1}^{p}\Gamma\left(a_k+r\zeta\right)}\cdot$$
$$\cdot\frac{\prod\limits_{k=1}^{n}\Gamma\left(1-a_k-r\zeta\right)}{\prod\limits_{k=m+1}^{q}\Gamma\left(1-b_k-r\zeta\right)}\cdot\left(w^{\frac{1}{r}}z^{\frac{g}{r}}\right)^{-r\zeta}r=$$
$$=\frac{(2\pi)^{\frac{1-g}{2}}}{(2\pi)^{\frac{1-g}{2}}}\cdot\frac{g^{\alpha-g\zeta-\frac{1}{2}}}{g^{\alpha+\beta-g\zeta-\frac{1}{2}}}\cdot\frac{\prod\limits_{j=0}^{g-1}\Gamma\left(\frac{\alpha+j}{g}-\zeta\right)}{\prod\limits_{j=0}^{g-1}\Gamma\left(\frac{\alpha+\beta+j}{g}-\zeta\right)}\cdot$$
$$\cdot\frac{\left(\prod\limits_{k=1}^{n}(2\pi)^{\frac{1-r}{2}}r^{-a_k-r\zeta+\frac{1}{2}}\right)\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{j-a_k+1}{r}-\zeta\right)}{\left(\prod\limits_{k=m+1}^{q}(2\pi)^{\frac{1-r}{2}}r^{-b_k-r\zeta+\frac{1}{2}}\right)\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{j-b_k+1}{r}-\zeta\right)}\cdot$$
$$\cdot\frac{\left(\prod\limits_{k=1}^{m}(2\pi)^{\frac{1-r}{2}}r^{b_k+r\zeta-\frac{1}{2}}\right)\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{b_k+j}{r}+\zeta\right)}{\left(\prod\limits_{k=n+1}^{p}(2\pi)^{\frac{1-r}{2}}r^{a_k+r\zeta-\frac{1}{2}}\right)\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{a_k+j}{r}+\zeta\right)}\left(wz^g\right)^{-\zeta}r.$$

Simplifying we obtain

$$\mathcal{A}(\alpha,\beta,r)=$$
$$=\frac{g^{-\beta}\prod\limits_{j=0}^{g-1}\Gamma\left(\frac{\alpha+j}{g}-\zeta\right)}{\prod\limits_{j=0}^{g-1}\Gamma\left(\frac{\alpha+\beta+j}{g}-\zeta\right)}\cdot$$
$$\cdot\frac{\prod\limits_{k=1}^{n}(2\pi)^{\frac{1-r}{2}}r^{-a_k-r\zeta+\frac{1}{2}}\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{1-a_k+j}{r}-\zeta\right)}{\prod\limits_{k=m+1}^{q}(2\pi)^{\frac{1-r}{2}}r^{-b_k-r\zeta+\frac{1}{2}}\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{1-b_k+j}{r}-\zeta\right)}\cdot$$
$$\cdot\frac{\prod\limits_{k=1}^{m}(2\pi)^{\frac{1-r}{2}}r^{b_k+r\zeta-\frac{1}{2}}\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{b_k+j}{r}+\zeta\right)}{\prod\limits_{k=n+1}^{p}(2\pi)^{\frac{1-r}{2}}r^{a_k+r\zeta-\frac{1}{2}}\prod\limits_{j=0}^{r-1}\Gamma\left(\frac{a_k+j}{r}+\zeta\right)}\left(wz^g\right)^{-\zeta}r=$$

$$= \frac{\prod_{k=1}^{n}(2\pi)^{\frac{1-r}{2}} r^{-a_k - r\zeta + \frac{1}{2}}}{\prod_{k=m+1}^{q}(2\pi)^{\frac{1-r}{2}} r^{-b_k - r\zeta + \frac{1}{2}}} \cdot \frac{\prod_{k=1}^{m}(2\pi)^{\frac{1-r}{2}} r^{b_k + r\zeta - \frac{1}{2}}}{\prod_{k=n+1}^{p}(2\pi)^{\frac{1-r}{2}} r^{a_k + r\zeta - \frac{1}{2}}} \cdot \frac{\prod_{j=0}^{g-1}\Gamma\left(\frac{\alpha+j}{g} - \zeta\right)}{\prod_{j=0}^{g-1}\Gamma\left(\frac{\alpha+\beta+j}{g} - \zeta\right)} \cdot$$

$$\cdot \frac{\prod_{k=1}^{n}\prod_{j=0}^{r-1}\Gamma\left(\frac{1-a_k+j}{r} - \zeta\right)}{\prod_{k=m+1}^{q}\prod_{j=0}^{r-1}\Gamma\left(\frac{1-b_k+j}{r} - \zeta\right)} \cdot \frac{\prod_{k=1}^{m}\prod_{j=0}^{r-1}\Gamma\left(\frac{b_k+j}{r} + \zeta\right)}{\prod_{k=n+1}^{p}\prod_{j=0}^{r-1}\Gamma\left(\frac{a_k+j}{r} + \zeta\right)} g^{-\beta}\left(wz^g\right)^{-\zeta} r.$$

Collecting powers we get

$$\mathcal{A}(\alpha, \beta, r) = \frac{\left((2\pi)^{\frac{1-r}{2}} r^{\frac{1}{2} - r\zeta}\right)^n}{\left((2\pi)^{\frac{1-r}{2}} r^{\frac{1}{2} - r\zeta}\right)^{q-m}} \cdot \frac{\left((2\pi)^{\frac{1-r}{2}} r^{\zeta r - \frac{1}{2}}\right)^m}{\left((2\pi)^{\frac{1-r}{2}} r^{\zeta r - \frac{1}{2}}\right)^{p-n}} \cdot$$

$$\cdot \frac{\prod_{k=1}^{n} r^{-a_k}}{\prod_{k=m+1}^{q} r^{-b_k}} \cdot \frac{\prod_{k=1}^{m} r^{b_k}}{\prod_{k=n+1}^{p} r^{a_k}} \cdot \frac{\prod_{j=0}^{g-1}\Gamma\left(\frac{\alpha+j}{g} - \zeta\right)}{\prod_{j=0}^{g-1}\Gamma\left(\frac{\alpha+\beta+j}{g} - \zeta\right)} \cdot$$

$$\cdot \frac{\prod_{k=1}^{n}\prod_{j=0}^{r-1}\Gamma\left(\frac{1-a_k+j}{r} - \zeta\right)}{\prod_{k=m+1}^{q}\prod_{j=0}^{r-1}\Gamma\left(\frac{1-b_k+j}{r} - \zeta\right)} \cdot \frac{\prod_{k=1}^{m}\prod_{j=0}^{r-1}\Gamma\left(\frac{b_k+j}{r} + \zeta\right)}{\prod_{k=n+1}^{p}\prod_{j=0}^{r-1}\Gamma\left(\frac{a_k+j}{r} + \zeta\right)} g^{-\beta}\left(wz^g\right)^{-\zeta} r =$$

$$= \frac{\prod_{j=0}^{g-1}\Gamma\left(\frac{\alpha+j}{g} - \zeta\right)}{\prod_{j=0}^{g-1}\Gamma\left(\frac{\alpha+\beta+j}{g} - \zeta\right)} \cdot \frac{\prod_{k=1}^{n}\prod_{j=0}^{r-1}\Gamma\left(\frac{1-a_k+j}{r} - \zeta\right)}{\prod_{k=m+1}^{q}\prod_{j=0}^{r-1}\Gamma\left(\frac{1-b_k+j}{r} - \zeta\right)} \cdot$$

$$\cdot \frac{\prod_{k=1}^{m}\prod_{j=0}^{r-1}\Gamma\left(\frac{b_k+j}{r} + \zeta\right)}{\prod_{k=n+1}^{p}\prod_{j=0}^{r-1}\Gamma\left(\frac{a_k+j}{r} + \zeta\right)} \cdot \frac{r^{\sum_{k=1}^{q} b_k - \sum_{k=1}^{p} a_k + \frac{p-q}{2} + 1}}{(2\pi)^{(r-1)\left(m+n-\frac{p+q}{2}\right)} g^{\beta}} \cdot \left(\frac{wz^g}{r^{r(q-p)}}\right)^{-\zeta}.$$

In above transformations we changed $\left(w^{\frac{1}{r}} z^{\frac{g}{r}}\right)^{-r\zeta}$ to $(wz^g)^{-\zeta}$, which is allowed under some restrictions. But in the result we arrived at products of Gamma functions with coefficients $+1$ or -1 at the variable of integration ζ.

This result allows us to write the following representation of Riemann-Liouville fractional order β integral of the generalized Meijer G-function through Meijer G-

function in the case when $\frac{g}{r}$ is the rational value with positive integers g and r:

$$\frac{1}{\Gamma(\beta)}\int_0^z (z-\tau)^{\beta-1}\tau^{\alpha-1}G_{p,q}^{m,n}\left(w\tau^g, r\,\middle|\,\begin{matrix}a_1,\dots,a_n,a_{n+1},\dots,a_p\\ b_1,\dots,b_m,b_{m+1},\dots,b_q\end{matrix}\right)d\tau$$

$$= z^{\alpha+\beta-1}\frac{r^{\sum\limits_{k=1}^{q}b_k-\sum\limits_{i=1}^{p}a_i+\frac{p-q}{2}+1}}{(2\pi)^{(r-1)\left(m+n-\frac{p+q}{2}\right)}g^{\beta}}G_{rp+g,rq+g}^{rm,rn+g}\left(\frac{wz^g}{r^{r(q-p)}}\,\middle|\,\begin{matrix}\Delta_1,\Delta_2\\ \Delta_3,\Delta_4\end{matrix}\right), \quad (3.42)$$

where

$$\begin{aligned}
\Delta_1 &= \frac{1-\alpha}{g},\dots,\frac{g-\alpha}{g},\frac{a_1}{r},\dots,\frac{a_1+r-1}{r},\dots,\frac{a_n}{r},\dots,\frac{a_n+r-1}{r},\\
\Delta_2 &= \frac{a_{n+1}}{r},\dots,\frac{a_{n+1}+r-1}{r},\dots,\frac{a_p}{r},\dots,\frac{a_p+r-1}{r},\\
\Delta_3 &= \frac{b_1}{r},\dots,\frac{b_1+r-1}{r},\dots,\frac{b_m}{r},\dots,\frac{b_m+r-1}{r},\\
\Delta_4 &= \frac{b_{m+1}}{r},\dots,\frac{b_{m+1}+r-1}{r},\dots,\frac{b_q}{r},\dots,\frac{b_q+r-1}{r},\frac{1-\alpha-\beta}{g},\dots,\frac{g-\alpha-\beta}{g}.
\end{aligned}$$

3.3.6 Big O representations for asymptotics of Meijer *G*-functions

If we use "Big O notation" [136] for description of asymptotic behaviour of Meijer G-function, we get the following generic formulas, describing Big O representation of Meijer G-function near its two or three singular points (see Wolfram Functions Site [138] and [4], p. 571):

$$G_{p,q}^{m,n}\left(z\,\middle|\,\begin{matrix}a_1,\dots,a_n,a_{n+1},\dots,a_p\\ b_1,\dots,b_m,b_{m+1},\dots,b_q\end{matrix}\right)$$
$$\leftrightarrow\begin{cases}\sum\limits_{k=1}^{m} z^{b_k} & \text{if } p\le q;\\ \sum\limits_{k=1}^{m} z^{b_k}+z^{\chi}e^{\frac{(-1)^{q-m-n}}{z}} & \text{if } p=q+1;\\ \sum\limits_{k=1}^{m} z^{b_k}+z^{\chi}\cos\left(2\sqrt{\frac{(-1)^{q-m-n-1}}{z}}\right) & \text{if } p=q+2;\\ \sum\limits_{k=1}^{m} z^{b_k}+z^{\chi}e^{(p-q)(-z)^{\frac{1}{q-p}}} & \text{if } p\ge q+3,\end{cases} \quad (3.43)$$

in (3.43) $|z|\to 0$,

$$\chi=\frac{1}{q-p}\left(\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j+\frac{p-q+1}{2}\right). \quad (3.44)$$

When $z \to (-1)^{m+n-p}$ we get

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right) \leftrightarrow \begin{cases} 1 + \left(1 - (-1)^{p-m-n} z\right)^{\psi_p} & \text{if} q = p \text{and} \psi_p \neq 0; \\ 1 + \log\left(1 - (-1)^{p-m-n} z\right) & \text{if} q = p \text{and} \psi_p = 0; \\ 1 & \text{if } q \neq p, \end{cases} \tag{3.45}$$

where

$$\psi_p = \sum_{j=1}^{p} \left(a_j - b_j\right) - 1. \tag{3.46}$$

Finally, for $|z| \to \infty$

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right) \leftrightarrow \begin{cases} \sum\limits_{k=1}^{n} z^{a_k - 1} & \text{if } q \leq p; \\ \sum\limits_{k=1}^{n} z^{a_k - 1} + z^{\chi} e^{(-1)^{p-m-n} z} & \text{if } q = p+1; \\ \sum\limits_{k=1}^{n} z^{a_k - 1} + z^{\chi} \cos\left(2\sqrt{(-1)^{p-m-n-1} z}\right) & \text{if } q = p+2; \\ \sum\limits_{k=1}^{n} z^{a_k - 1} + z^{\chi} e^{(q-p)(-z)^{\frac{1}{q-p}}} & \text{if } q \geq p+3, \end{cases} \tag{3.47}$$

where χ is defined by (3.44).

3.3.7 Conditions of convergence for Riemann-Liouville integrals with Meijer *G*-functions

Above relations include power functions like z^{b_k} or z^{χ} or $z^{a_k - 1}$ or 1 or $\log(1 - (-1)^{p-m-n} z)$ which match "Big O terms" [136] and used here instead of $O(z^{b_k})$ or $O(z^{\chi})$ or $O(z^{a_k - 1})$ or $O(1)$ or $O(\log(1 - (-1)^{p-m-n} z))$ inside of asymptotic expansions. It allows for a very simple way to establish conditions of convergence integrals involving Meijer G-functions. For example, the classical Riemann-Liouville integral

$$\frac{1}{\Gamma(\beta)} \int_0^z (z-\tau)^{\beta-1} \tau^{\alpha-1} G^{m,n}_{p,q}\left(w\tau^g, r \,\middle|\, \begin{matrix} a_1, ..., a_n, a_{n+1}, ..., a_p \\ b_1, ..., b_m, b_{m+1}, ..., b_q \end{matrix}\right) d\tau \tag{3.48}$$

has Meijer G-function $G^{m,n}_{p,q}[w\tau^g, r|...]$ with parameter r, which under some conditions can be written as classical G-function without parameter $r = 1$ as

$G_{p,q}^{m,n}[w^{\frac{1}{r}}\tau^{\frac{g}{r}}|...]$. The interval of integration here is finite andwe do not need conditions for convergence at infinity if $\frac{g}{r} > 0$ (then $\tau^{\frac{g}{r}} \to \infty$ for $\tau \to \infty$). But for $q < p$ and $\frac{g}{r} > 0$ the point $\tau = 0$ becomes essentially singular and this integral will converge at zero if and only if the following "Big O-equivalent" integrals will converge:

$$\int\limits_0^z \tau^{\alpha-1}\sum_{k=1}^m z^{b_k}d\tau, \qquad p \le q, \tag{3.49}$$

$$\int\limits_0^z \left(\tau^{\alpha-1}\sum_{k=1}^m z^{b_k} + \tau^{\alpha-1}z^{\chi}e^{\frac{(-1)^{q-m-n}}{z}}\right)d\tau, \qquad p = q+1, \tag{3.50}$$

$$\int\limits_0^z \left(\tau^{\alpha-1}\sum_{k=1}^m z^{b_k} + \tau^{\alpha-1}z^{\chi}\cos\left(2\sqrt{\frac{(-1)^{q-m-n}}{z}}\right)\right)d\tau, \qquad p = q+2, \tag{3.51}$$

$$\int\limits_0^z \left(\tau^{\alpha-1}\sum_{k=1}^m z^{b_k} + \tau^{\alpha-1}z^{\chi}e^{(p-q)(-z)^{\frac{1}{q-p}}}\right)d\tau, \qquad p \ge q+3, \tag{3.52}$$

where $z = w^{\frac{1}{r}}\tau^{\frac{g}{r}}$ and $\frac{g}{r} > 0$, χ is defined by (3.44). The convergence of the integral (3.49) takes place when the integral (3.52) in the above chain converges. It can happened under condition

$$\mathrm{M} = \mathrm{Re}(\alpha) + \frac{g}{r}\min_{1\le k\le m}\{\mathrm{Re}(b_k)\} > 0. \tag{3.53}$$

Evidently, that this condition appears in other three situation with $p > q$. Let

$$\theta(\delta) = \begin{cases} 0, \ \delta < 0; \\ 1, \ \delta \ge 0. \end{cases}$$

There we see three additional integrals with exponential and cosine functions, for which one can write and evaluate the following “convergent equivalents”:

$$\int\limits_0^1 x^{\gamma}e^{ax^{\delta}}dx = \frac{(-a)^{-\frac{\gamma+1}{\delta}}}{\delta}\left(\theta(\delta)\Gamma\left(\frac{\gamma+1}{\delta}\right) - \Gamma\left(\frac{\gamma+1}{\delta}, -a\right)\right),$$

where $\mathrm{Re}(\gamma) > -1$ when $\delta > 0$ and $\mathrm{Re}(a) \le 0$ when $\delta < 0$;

$$\int_0^1 x^{\gamma}\cos(ax^{\delta})dx = \frac{1}{\gamma+1}\,{}_1F_2\left(\frac{\gamma+1}{2\delta};\frac{1}{2},\frac{\gamma+1}{2\delta}+1;-\frac{a^2}{4}\right)-$$
$$-\frac{\theta(-\delta)}{\delta}\cos\left(\frac{\pi(\gamma+1)}{2\delta}\right)\Gamma\left(\frac{\gamma+1}{\delta}\right)|a|^{-\frac{\gamma+1}{\delta}},$$

where $a \in \mathbb{R}$, $\delta \in \mathbb{R}$ and $\mathrm{Re}\left(\frac{\gamma+1}{\delta}\right) < 1$.

If we apply above conditions of convergence to mentioned three integrals, re-written in the form

$$\int_0^1 \tau^{\alpha+\frac{g}{r}\chi-1}e^{(-1)^{q-m-n}w^{-\frac{1}{r}}\tau^{-\frac{g}{r}}}d\tau,$$
$$\int_0^1 \tau^{\alpha+\frac{g}{r}\chi-1}\cos\left(2\tau^{\frac{g}{2r}}\sqrt{(-1)^{q-m-n-1}w^{-\frac{1}{r}}}\right)d\tau,$$
$$\int_0^1 \tau^{\alpha+\frac{g}{r}\chi-1}e^{(p-q)\left(-w^{\frac{1}{r}}\right)^{\frac{1}{q-p}}\tau^{\frac{g}{r(q-p)}}}d\tau$$

we come to the following three groups of conditions

$$\begin{cases} \mathrm{Re}\left(\alpha+\frac{g}{r}\chi\right) > 0, & gr < 0; \\ \mathrm{Re}\left(w^{-\frac{1}{r}}(-1)^{q-m-n}\right) \le 0, & gr > 0, \end{cases} \tag{3.54}$$

$$\sqrt{w^{-\frac{1}{r}}(-1)^{q-m-n-1}} \in \mathbb{R} \quad \text{and} \quad \frac{g}{r} \in \mathbb{R} \quad \text{and} \quad 2\mathrm{Re}\left(\frac{\alpha r}{g}+\chi\right) < 1, \tag{3.55}$$

$$\begin{cases} \mathrm{Re}\left(\alpha+\frac{g}{r}\chi\right) > 0, & \frac{g}{r(q-p)} > 0; \\ \mathrm{Re}\left((p-q)\left(-w^{\frac{1}{r}}\right)^{\frac{1}{q-p}}\right) \le 0, & \frac{g}{r(q-p)} < 0. \end{cases} \tag{3.56}$$

After incorporating these conditions into "Big O-equivalent" integrals above we come to the following conditions of convergence of the initial Riemann-Liouville integral at the point $\tau = 0$:

$$\begin{cases} \mathrm{M} > 0, & & p \le q; \\ \mathrm{M} > 0, & \begin{cases} \mathrm{Re}\left(\alpha+\frac{g}{r}\chi\right) > 0, & \frac{g}{r} < 0; \\ \mathrm{Re}\left((-1)^{q-m-n}w^{-\frac{1}{r}}\right) \le 0, & \frac{g}{r} > 0, \end{cases} & p = q+1; \\ \mathrm{M} > 0, & \sqrt{w^{-\frac{1}{r}}(-1)^{q-m-n-1}} \in \mathbb{R},\ \frac{g}{r} \in \mathbb{R},\ \mathrm{Re}\left(\frac{\alpha r}{g}+\chi\right) < \frac{1}{2}, & p = q+2; \\ \mathrm{M} > 0, & \begin{cases} \mathrm{Re}\left(\alpha+\frac{g}{r}\chi\right) > 0, & \frac{g}{r(q-p)} > 0; \\ \mathrm{Re}\left((p-q)\left(-w^{\frac{1}{r}}\right)^{\frac{1}{q-p}}\right) \le 0, & \frac{g}{r(q-p)} < 0, \end{cases} & p \ge q+3, \end{cases} \tag{3.57}$$

with $\frac{g}{r} > 0$, χ is given by (3.44).

Evidently, that we need to add the restriction $\mathrm{Re}(\beta) > 0$ for convergence of the integral at point $\tau = z$ and maybe we can have third singular point of Meijer G-function $w^{\frac{1}{r}}\tau^{\frac{g}{r}} = (-1)^{m+n-p}$, arising for $q = p$.

Let

$$\tau_0 = ((-1)^{m+n-p} w^{\frac{1}{r}})^{\frac{r}{g}}. \tag{3.58}$$

Then under conditions $\tau_0 \in \mathbb{R}$ and $0 < \tau_0 < z$ we should add the restriction

$$\mathrm{Re}\left(\sum_{j=1}^{p}(a_j - b_j)\right) > 0 \tag{3.59}$$

(it is $\mathrm{Re}(\psi_p + 1) > 0$ for $q = p$) which becomes weaker as

$$\mathrm{Re}\left(\sum_{j=1}^{p}(a_j - b_j) + \beta - 1\right) > 0 \tag{3.60}$$

if point τ_0 coincides with z: $\tau_0 = z$ (here restriction $\mathrm{Re}(\beta) > 0$ should be annulated). It provides convergence at the point τ_0 for $0 < \tau_0 \le z$.

Using notations (3.44), (3.46), (3.53) and (3.58) for χ, ψ_p, M and τ_0 we can write the following conditions of convergence for integral (3.48):

$$\begin{cases} \mathrm{Re}(\beta) > 0,\ \ \mathrm{M} > 0, & p < q; \\ \mathrm{M} > 0,\ \ \tau_0 \in \mathbb{R} \text{ and} & \\ \begin{cases} 0 < \tau_0 < z,\ \ \mathrm{Re}\left(\psi_p\right) > -1 & \text{if } \mathrm{Re}(\beta) > 0; \\ \tau_0 = z & \text{if } \mathrm{Re}\left(\beta + \psi_p\right) > 0, \end{cases} & p = q; \\ \mathrm{Re}(\beta) > 0,\ \ \mathrm{M} > 0 \text{ and} & \\ \begin{cases} \mathrm{Re}\left(\alpha + \frac{g}{r}\chi\right) > 0 & \text{if } \frac{g}{r} < 0; \\ \mathrm{Re}\left((-1)^{q-m-n} w^{-\frac{1}{r}}\right) \le 0 & \text{if } \frac{g}{r} > 0, \end{cases} & p = q+1; \\ \mathrm{Re}(\beta) > 0,\ \ \mathrm{M} > 0,\ \ \sqrt{(-1)^{q-m-n-1} w^{-\frac{1}{r}}} \in \mathbb{R}, & \\ \frac{g}{r} \in \mathbb{R},\ \mathrm{Re}\left(\frac{\alpha r}{g} + \chi\right) < \frac{1}{2}, & p = q+2; \\ \mathrm{Re}(\beta) > 0,\ \ \mathrm{M} > 0,\ \text{ and} & \\ \begin{cases} \mathrm{Re}\left(\alpha + \frac{g}{r}\chi\right) > 0 & \text{if } \frac{g}{r(q-p)} > 0; \\ \mathrm{Re}\left((p-q)\left(-w^{\frac{1}{r}}\right)^{\frac{1}{q-p}}\right) \le 0 & \text{if } \frac{g}{r(q-p)} < 0, \end{cases} & p > q+2, \end{cases} \tag{3.61}$$

3.4 Support of differential constants. Generic formulas for fractional differentiation

With the advent of computer algebra systems, such as Wolfram Mathematics, Maple, Matlab, etc., it became necessary to revise approaches to well-known functions.

Computer systems demanded to provide correct numerical evaluations of analytical functions everywhere in complex plane, including on branch lines like $(-\infty, 0)$ for $\sqrt{z}$. It stimulated developers of the Mathematica system to revise mathematical formulas, where the behavior of functions on branch cuts was accurately described not only theoretically, but supported by numerical evaluations in Mathematica. As a result, Wolfram Mathematics uses a simple axiom: "The argument of all complex numbers z satisfies the inequality $-\pi < \arg(z) \leq \pi$." As a result functions are internally consistent with each to other and can be described even on branch cut lines. As a result `Mathematica` makes operations in the full complex plane with all functions involving so-called differential constants (like $\frac{\sqrt{z^2}}{z}$, $\log\left(z^2\right) - 2\log(z)$) and piecewise construction including logarithmic situations. The question of what happens to the fractional integro-differentiation in such cases previously ignored in the literature. Operation `FractionalOrderD` fills this gap.

By definition in Mathematica we have

$$\frac{\sqrt{z^2}}{z} = \begin{cases} 1, & -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}; \\ -1, & \text{in other cases} \end{cases} \tag{3.62}$$

and

$$\log\left(z^2\right) - 2\log(z) = \begin{cases} 0, & -\frac{\pi}{2} < \arg(z) \leq \frac{\pi}{2}; \\ 2i\pi, & -\pi < \arg(z) \leq -\frac{\pi}{2}; \\ -2i\pi, & \text{in other cases.} \end{cases} \tag{3.63}$$

The built-in Mathematica derivative operator `D` misses discontinuities at branch cuts:

$$\mathbf{D}\left[\frac{\sqrt{z^2}}{z}, z\right] = 0 \quad \text{and} \quad \mathbf{D}\left[\log\left(z^2\right) - 2\log(z), z\right] = 0. \tag{3.64}$$

`FractionalOrderD` provides the following results, which saves components $\sqrt{z^2}$ and $\log(z^2) = \log(-iz) + \log(iz)$:

$$\frac{d^\alpha}{dz^\alpha}\frac{\sqrt{z^2}}{z} = \frac{\sqrt{z^2}z^{-\alpha-1}}{\Gamma(1-\alpha)} \tag{3.65}$$

and

$$\frac{d^\alpha}{dz^\alpha}[\log\left(z^2\right) - 2\log(z)] = \frac{z^{-\alpha}(\log(-iz) + \log(iz) - 2\log(z))}{\Gamma(1-\alpha)}. \tag{3.66}$$

Classical (order 1 or 2, etc) differentiation and integration can operate with abstract functions $f(z), g(z), h(z)$ and some their constructions (product, composition, ratio, inverse function, series, etc.). Similar formulas can be derived for fractional integro-differentiation. Many of such formulas we already have in

`ResourceFunction["FractionalOrderD"]`. For example, we have generic formulas for

$$f^{(-1)}(z), \quad f(z)\cdot g(z), \quad f(z)\cdot g(z)\cdot h(z), \quad \frac{1}{f(z)}, \quad \frac{f(z)}{g(z)},$$
$$(f(z))^a, \quad (f(z))^{g(z)}, \quad f(z)^{g(z)^{h(z)}}, \quad (az)^b \log^c(d\cdot f(z)),$$
$$c^{bf(z)^a+d}, \quad c^{bf(z^a)+d}, \quad c^{bf(a^z)+d}, \quad f(g(z)), \quad f(g(h(z))), \quad \text{etc.}$$

Here $f^{(-1)}(z)$ is an inverse for $f(z)$ function. Below we give one rather simple example for product $f(z)\cdot g(z)$:

$$\frac{d^\alpha}{dz^\alpha}(f(z)g(z)) = \begin{cases} \sum\limits_{k=0}^{\alpha} \binom{\alpha}{k} f^{(\alpha-k)}(z)g^{(k)}(z) & \text{if } z\in\mathbb{Z}, \alpha\geq 0; \\ \sum\limits_{k=0}^{\alpha} \binom{\alpha}{k} \frac{d^{\alpha-k}f(z)}{dz^{\alpha-k}}\frac{d^k g(z)}{dz^k} & \text{in other cases.} \end{cases} \tag{3.67}$$

Of course, most formulas of fractional integro-differentiation of abstract functions are very complicated and large. But in the case of specific formulas, we get simpler expressions. For example,

$$\frac{d^\alpha}{dz^\alpha}e^{z^2} = \frac{\sqrt{\pi}2^\alpha z^{-\alpha}}{\Gamma\left(1-\frac{\alpha+1}{2}\right)\Gamma\left(1-\frac{\alpha}{2}\right)}\,{}_2F_2\left(\frac{1}{2},1;1-\frac{\alpha+1}{2},1-\frac{\alpha}{2};z^2\right). \tag{3.68}$$

In recent years Wolfram Language has added some new generic functions

- 10 Heun functions,
- 4 Lamé functions,
- 8 Carlson elliptic integrals,
- Fox H–function,
- 4 Coulomb functions.

For these functions corresponding formulas of fractional order integro-differentiation can be obtained. For example, we consider Carlson's elliptic integral (see [145])

$$R_D(x,y,z) = \frac{3}{2}\int_0^\infty (t+x)^{-1/2}(t+y)^{-1/2}(t+z)^{-3/2}dt, \quad x>0, \quad y>0, \quad z>0 \tag{3.69}$$

and Heun G-function $H(a,b,c,d,p,q,z)$ (see [146]).

Heun G-function $H(a,b,c,d,p,q,z)$ specializes to ${}_2F_1(c,d,p;z)$ if $b=a\cdot c\cdot d$ and $q=c+d-p+1$ or $a=1$ and $b=c\cdot d$.

We obtain

$$\frac{d^\alpha}{dz^\alpha} R_D\left(\frac{1}{2}, z, 1\right) =$$

$$\begin{cases} \frac{3(-1)^\alpha}{2\alpha+3}\left(\frac{1}{2}\right)_\alpha F_1\left(\alpha+\frac{3}{2}; \frac{1}{2}, 1; \alpha+\frac{5}{2}; \frac{1}{2}, 1-z\right), & \alpha \in \mathbb{Z}, \alpha \geq 0; \\ \frac{3\pi z^{-\alpha}}{4\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \frac{{}_2F_1\left(\frac{1}{2}, k+\frac{3}{2}; 2; \frac{1}{2}\right)\left(\frac{3}{2}\right)_k z^k}{(1-\alpha)_k} & \\ -\frac{3\sqrt{\pi} z^{\frac{1}{2}-\alpha}}{2\Gamma\left(\frac{3}{2}-\alpha\right)} \sum_{k=0}^{\infty} \frac{(k+1)!\, {}_2F_1\left(\frac{1}{2}, k+2; 2; \frac{1}{2}\right) z^k}{\left(\frac{3}{2}-\alpha\right)_k}, & \text{in other cases} \end{cases} \tag{3.70}$$

and

$$\frac{d^\alpha}{dz^\alpha} H(a, b, c, d, p, q, z) = \sum_{k=0}^{\infty} \frac{k! c_k z^{k-\alpha}}{\Gamma(k-\alpha+1)}, \tag{3.71}$$

where $|z| < \min(1, |a|)$, $c_0 = 1$, $c_1 = \frac{b}{ap}$, $c_j = -\frac{c_{j-2}P_{j-2}+c_{j-1}Q_{j-1}}{R_j}$, $P_j = (c + j)(d + j)$, $Q_j = -j(a(j + p + q - 1) + c + d + j - q) - b$, $R_j = aj(j + p - 1)$.

Heun G-function, in general, is not hypergeometric type function and its series representation has coefficients $c_k = \frac{k! c_k}{\Gamma(k-\alpha+1)}$, $\alpha = 0$, which satisfies three-terms recurrent relations, which were described in `ConditionalExpression`.

Currently we have general rules for: $a^{g(z)}$, $\left(a^{bf(z)}\right)^c$, $c^{ba^{f(z)}+d}$, $c^{bf(a^z)+d}$, $c^{bf(z)^a+d}$, $c^{bf(z^a)}$, $e^{g(z)}$, $e^{g(h(z))}$, $f(a^z)$, $(f(a^{bz}))^c$, $f\left(a^{bz^c}\right)$, $\frac{1}{f(z)}$, $zf(z)$, $z^2 f(z)$, $(f(z))^a$, $(f(z))^{g(z)}$, $(f(z))^{(g(z))^{h(z)}}$, $f(z^a)$, $f(g(z))$, $f(g(h(z)))$, $\frac{f(z)}{g(z)}$, $f(z)\cdot g(z)$, $f(z)\cdot g(z)\cdot h(z)$, $f^{(-1)}(z)$.

4 Conclusion

Having considered various approaches to introducing an arbitrary power of a differential operator $\frac{d}{dx}$, we come to the conclusion that these approaches are not so different. So, when calculating various fractional derivatives of a power function x^p, we almost always get the same result. This makes the considered derivatives coincide on the class of analytic functions. On the other hand, if there are additional parameters in the fractional derivative, then the result of applying such an operator to a power function will depend on these parameters. In some applied problems, this makes sense. Another reason for introducing different definitions of fractional derivatives is to apply regularizations of divergent integrals. Wolfram Mathematica uses the Hadamard regularization of divergent integrals. In addition, Mathematics systematizes formulas for fractional derivatives of general form and for general functions.

Finally, returning to main question: "How maximally naturally make extensions of differentiation $\frac{d^n}{dx^n}$ from natural order $n = 1, 2, 3, \ldots$ to arbitrary symbolical (or complex) order α" we can say the following: "Like in the case of extension factorial $n!$ from natural n to Gamma function $\Gamma(\alpha+1)$ $(\alpha = n)$ we do not have unique solution in such extension (corresponding details for $n!$ see in [73], [74] page 43.). As result we see in literature the numerous approaches to definitions of fractional order differentiation by variable x. Corresponding inversions define fractional order integrations, which in majority cases include additional point a at the end of integral from a to x, which demands additional restrictions for convergence. Existence of this point distracts attention and it is naturally to use $a = 0$ as the end of integral from 0 to x and allow x to be negative or complex. Such construction leads to Riemann-Liouville left-sided fractional integral with beginning at 0 and Riemann-Liouville fractional derivative described at formula (3.1). After involving Hadamard's concept of the "finite part" we avoid influence of other "aside" point $(x = 0)$ on procedure of fractional integro-differentiation in our main point x, which effectively works in all complex x-plane for any analytical functions in their regular or branch points, including logarithmic cases ($\log(x)$, $\frac{1}{x}$, x^c, $(x^n)^{1/m}$, etc.). It allowed to form described above Riemann-Liouville-Hadamard fractional order integro-differentiation, which looks as maximally natural extension of integer order differentiation and its inversion (integer order repeatable indefinite integration without arbitrary polynomials, that arises). This statement follows from comparison of different approaches to fractional order integro-differentiation between themself and how they realized on the basic simplest functions z^λ, e^z, etc. which is widely used in Taylor and Fourier series."

Additional information about Fractional Differentiation one can find in the talk of Oleg Marichev, Paco Jain "New in Fractional Differentiation" at Wolfram virtual Technology Conference (October 12-15, 2021) [147], which was extended to the next talk of Oleg Marichev "Compositional Structure of Classical Integral Transforms" at the Wolfram Technology Conference (October 18-21, 2022) [148]. Later these results got development at the talk of Marichev O., Shishkina E. [75] and set of Internet presentations of Oleg Marichev [149–151].

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Declarations

Conflict of interest The authors declare that they have no conflict of interest.

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